Discrete Mathematics

Lecture 4: Propositional Logic and Predicate Logic

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- A statement can be formed using other statements connected to each other by 5 kinds of connectives: AND, OR, NOT, IMPLIES and IFF.
- A statement can have unspecified terms, called variables. All the variables has to be properly quantified using the two quantifiers FOR-ALL and THERE EXISTS.

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A	B	$A \implies B$	$(\neg B \wedge A) = F$
F	F		
F	T		
T	F		
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F	T	T	T
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Problems on Propositional Logic

- Prove that $(a \implies b)$ is equivalent to $(\neg a \vee b)$.
- Prove that $(a \iff b)$ is equivalent to $(a \implies b) \wedge (b \implies a)$.
- Prove that $(p \vee q) \implies r$ is equivalent to $(p \implies r) \wedge (q \implies r)$.
- Prove that $(a \implies b)$ is equivalent to $(\neg b \implies \neg a)$.

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Another approach is to use some already proved rules to simplify the formulas before using the brute force truthtable approach.

Rules of Propositional Logic

Let p, q and r be propositions.

1. Commutative law:

$$(p \vee q) = (q \vee p) \text{ and } (p \wedge q) = (q \wedge p)$$

2. Associative law:

$$(p \vee (q \vee r)) = ((p \vee q) \vee r) \text{ and } (p \wedge (q \wedge r)) = ((p \wedge q) \wedge r)$$

3. Distributive law:

$$(p \vee (q \wedge r)) = (p \vee q) \wedge (p \vee r) \text{ and}$$
$$(p \wedge (q \vee r)) = (p \wedge q) \vee (p \wedge r)$$

4. De Morgan's Law:

$$\neg(p \vee q) = (\neg p \wedge \neg q) \text{ and } \neg(p \wedge q) = (\neg p \vee \neg q)$$

Rules for Negation

- $\neg(\forall x\ P(x))$ is same as $\exists x\ (\neg P(x))$
- $\neg(\exists x\ P(x))$ is same as $\forall x\ (\neg P(x))$

Problems on Propositional Logic

The function

$$\Big(\ (p \vee (r \vee q))\ \ \wedge \neg\big(p \wedge (\neg q \wedge \neg r)\big)\Big)$$

is equal to which of the following functions:

A. $q \vee r$

B. $\neg p \vee (r \wedge q)$

C. $(p \vee q) \vee r$

D. $(p \vee q) \wedge \neg(p \vee r)$

E. $(p \wedge r) \vee (p \wedge q)$

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Negating a sentence

What is the negation of the sentence: "There is an university in USA where every department has at least 20 faculty and at least one noble laureate."

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What is the negation of the sentence: "There is an university in USA where every department has at least 20 faculty and at least one noble laureate."

A. There is an university in USA where every department has less than 20 faculty and at least one noble laureate.

B. All universities in USA where every department has at least 20 faculty and at least one noble laureate.

C. For all universities in USA there is a department has less than 20 faculty or at most one noble laureate.

D. For all universities in USA there is a department has less than 20 faculty and at least one noble laureate.

Propositional Logic and Predicate Logic

- Every statement is either TRUE or FALSE
- There are logical connectives $\vee$, $\wedge$, $\neg$, $\implies$ and $\iff$.
- Two logical statements can be equivalent if the two statements answer exactly in the same way on every input.
- To check whether two logical statements are equivalent one can do one of the following:
  - Checking the Truthtable of each statement
  - Reducing one to the other using reductions

Propositional Logic and Predicate Logic

- There are two important symbols: $\forall$ and $\exists$.
- Some statements can be defined using a variable.
- For example: $P_x =$ "$4x^2 + 3$ is divisible by 5"
- We can have statements like: $\forall x \in \mathbb{Z}$, $4x^2 + 3$ is divisible by 5.
- Or $\exists x \in \mathbb{Z}$, $4x^2 + 3$ is divisible by 5.

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- Depending on whether A or B (or both) can be split into smaller statements and how the smaller statements are connected we can design different techniques for proving the overall statement of $A \implies B$.
- If indeed we can proof that the statement is correct then we can call it a Theorem.

Proof Techniques

To prove statement B from A.

- Constructive Proofs
- Proof by Contradiction
- Proof by Contrapositive
- Induction
- Counter example
- Existential Proof