Discrete Mathematics

Lecture 12: Proof Technique (Contrapositive)

Instructor: Sourav Chakraborty

Proof Techniques

To prove statement $A \implies B$.

There are different proof techniques:

- Constructive Proofs
- Proof by Contradiction
- Proof by Contrapositive
- Induction
- Counter example
- Existential Proof

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- Sometimes viewing the problem is a different way can also help in tackling the problem easily.
- Whether to split a problem or how to split a problem or how to look at a problem is an ART that has to be developed.
- There are some thumb rules but at the end it is a skill you develop using a lot of practice.

Tricks for solving problems

- (Splitting into smaller problem) If the problem is to prove $A \implies B$ and B can be written as $B = C \wedge D$ then note that

$$(A \implies B) \equiv (A \implies C \wedge D) \equiv (A \implies C) \wedge (A \implies D).$$

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- (Sometimes proving something stronger is easier) If $C \implies B$ then

$$(A \implies C) \implies (A \implies B).$$

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- A simpler technique is to have a backward proof.
- If we have to prove $(A \implies B)$ then the idea is to simplify B.
- And if $C \iff B$ then $(A \implies B) \equiv (A \implies C)$.

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$$(A \implies B) \equiv (C \implies B) \wedge (D \implies B).$$

Proof by Contradiction

- Note that

$$(A \implies B) \equiv (\neg B \wedge A = \quad \text{False})$$

  This is called "proof by contradiction"

- To proof $A \implies B$ sometimes its easier to prove that

$$\neg B \wedge A = \quad \text{False}.$$

- A similar statement is

$$(A \implies B) \equiv (\neg B \implies \neg A)$$

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$$(A \implies B) \equiv (\neg B \implies \neg A) \equiv ((\neg C \wedge \neg D) \implies \neg A)$$

Example 1

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If a and b are two positive integers and $a^2 + b^2$ is even then either both a and b are odd or both a and b are even.

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Complete the proof by yourself.

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- $A =$ "$n \equiv 2 (\bmod 3)$"
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This is an exercise. We will discuss it in the problem solving video.