Discrete Mathematics

Lecture 2: Sets, Relations and Functions

Instructor: Sourav Chakraborty

Definition of Sets

- A collection of objects in called a set .

Definition of Sets

- A collection of objects in called a set .
- The objects that comprises of the set are called elements .
- Number of objects in a set can be finite or infinite.

Definition of Sets

- A collection of objects in called a set .
- The objects that comprises of the set are called elements .
- Number of objects in a set can be finite or infinite.
- For Example:
  - a set of chairs,
  - the set of nobel laureates in the worlds,
  - the set of integers,
  - the set of natural numbers less than 10,
  - the set of points in the plane $\mathbb{R}^2$.

Definition of Sets

- A collection of objects in called a set .
- The objects that comprises of the set are called elements .
- Number of objects in a set can be finite or infinite.
- For Example:
  - a set of chairs,
  - the set of nobel laureates in the worlds,
  - the set of integers,
  - the set of natural numbers less than 10,
  - the set of points in the plane $\mathbb{R}^2$.
- The number of elements in a set is called the cardinality of the set. (If S is a set the cardinality is denoted by $|S|$)

Notations related to set

- Usually a set is represented by its list of elements separated by comma, between two curly brackets. For example $\{1, 2, 3, 4, 5\}$ is the list of integers bigger than 0 and lesser than or equal to 5.

Notations related to set

- Usually a set is represented by its list of elements separated by comma, between two curly brackets. For example $\{1, 2, 3, 4, 5\}$ is the list of integers bigger than 0 and lesser than or equal to 5.
- If S is a set and we want to denote that x is an element of the set we write as $x \in S$.

Notations related to set

- Usually a set is represented by its list of elements separated by comma, between two curly brackets. For example $\{1, 2, 3, 4, 5\}$ is the list of integers bigger than 0 and lesser than or equal to 5.
- If S is a set and we want to denote that x is an element of the set we write as $x \in S$.
- If S is a set and T is another set such that all the elements of T is contained in the set set S then T is called a subset of the set S and is denoted as $T \subseteq S$ or $T \subset S$ (depending on whether the containment is strict of not). Conversely, in this case S is called a super-set of T.

Kinds of Sets

- Usually by a set we mean a collection of elements where the ordering of the elements in the set does not matter and no element is repeated. For example: the set $\{3, 1, 2, 2, 4, 4\}$ is actually thought of as $\{1, 2, 3, 4\}$.
- But in some context we may have to allow repetitions. We call them multisets . Thus in the multiset $\{1, 1, 2\}$ is different from $\{1, 2\}$ is different from $\{1, 1, 1, 2\}$.
- Sometimes we care about the ordering of the elements in the set. We call them ordered sets . They are sometime referred as lists or strings or vectors. For example: the ordered set $\{1, 2, 3\}$ is different from $\{2, 1, 3\}$.
- We can also have ordered multi-sets.

Operations on Sets

- Union, $\cup$.
  $A \cup B$ is the set of all elements that are in A OR B.

Operations on Sets

- Union, $\cup$.
  $A \cup B$ is the set of all elements that are in A OR B.
- Intersection, $\cap$
  $A \cap B$ is the set of all elements that are in A AND B.

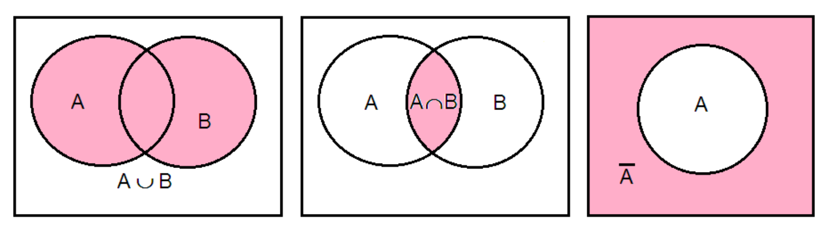
Operations on Sets

- Union, $\cup$.
  $A \cup B$ is the set of all elements that are in A OR B.
- Intersection, $\cap$
  $A \cap B$ is the set of all elements that are in A AND B.
- Complement, A^c or $\overline{A}$
  A^c is the set of elements NOT in A.

Operations on Sets

- Union, $\cup$.
  $A \cup B$ is the set of all elements that are in A OR B.
- Intersection, $\cap$
  $A \cap B$ is the set of all elements that are in A AND B.
- Complement, A^c or $\overline{A}$
  A^c is the set of elements NOT in A.
- Cartesian Product.

Union, Intersection and Complement



Rules of Set Theory

Let p, q and r be sets.

1. Commutative law:

$$(p \cup q) = (q \cup p) \text{ and } (p \cap q) = (q \cap p)$$

2. Associative law:

$$(p \cup (q \cup r)) = ((p \cup q) \cup r) \text{ and } (p \cap (q \cap r)) = ((p \cap q) \cap r)$$

3. Distributive law:

$$(p \cup (q \cap r)) = (p \cup q) \cap (p \cup r) \text{ and}$$
$$(p \cap (q \cup r)) = (p \cap q) \cup (p \cap r)$$

4. De Morgan's Law:

$$(p \cup q)^c = (p^c \cap q^c) \text{ and } (p \cap q)^c = (p^c \cup q^c)$$

Cartesian Product

Cartesian Product

- Let A be a set. $A \times A$ is a the set of ordered pairs (x, y) where $x, y \in A$.

Cartesian Product

- Let A be a set. $A \times A$ is a the set of ordered pairs (x, y) where $x, y \in A$.
- Similarly, $A \times A \times \cdots \times A$ (n times) (also denoted as A^n) is the set of all ordered subsets (with repetitions) of A of size n

Cartesian Product

- Let A be a set. $A \times A$ is a the set of ordered pairs (x, y) where $x, y \in A$.
- Similarly, $A \times A \times \cdots \times A$ (n times) (also denoted as A^n) is the set of all ordered subsets (with repetitions) of A of size n
- For example: $\{0, 1\}^n$ the set of all "strings" of 0 and 1 of length n.

Some problems on set theory

- If $|A| = 5$ and $|B| = 8$ and $|A \cup B| = 11$ what is the size of $A \cap B$?

Some problems on set theory

- If $|A| = 5$ and $|B| = 8$ and $|A \cup B| = 11$ what is the size of $A \cap B$?

- If $|A^c \cap B| = 10$ and $|A \cap B^c| = 8$ and $|A \cap B| = 5$ then how many elements are there is $A \cup B$?

Some problems on set theory

- If $|A| = 5$ and $|B| = 8$ and $|A \cup B| = 11$ what is the size of $A \cap B$?
- If $|A^c \cap B| = 10$ and $|A \cap B^c| = 8$ and $|A \cap B| = 5$ then how many elements are there is $A \cup B$?
- How many elements are there in the set $\{0, 1\}^n$?

Some problems on set theory

- If $|A| = 5$ and $|B| = 8$ and $|A \cup B| = 11$ what is the size of $A \cap B$?

- If $|A^c \cap B| = 10$ and $|A \cap B^c| = 8$ and $|A \cap B| = 5$ then how many elements are there is $A \cup B$?

- How many elements are there in the set $\{0, 1\}^n$?

Definition of Binary Relations

- Let S be a set. A binary relation is a subset of $S \times S$. (Usually we will say relation instead of binary relation)

Definition of Binary Relations

- Let S be a set. A binary relation is a subset of $S \times S$. (Usually we will say relation instead of binary relation)
- If $\mathcal{R}$ is a relation on the set S (that is, $\mathcal{R} \subseteq S \times S$) and $(x, y) \in \mathcal{R}$ we say "x is related to y".

Definition of Binary Relations

- Let S be a set. A binary relation is a subset of $S \times S$. (Usually we will say relation instead of binary relation)
- If $\mathcal{R}$ is a relation on the set S (that is, $\mathcal{R} \subseteq S \times S$) and $(x, y) \in \mathcal{R}$ we say "x is related to y".
- Sometimes it is denoted as "$x \sim y$" and sometime by abuse of notation we will say "$\sim$" is the relation.

Definition of Binary Relations

- Let S be a set. A binary relation is a subset of $S \times S$. (Usually we will say relation instead of binary relation)
- If $\mathcal{R}$ is a relation on the set S (that is, $\mathcal{R} \subseteq S \times S$) and $(x, y) \in \mathcal{R}$ we say "x is related to y".
- Sometimes it is denoted as "$x \sim y$" and sometime by abuse of notation we will say "$\sim$" is the relation.
- Binary relation is a simple yet powerful tool to represent complicated situations and hence is heavily used for modeling of problems.

Types of Relations

There are three main types of relations:

- [Reflexive] "x related to x", that is, the relation $\mathcal{R}$ on the set S is reflexive is for all $x \in S$, $(x, x) \in \mathcal{R}$.
- [Symmetric] "x related to y implies y related to x", that is, the relation $\mathcal{R}$ on the set S is symmetric is for all $x, y \in S$, $(x, y) \in \mathcal{R}$ implies $(y, x) \in \mathcal{R}$.
- [Transitive] "x related to y and y related to z implies x is related to z", that is, the relation $\mathcal{R}$ on the set S is symmetric is for all $x, y, z \in S$, $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$ implies $(x, z) \in \mathcal{R}$.

If a relation is reflexive, symmetric and transitive then it is called an equivalence relation. An equivalence relation is often denoted by "$\equiv$".

Properties of Equivalence Relation

- Let "$\equiv$" be an equivalence relation on the set S. An equivalence class is a maximal subset E of the set S such that any two element in the set E is related.
- There can be multiple equivalence class corresponding to the relation $\equiv$.

Theorem

If $\equiv$ is an equivalence relation on the set S then the equivalence classes for $\equiv$ forms a partition of the set S. That is, the union of the equivalence classes is the whole set and no two equivalence class intersect.

Problems on Binary Relations

- Prove that the relation "$=$" is an equivalence relation.
- Which of the properties of reflexive, symmetric and transitive does the relation "$\leq$" satisfy.
- A binary relation can have the properties of reflexive, symmetric and transitive. thus we have 8 possible different subsets of these properties a binary relation can have. Give an example of a relation in each of the eight cases.
- Prove the proposition on the equivalence classes.

Functions

- Functions are mappings from a set (called domain) to another set (called range).

Functions

- Functions are mappings from a set (called domain) to another set (called range).
- $f : D \rightarrow R$

Functions

- Functions are mappings from a set (called domain) to another set (called range).
- $f : D \rightarrow R$
- For all element $x \in D$, $f(x) \in R$.

Functions

- Functions are mappings from a set (called domain) to another set (called range).
- $f : D \to R$
- For all element $x \in D$, $f(x) \in R$.
- For every element in D there is a uniquely defined element $f(x)$.

Example of a Function

$f : \{0, 1, 2, 3, 4, 5\} \to \mathbb{R}$
$f(0) = 0$
$f(1) = 1$
$f(2) = 4$
$f(3) = 9$
$f(4) = 16$
$f(5) = 25$

How to represent the function f?

Example of a Function

$f : \{0, 1, 2, 3, 4, 5\} \to \mathbb{R}$
$f(0) = 0$
$f(1) = 1$
$f(2) = 4$
$f(3) = 9$
$f(4) = 16$
$f(5) = 25$

How to represent the function f?
Either explicitly give the function

Example of a Function

$f : \{0, 1, 2, 3, 4, 5\} \to \mathbb{R}$
$f(0) = 0$
$f(1) = 1$
$f(2) = 4$
$f(3) = 9$
$f(4) = 16$
$f(5) = 25$

How to represent the function f?
Either explicitly give the function OR say $f(x) = x^2$

Truth-table

Functions are represented using truth-table. Let $f : D \to R$.

Truth-table

Functions are represented using truth-table. Let $f : D \to R$.

- Order the elements in the domain in a mutually agreeable way.
  Say $D = \{x_1, x_2, \ldots, x_d\}$

Truth-table

Functions are represented using truth-table. Let $f : D \to R$.

- Order the elements in the domain in a mutually agreeable way.
  Say $D = \{x_1, x_2, \ldots, x_d\}$
- For each elements in the particular order write down the function value of the element in the same order.
  $\{f(x_1), f(x_2), \ldots, f(x_d)\}$

Boolean Functions

- $f : \{0,1\}^n \rightarrow \{0,1\}$
- Sometimes $\{0,1\}$ can be viewed as $\{True,\ False\}$ or $\{Left, Right\}$ or $\{+1, -1\}$.

Some problems on functions

- What is the size of the truth-table for a function $f : \{0,1\}^n \rightarrow \{0,1\}$?

Some problems on functions

- What is the size of the truth-table for a function $f : \{0,1\}^n \to \{0,1\}$?
- How many different functions are there from $\{1,2,3,4,5,6\}$ to $\{a,\ e,\ i,\ o,\ u\}$?

Some problems on functions

- What is the size of the truth-table for a function $f : \{0,1\}^n \rightarrow \{0,1\}$?
- How many different functions are there from $\{1,2,3,4,5,6\}$ to $\{a,\ e,\ i,\ o,\ u\}$?
- How many different Boolean functions of the for $\{0,1\}^n \rightarrow \{0,1\}$ are possible?