Discrete Mathematics

Lecture 3: Propositional and Predicate Logic

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- A deduction

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- A premise or set of assumptions, and
- A deduction

The premise can be composed of multiple statements (that can individually be true or false, or may depend on each other) and the statements can be connected using connectives like “and” or “or”.

Similarly, the deduction can be composed of multiple statements (that can individually be true or false, or may depend on each other) and the statements can be also connected using connectives.

Examples

- When it is cloudy it rains. Today its cloudy so it would rain today.

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- When it is cloudy it rains. Today its cloudy so it would rain today.
- Every city in India has horrible traffic. Chennai is an Indian city. So Chennai has horrible traffic.

Propositional logic and Predicate Logic

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- Every statement (or proposition) is either TRUE or FALSE.
- A statement can be formed using other statements connected to each other by 5 kinds of connectives: AND, OR, NOT, IMPLIES and IFF.
- A statement can have unspecified terms, called variable. Every variable has to be quantified properly.

The Connectives AND ($\wedge$), OR ($\vee$), NOT ($\neg$), IMPLIES ($\implies$) and IFF ($\iff$)

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- The connectives AND, OR, IMPLIES and IFF takes two statements (that are either TRUE or FALSE) and combines them to produce a single statement (that is either TRUE or FALSE depending upon the input statements). So these connectives are functions of the form $\{True,\ False\}^2 \to \{True,\ False\}$.
- The connective NOT takes a single statement and outputs a single statement. So the connective NOT is a function of the form $\{True,\ False\} \to \{True,\ False\}$.

Truthtable of the AND ($p \wedge q$)

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

Truthtable of the OR ($p \vee q$)

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

Truthtable of the NOT ($\neg p$)

p	$\neg p$
F	T
T	F

The IMPLIES ($p \implies q$)

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- TRUE statements cannot proves a FALSE statement.

The IMPLIES ($p \implies q$)

- TRUE statements proves a TRUE statement.
- TRUE statements cannot proves a FALSE statement.
- FALSE statement can prove any statement.

Example of False implying anything

Example by famous mathematician G.H.Hardy:
"If $2 + 2 = 5$ then you are pope."

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Let $2+2=5$.

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"If $2 + 2 = 5$ then you are pope."

Let $2 + 2 = 5$.
But we know $2 + 2 = 4$.

Example of False implying anything

Example by famous mathematician G.H.Hardy:
"If $2 + 2 = 5$ then you are pope."

Let $2 + 2 = 5$.
But we know $2 + 2 = 4$.
So $5 = 4$

Example of False implying anything

Example by famous mathematician G.H.Hardy:
"If $2+2=5$ then you are pope."

Let $2+2=5$.
But we know $2+2=4$.
So $5=4$ and so subtracting 3 from both sides $2=1$

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Let $2 + 2 = 5$.
But we know $2 + 2 = 4$.
So $5 = 4$ and so subtracting 3 from both sides $2 = 1$
So 2 person $= 1$ person.

Example of False implying anything

Example by famous mathematician G.H.Hardy:
"If $2+2=5$ then you are pope."

Let $2+2=5$.
But we know $2+2=4$.
So $5=4$ and so subtracting 3 from both sides $2=1$
So 2 person $=$ 1 person.
So YOU and POPE are 1 person and hence you are pope.

The IMPLIES ($p \implies q$)

p	q	$p \implies q$
F	F	T
F	T	T
T	F	F
T	T	T

Truthtable of the IFF ($p \iff q$)

p	q	$p \iff q$
F	F	T
F	T	F
T	F	F
T	T	T

Quantifiers

- "For all" $\forall$

$$\exists x \; P(x)$$

- "There exists" $\exists$

$$\forall x \; P(x)$$

Universality

Every logical sentence can be written using the AND, OR, NOT, IMPLIES, IFF and two more symbols:

There exists, $\exists$

For all, $\forall$

Converting a normal logical statement to a mathematical logic statement

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If you did not know the material earlier and you don't study hard then you will not get a A in this course. Therefore if you get a A grade in this course then you knew this material earlier and you studied hard.

Every statement (proposition) is either TRUE or FALSE.

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A statement is true if under any condition satisfying the premise (or assumptions) the statement holds true.

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Is the above sentence True or False?

A statement can be formed using other statements ...

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Statements are connected to each other by 5 kinds of connectives: AND, OR, NOT, IMPLIES and IFF.

If you did not know the material earlier and you don't study hard then you will not get a A in this course. Therefore if you get a A grade in this course then you knew this material earlier and you studied hard.

Writing a sentence as a proposition

If you did not know the material earlier and you don't study hard then you will not get a A in this course. Therefore if you get a A grade in this course then you knew this material earlier and you studied hard.

- Variable:
    - you did not know the material earlier

Writing a sentence as a proposition

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- Variable:
  - you did not know the material earlier $= p$

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If you did not know the material earlier and you don't study hard then you will not get a A in this course. Therefore if you get a A grade in this course then you knew this material earlier and you studied hard.

- Variable:
  - you did not know the material earlier $= p$
  - you don't study hard

Writing a sentence as a proposition

If you did not know the material earlier and you don't study hard then you will not get a A in this course. Therefore if you get a A grade in this course then you knew this material earlier and you studied hard.

- Variable:
  - you did not know the material earlier $= p$
  - you don't study hard $= q$

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- Variable:
  - you did not know the material earlier $= p$
  - you don't study hard $= q$
  - you will not get a A in this course $= r$

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- Variable:
  - you did not know the material earlier $= p$
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  - you will not get a A in this course $= r$
- What is "you knew this material earlier"?

Writing a sentence as a proposition

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- Variable:
  - you did not know the material earlier $= p$
  - you don't study hard $= q$
  - you will not get a A in this course $= r$
- What is "you knew this material earlier"?
  you knew this material earlier$= \neg p$

Writing a sentence as a proposition

If you did not know the material earlier and you don't study hard then you will not get a A in this course. Therefore if you get a A grade in this course then you knew this material earlier and you studied hard.

- you did not know the material earlier $= p$

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- you knew this material earlier $= \neg p$

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- you knew this material earlier $= \neg p$
- you studied hard $= \neg q$

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- you will not get a A in this course $= r$

- you knew this material earlier $= \neg p$
- you studied hard $= \neg q$
- you get a A grade in this course $= \neg r$

Writing a sentence as a proposition

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- you did not know the material earlier $= p$
- you don't study hard $= q$
- you will not get a A in this course $= r$

So the sentence is

$$((p \wedge q) \implies r) \quad \implies \quad (\neg r \implies (\neg p \wedge \neg q))$$

Writing a sentence as a proposition

The following sentence is logically correct:
If you did not know the material earlier and you don't study hard then you will not get a A in this course. Therefore if you get a A grade in this course then you knew this material earlier and you studied hard.

Writing a sentence as a proposition

The following sentence is logically correct:
If you did not know the material earlier and you don't study hard then you will not get a A in this course. Therefore if you get a A grade in this course then you knew this material earlier and you studied hard.

if and only if the following expression evaluates to TRUE under any setting of the values of the variables to True or False:

$$((p \wedge q) \implies r) \implies (\neg r \implies (\neg p \wedge \neg q))$$

Checking the correctness of the statement

To check if an expression is correct/consistent we try all possible values of the input and see if it always evaluate to TRUE.

We create a table with all the possible input and the evaluations. That is, we write the truthtable explicitly.

Truthtable

$$f = [((p \wedge q) \implies r) \quad \implies \quad (\neg r \implies (\neg p \wedge \neg q))]$$

p	q	r	f
F	F	F	
F	F	T	
F	T	F	
F	T	T	
T	F	F	
T	F	T	
T	T	F	
T	T	T	

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$$f = [s \implies t]$$

Truthtable

$$f = [((p \wedge q) \implies r) \implies (\neg r \implies (\neg p \wedge \neg q))]$$

$$f = [s \implies t]$$

p	q	r	$p \wedge q$	$g \to r$	$\neg p$	$\neg q$	$\neg r$	$p' \wedge q'$	$r' \to h$	f
			(g)	(s)	p'	q'	r'	(h)	(t)	
F	F	F								
F	F	T								
F	T	F								
F	T	T								
T	F	F								
T	F	T								
T	T	F								
T	T	T								

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$$f = [s \implies t]$$

p	q	r	$p \wedge q$	$g \rightarrow r$	$\neg p$	$\neg q$	$\neg r$	$p' \wedge q'$	$r' \rightarrow h$	f
			(g)	(s)	p'	q'	r'	(h)	(t)	
F	F	F	F							
F	F	T	F							
F	T	F	F							
F	T	T	F							
T	F	F	F							
T	F	T	F							
T	T	F	T							
T	T	T	T							

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p	q	r	$p \wedge q$	$g \to r$	$\neg p$	$\neg q$	$\neg r$	$p' \wedge q'$	$r' \to h$	f
			(g)	(s)	p'	q'	r'	(h)	(t)	
F	F	F	F	T						
F	F	T	F	T						
F	T	F	F	T						
F	T	T	F	T						
T	F	F	F	T						
T	F	T	F	T						
T	T	F	T	F						
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			(g)	(s)	p'	q'	r'	(h)	(t)	
F	F	F	F	T	T					
F	F	T	F	T	T					
F	T	F	F	T	T					
F	T	T	F	T	T					
T	F	F	F	T	F					
T	F	T	F	T	F					
T	T	F	T	F	F					
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p	q	r	$p \wedge q$	$g \to r$	$\neg p$	$\neg q$	$\neg r$	$p' \wedge q'$	$r' \to h$	f
			(g)	(s)	p'	q'	r'	(h)	(t)	
F	F	F	F	T	T	T	T			
F	F	T	F	T	T	T	F			
F	T	F	F	T	T	F	T			
F	T	T	F	T	T	F	F			
T	F	F	F	T	F	T	T			
T	F	T	F	T	F	T	F			
T	T	F	T	F	F	F	T			
T	T	T	T	T	F	F	F			

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p	q	r	$p \wedge q$	$g \to r$	$\neg p$	$\neg q$	$\neg r$	$p' \wedge q'$	$r' \to h$	f
			(g)	(s)	p'	q'	r'	(h)	(t)	
F	F	F	F	T	T	T	T	T		
F	F	T	F	T	T	T	F	T		
F	T	F	F	T	T	F	T	F		
F	T	T	F	T	T	F	F	F		
T	F	F	F	T	F	T	T	F		
T	F	T	F	T	F	T	F	F		
T	T	F	T	F	F	F	T	F		
T	T	T	T	T	F	F	F	F		

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p	q	r	$p \wedge q$	$g \to r$	$\neg p$	$\neg q$	$\neg r$	$p' \wedge q'$	$r' \to h$	f
			(g)	(s)	p'	q'	r'	(h)	(t)	
F	F	F	F	T	T	T	T	T	T	
F	F	T	F	T	T	T	F	T	T	
F	T	F	F	T	T	F	T	F	F	
F	T	T	F	T	T	F	F	F	T	
T	F	F	F	T	F	T	T	F	F	
T	F	T	F	T	F	T	F	F	T	
T	T	F	T	F	F	F	T	F	F	
T	T	T	T	T	F	F	F	F	T	

Truthtable

$$f = [((p \wedge q) \implies r) \implies (\neg r \implies (\neg p \wedge \neg q))]$$

$$f = [s \implies t]$$

p	q	r	$p \wedge q$	$g \rightarrow r$	$\neg p$	$\neg q$	$\neg r$	$p' \wedge q'$	$r' \rightarrow h$	f
			(g)	(s)	p'	q'	r'	(h)	(t)	
F	F	F	F	T	T	T	T	T	T	T
F	F	T	F	T	T	T	F	T	T	T
F	T	F	F	T	T	F	T	F	F	F
F	T	T	F	T	T	F	F	F	T	T
T	F	F	F	T	F	T	T	F	F	F
T	F	T	F	T	F	T	F	F	T	T
T	T	F	T	F	F	F	T	F	F	T
T	T	T	T	T	F	F	F	F	T	T

Consistency/correctness of the expression

$$f = [((p \wedge q) \implies r) \quad \implies \quad (\neg r \implies (\neg p \wedge \neg q))]$$

Since the expression does not evaluate to true always so the expression is not correct.

Checking Equivalence

Definition

Two statements are equivalent if their TRUTHTABLES are the same.

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A	B	$A \implies B$	$(\neg B \wedge A) = F$
F	F		
F	T		
T	F		
T	T		

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Two statements are equivalent if their TRUTHTABLES are the same.

Is: $A \implies B$ is equivalent to $(\neg B \wedge A) = FALSE$

A	B	$A \implies B$	$(\neg B \wedge A) = F$
F	F	T	
F	T	T	
T	F	F	
T	T	T	

Problems on Propositional Logic

- Let $r =$ "she registered to vote" and $v =$ "she voted". Write the following statement in symbolic form: She registered to vote but she did not vote.
- Make a truth table for $(p \vee (\sim p \vee q)) \wedge \sim (q \wedge \sim r)$
- A tautology is a statement that is always true and a contradiction is a statement that is always false. Now is the statement form $(p \wedge q) \vee (\neg p \vee (p \wedge \neg q))$ a tautology or a contradiction or none.

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- A statement can have an unspecified term, called variable.

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Is: $A \implies B$ is equivalent to $(\neg B \wedge A) = FALSE$

A	B	$A \implies B$	$(\neg B \wedge A) = F$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	T	T

Problems on Propositional Logic

- Prove that $(a \implies b)$ is equivalent to $(\neg a \vee b)$.
- Prove that $(a \iff b)$ is equivalent to $(a \implies b) \wedge (b \implies a)$.
- Prove that $(p \vee q) \implies r$ is equivalent to $(p \implies r) \wedge (q \implies r)$.
- Prove that $(a \implies b)$ is equivalent to $(\neg b \implies \neg a)$.

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Another approach is to use some already proved rules to simplify the formulas before using the brute force truthtable approach.

Rules of Propositional Logic

Let p, q and r be propositions.

1. Commutative law:

$$(p \vee q) = (q \vee p) \text{ and } (p \wedge q) = (q \wedge p)$$

2. Associative law:

$$(p \vee (q \vee r)) = ((p \vee q) \vee r) \text{ and } (p \wedge (q \wedge r)) = ((p \wedge q) \wedge r)$$

3. Distributive law:

$$(p \vee (q \wedge r)) = (p \vee q) \wedge (p \vee r) \text{ and}$$
$$(p \wedge (q \vee r)) = (p \wedge q) \vee (p \wedge r)$$

4. De Morgan's Law:

$$\neg(p \vee q) = (\neg p \wedge \neg q) \text{ and } \neg(p \wedge q) = (\neg p \vee \neg q)$$

Rules of Quantifiers

- $\neg(\forall x\ P(x))$ is same as $\exists x\ (\neg P(x))$
- $\neg(\exists x\ P(x))$ is same as $\forall x\ (\neg P(x))$

Problems on Propositional Logic

- Prove that $(a \implies b)$ is equivalent to $(\neg a \vee b)$.
- Prove that $(a \iff b)$ is equivalent to $(a \implies b) \wedge (b \implies a)$.
- Prove that $(p \vee q) \implies r$ is equivalent to $(p \implies r) \wedge (q \implies r)$.
- Prove that $(a \implies b)$ is equivalent to $(\neg b \implies \neg a)$.

Using Propositional Logic for designing proofs

- A mathematical statement comprises of a premise (or assumptions). And when the assumptions are satisfied the statement deduces something.
- If A is the set of assumptions and B is the deduction then a mathematical statement is of the form

$$A \implies B$$

- Now how to check if the statement if correct? And if it is indeed correct how to prove the statement?
- Depending on whether A or B (or both) can be split into smaller statements and how the smaller statements are connected we can design different techniques for proving the overall statement of $A \implies B$.
- If indeed we can proof that the statement is correct then we can call it a Theorem.