Discrete Mathematics

Lecture 11: Proof Technique (Contradiction)

Instructor: Sourav Chakraborty

Proof Techniques

To prove statement $A \implies B$.

There are different proof techniques:

- Constructive Proofs
- Proof by Contradiction
- Proof by Contrapositive
- Induction
- Counter example
- Existential Proof

Which approach to apply

Which approach to apply

- It depends on the problem.

Which approach to apply

- It depends on the problem.
- Sometimes the problem can be split into smaller problems that can be easier to tackle individually.

Which approach to apply

- It depends on the problem.
- Sometimes the problem can be split into smaller problems that can be easier to tackle individually.
- Sometimes viewing the problem is a different way can also help in tackling the problem easily.

Which approach to apply

- It depends on the problem.
- Sometimes the problem can be split into smaller problems that can be easier to tackle individually.
- Sometimes viewing the problem is a different way can also help in tackling the problem easily.
- Whether to split a problem or how to split a problem or how to look at a problem is an ART that has to be developed.

Which approach to apply

- It depends on the problem.
- Sometimes the problem can be split into smaller problems that can be easier to tackle individually.
- Sometimes viewing the problem is a different way can also help in tackling the problem easily.
- Whether to split a problem or how to split a problem or how to look at a problem is an ART that has to be developed.
- There are some thumb rules but at the end it is a skill you develop using a lot of practice.

Tricks for solving problems

- (Splitting into smaller problem) If the problem is to prove $A \implies B$ and B can be written as $B = C \wedge D$ then note that

$$(A \implies B) \equiv (A \implies C \wedge D) \equiv (A \implies C) \wedge (A \implies D).$$

- (Remove Redundant Assumptions) If $A \implies B$ then $A \wedge C$ also implies B.

Tricks for solving problems

- (Splitting into smaller problem) If the problem is to prove $A \implies B$ and B can be written as $B = C \wedge D$ then note that

$$(A \implies B) \equiv (A \implies C \wedge D) \equiv (A \implies C) \wedge (A \implies D).$$

- (Remove Redundant Assumptions) If $A \implies B$ then $A \wedge C$ also implies B.

$$(A \implies B) \implies (A \wedge C \implies B) = True$$

Tricks for solving problems

- (Splitting into smaller problem) If the problem is to prove $A \implies B$ and B can be written as $B = C \wedge D$ then note that

$$(A \implies B) \equiv (A \implies C \wedge D) \equiv (A \implies C) \wedge (A \implies D).$$

- (Remove Redundant Assumptions) If $A \implies B$ then $A \wedge C$ also implies B.

$$(A \implies B) \implies (A \wedge C \implies B) = True$$

- (Sometimes proving something stronger is easier) If $C \implies B$ then

$$(A \implies C) \implies (A \implies B).$$

Constructive Proof: Direct Proof

- For proving $A \implies B$ we can start with the assumption A and step-by-step prove that B is true.

Constructive Proof: Direct Proof

- For proving $A \implies B$ we can start with the assumption A and step-by-step prove that B is true.
- Sometimes a direct proof (as in the previous example) can be magical and hard to understand how to obtain.

Constructive Proof: Direct Proof

- For proving $A \implies B$ we can start with the assumption A and step-by-step prove that B is true.
- Sometimes a direct proof (as in the previous example) can be magical and hard to understand how to obtain.
- A simpler technique is to have a backward proof.

Constructive Proof: Direct Proof

- For proving $A \implies B$ we can start with the assumption A and step-by-step prove that B is true.
- Sometimes a direct proof (as in the previous example) can be magical and hard to understand how to obtain.
- A simpler technique is to have a backward proof.
- If we have to prove $(A \implies B)$ then the idea is to simplify B.

Constructive Proof: Direct Proof

- For proving $A \implies B$ we can start with the assumption A and step-by-step prove that B is true.
- Sometimes a direct proof (as in the previous example) can be magical and hard to understand how to obtain.
- A simpler technique is to have a backward proof.
- If we have to prove $(A \implies B)$ then the idea is to simplify B.
- And if $C \iff B$ then $(A \implies B) \equiv (A \implies C)$.

Constructive Proof: Case Studies

Constructive Proof: Case Studies

- Sometimes the assumption or the premise can be split into different cases. In that case we can split the problem according to cases.

Constructive Proof: Case Studies

- Sometimes the assumption or the premise can be split into different cases. In that case we can split the problem according to cases.
- If $A = C \vee D$ then

$$(A \implies B) \equiv (C \implies B) \wedge (D \implies B).$$

Proof by Contradiction

- Note that

$$(A \implies B) \equiv (\neg B \wedge A = \quad \text{False})$$

  This is called "proof by contradiction"

- To proof $A \implies B$ sometimes its easier to prove that

$$\neg B \wedge A = \quad \text{False}.$$

- A similar statement is

$$(A \implies B) \equiv (\neg B \implies \neg A)$$

  This is called "proof by contra-positive"

Proof by Contradiction

Proof by Contradiction

Example: **Prove that earth is not flat.**

Proof by Contradiction

Example: **Prove that earth is not flat.**

Attempt 1:

Proof by Contradiction

Example: **Prove that earth is not flat.**

Attempt 1: If a ship is coming from the horizon we first see the mast (top) of the ship and slowly the complete ship. So the earth must be round hence not flat.

Proof by Contradiction

Example: **Prove that earth is not flat.**

Attempt 1: If a ship is coming from the horizon we first see the mast (top) of the ship and slowly the complete ship. So the earth must be round hence not flat.

Attempt 2:

Proof by Contradiction

Example: **Prove that earth is not flat.**

Attempt 1: If a ship is coming from the horizon we first see the mast (top) of the ship and slowly the complete ship. So the earth must be round hence not flat.

Attempt 2: Lets assume the earth is flat. Then when a ship came from the horizon the whole ship would appear at the same time.

Proof by Contradiction

Example: **Prove that earth is not flat.**

Attempt 1: If a ship is coming from the horizon we first see the mast (top) of the ship and slowly the complete ship. So the earth must be round hence not flat.

Attempt 2: Lets assume the earth is flat. Then when a ship came from the horizon the whole ship would appear at the same time.
But that does not happen - first the mast is seen then the whole ship. So a contradiction.

Proof by Contradiction

Example: **Prove that earth is not flat.**

Attempt 1: If a ship is coming from the horizon we first see the mast (top) of the ship and slowly the complete ship. So the earth must be round hence not flat.

Attempt 2: Lets assume the earth is flat. Then when a ship came from the horizon the whole ship would appear at the same time.
But that does not happen - first the mast is seen then the whole ship. So a contradiction.
Hence initial assumption that earth is flat does not hold.

Problem

- A real number is rational if it can be written as p/q where p and q are two integers.
- For example: $1, 2, 3, 2/3, 49/99$ are rational numbers.

Problem

- A real number is rational if it can be written as p/q where p and q are two integers.
- For example: $1, 2, 3, 2/3, 49/99$ are rational numbers.

Problem

Prove that $\sqrt{3}$ is not a rational number.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.
- Case 1: Both p and q are not divisible by 3.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.
- Case 1: Both p and q are not divisible by 3.
- Case 2: p is not-divisible by 3 and q is divisible by 3.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.
- Case 1: Both p and q are not divisible by 3.
- Case 2: p is not-divisible by 3 and q is divisible by 3.
- Case 3: p is divisible by 3 and q is not divisible by 3.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.
- Case 1: Both p and q are not divisible by 3.
- Case 2: p is not-divisible by 3 and q is divisible by 3.
- Case 3: p is divisible by 3 and q is not divisible by 3.

If for all the above cases we prove that $3q^2 = p^2$ is not a possibility then we are done.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.
- Case 1: Both p and q are not divisible by 3.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.
- Case 1: Both p and q are not divisible by 3.

$3q^2$ is divisible by 3.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.
- Case 1: Both p and q are not divisible by 3.

$3q^2$ is divisible by 3.
p^2 is not divisible by 3.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.
- Case 1: Both p and q are not divisible by 3.

  $3q^2$ is divisible by 3.
  p^2 is not divisible by 3.
  So $3q^2$ cannot be equal to p^2.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.
- Case 2: p is not-divisible by 3 and q is divisible by 3.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.
- Case 2: p is not-divisible by 3 and q is divisible by 3.

$3q^2$ is divisible by 3.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.
- Case 2: p is not-divisible by 3 and q is divisible by 3.

$3q^2$ is divisible by 3.
p^2 is not divisible by 3.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.
- Case 2: p is not-divisible by 3 and q is divisible by 3.

$3q^2$ is divisible by 3.
p^2 is not divisible by 3.
So $3q^2$ cannot be equal to p^2.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.
- Case 3: p is divisible by 3 and q is not-divisible by 3.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.
- Case 3: p is divisible by 3 and q is not-divisible by 3.

  Let $p = 3k$.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.
- Case 3: p is divisible by 3 and q is not-divisible by 3.

Let $p = 3k$. So $3q^2 = p^2 \iff 3q^2 = 9k^2 \iff q^2 = 3k^2$
$3k^2$ is divisible by 3.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.
- Case 3: p is divisible by 3 and q is not-divisible by 3.

  Let $p = 3k$. So $3q^2 = p^2 \iff 3q^2 = 9k^2 \iff q^2 = 3k^2$
  $3k^2$ is divisible by 3.
  q^2 is not divisible by 3.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.
- Case 3: p is divisible by 3 and q is not-divisible by 3.

  Let $p = 3k$. So $3q^2 = p^2 \iff 3q^2 = 9k^2 \iff q^2 = 3k^2$
  $3k^2$ is divisible by 3.
  q^2 is not divisible by 3.
  So $3k^2$ cannot be equal to q^2.

Prove $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.
- Case 3: p is divisible by 3 and q is not-divisible by 3.

Let $p = 3k$. So $3q^2 = p^2 \iff 3q^2 = 9k^2 \iff q^2 = 3k^2$
$3k^2$ is divisible by 3.
q^2 is not divisible by 3.
So $3k^2$ cannot be equal to q^2.
So $3q^2$ cannot be equal to p^2.

Overview of the proof that $\sqrt{3}$ is not rational

We prove by contradiction.

Overview of the proof that $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$

Overview of the proof that $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We can assume p and q has no common factors else we can factor it out.
- In other words we can assume both p and q cannot be divisible by 3.

Overview of the proof that $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We can assume p and q has no common factors else we can factor it out.
- In other words we can assume both p and q cannot be divisible by 3.
- Now $\sqrt{3} = p/q \iff 3 = p^2/q^2 \iff 3q^2 = p^2$

Overview of the proof that $\sqrt{3}$ is not rational

We prove by contradiction.

- Let $\sqrt{3} = p/q$
- We can assume p and q has no common factors else we can factor it out.
- In other words we can assume both p and q cannot be divisible by 3.
- Now $\sqrt{3} = p/q \iff 3 = p^2/q^2 \iff 3q^2 = p^2$
- We prove by case by case analysis that if p and q are integers, not both divisible by 3 then $3q^2$ cannot be equal to p^2 and hence we get a contradiction.

Problems for practice

- Prove that $\sqrt{2}$ is not rational.
- Prove that $\sqrt{5}$ is not rational.
- Prove that $\sqrt{6}$ is not rational.

Rational Numbers

A number x is rational if it can be written as p/q where p and q are integers.

Rational Numbers

A number x is rational if it can be written as p/q where p and q are integers.

Prove that:

- Rational $\times$ Rational $=$ Rational

Rational Numbers

A number x is rational if it can be written as p/q where p and q are integers.

Prove that:

- Rational $\times$ Rational $=$ Rational
- Rational $\times$ Not Rational $=$ Not Rational.

Rational Numbers

A number x is rational if it can be written as p/q where p and q are integers.

Prove that:

- Rational $\times$ Rational $=$ Rational
- Rational $\times$ Not Rational $=$ Not Rational.
  So $(-\sqrt{3})$ is not rational.

Rational Numbers

A number x is rational if it can be written as p/q where p and q are integers.

Prove that:

- Rational $\times$ Rational $=$ Rational
- Rational $\times$ Not Rational $=$ Not Rational.
  So $(-\sqrt{3})$ is not rational.
- $1/Rational$ is rational.

Rational Numbers

A number x is rational if it can be written as p/q where p and q are integers.

Prove that:

- Rational $\times$ Rational $=$ Rational
- Rational $\times$ Not Rational $=$ Not Rational.
  So $(-\sqrt{3})$ is not rational.
- $1/Rational$ is rational.
- $1/(not\ rational)$ is not rational.

Rational Numbers

A number x is rational if it can be written as p/q where p and q are integers.

Prove that:

- Rational $\times$ Rational $=$ Rational
- Rational $\times$ Not Rational $=$ Not Rational.
  So $(-\sqrt{3})$ is not rational.
- $1/Rational$ is rational.
- $1/(not\ rational)$ is not rational. $1/\sqrt{3}$ is not rational.

Rational Numbers

A number x is rational if it can be written as p/q where p and q are integers.

Prove that:

- Rational $\times$ Rational $=$ Rational
- Rational $\times$ Not Rational $=$ Not Rational.
  So $(-\sqrt{3})$ is not rational.
- $1/\mathit{Rational}$ is rational.
- $1/(\mathit{not\ rational})$ is not rational. $1/\sqrt{3}$ is not rational.
- Not Rational $\times$ Not Rational $=$?

Rational Numbers

A number x is rational if it can be written as p/q where p and q are integers.

Prove that:

- Rational $\times$ Rational $=$ Rational
- Rational $\times$ Not Rational $=$ Not Rational.
  So $(-\sqrt{3})$ is not rational.
- $1/\mathit{Rational}$ is rational.
- $1/(\mathit{not\ rational})$ is not rational. $1/\sqrt{3}$ is not rational.
- Not Rational $\times$ Not Rational $=$?

Is $\sqrt{2}+\sqrt{3}$ a rational?

Prove that $\sqrt{2}+\sqrt{3}$ is not rational.

Is $\sqrt{2} + \sqrt{3}$ a rational?

Prove that $\sqrt{2} + \sqrt{3}$ is not rational.
To prove by contradiction what do have to prove:

Is $\sqrt{2}+\sqrt{3}$ a rational?

Prove that $\sqrt{2}+\sqrt{3}$ is not rational.
To prove by contradiction what do have to prove:

- Let $\sqrt{2}+\sqrt{3}$ be a rational number

Is $\sqrt{2} + \sqrt{3}$ a rational?

Prove that $\sqrt{2} + \sqrt{3}$ is not rational.
To prove by contradiction what do have to prove:

- Let $\sqrt{2} + \sqrt{3}$ be a rational number
- $\sqrt{2} + \sqrt{3}$ can be written as $\frac{p}{q}$ for any posetive integer p and q.

Is $\sqrt{2}+\sqrt{3}$ a rational?

Prove that $\sqrt{2}+\sqrt{3}$ is not rational.
To prove by contradiction what do have to prove:

- Let $\sqrt{2}+\sqrt{3}$ be a rational number
- $\sqrt{2}+\sqrt{3}$ can be written as $\frac{p}{q}$ for any posetive integer p and q.
- If $\sqrt{2}+\sqrt{3}=\frac{p}{q}$ for some posetive integers p and q then there is some problem

Is $\sqrt{2}+\sqrt{3}$ a rational?

Prove that $\sqrt{2}+\sqrt{3}$ is not rational.

- Let $\sqrt{2}+\sqrt{3}=p/q$

Is $\sqrt{2}+\sqrt{3}$ a rational?

Prove that $\sqrt{2}+\sqrt{3}$ is not rational.

- Let $\sqrt{2}+\sqrt{3}=p/q$
- $\iff \sqrt{3}=p/q-\sqrt{2}$

Is $\sqrt{2} + \sqrt{3}$ a rational?

Prove that $\sqrt{2} + \sqrt{3}$ is not rational.

- Let $\sqrt{2} + \sqrt{3} = p/q$
- $\iff \sqrt{3} = p/q - \sqrt{2}$
- $\iff 3 = (p^2/q^2) - 2\sqrt{2}p/q + 2$

Is $\sqrt{2} + \sqrt{3}$ a rational?

Prove that $\sqrt{2} + \sqrt{3}$ is not rational.

- Let $\sqrt{2} + \sqrt{3} = p/q$
- $\iff \sqrt{3} = p/q - \sqrt{2}$
- $\iff 3 = (p^2/q^2) - 2\sqrt{2}p/q + 2$
- $\iff 2\sqrt{2}p/q = (p^2/q^2) - 1 = (p^2 - q^2)/q^2$

Is $\sqrt{2}+\sqrt{3}$ a rational?

Prove that $\sqrt{2}+\sqrt{3}$ is not rational.

- Let $\sqrt{2}+\sqrt{3}=p/q$
- $\iff \sqrt{3}=p/q-\sqrt{2}$
- $\iff 3=(p^2/q^2)-2\sqrt{2}p/q+2$
- $\iff 2\sqrt{2}p/q=(p^2/q^2)-1=(p^2-q^2)/q^2$
- $$\iff \sqrt{2}=\frac{(p^2-q^2)}{2pq}=\frac{p'}{q'}$$

Is $\sqrt{2} + \sqrt{3}$ a rational?

Prove that $\sqrt{2} + \sqrt{3}$ is not rational.

- Let $\sqrt{2} + \sqrt{3} = p/q$
- $\iff \sqrt{3} = p/q - \sqrt{2}$
- $\iff 3 = (p^2/q^2) - 2\sqrt{2}p/q + 2$
- $\iff 2\sqrt{2}p/q = (p^2/q^2) - 1 = (p^2 - q^2)/q^2$
- $$\iff \sqrt{2} = \frac{(p^2 - q^2)}{2pq} = \frac{p'}{q'}$$

So $\sqrt{2}$ is a rational since $(p^2 - q^2)$ and $2pq$ are integers.

Is $\sqrt{2}+\sqrt{3}$ a rational?

Prove that $\sqrt{2}+\sqrt{3}$ is not rational.

So If $\sqrt{2}+\sqrt{3}$ is rational then $\sqrt{2}$ is rational which is a contradiction.

Is $\sqrt{2}+\sqrt{3}$ a rational?

Prove that $\sqrt{2}+\sqrt{3}$ is not rational.

So If $\sqrt{2}+\sqrt{3}$ is rational then $\sqrt{2}$ is rational which is a contradiction.

Thus our initial assumption was wrong. Thus $\sqrt{2}+\sqrt{3}$ is not a rational number.