Expected revenue of the first price auction:

Nash equilibrium

$$b_1 = \frac{1}{2} v_1$$

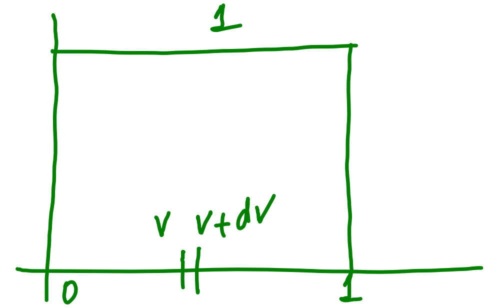
$$b_2 = \frac{1}{2} v_2$$

Since the player with the maximum bid wins the auction and pays an amount equal to the bid,

revenue: $\max\{b_1, b_2\}$.

$$\text{revenue} = \max\{b_1, b_2\}$$
$$= \max\left\{\frac{1}{2}V_1, \frac{1}{2}V_2\right\}$$
$$= \frac{1}{2}\max\{V_1, V_2\}$$

V_1, V_2 are independent valuations uniformly distributed in $[0, 1]$.



What is the probability that $\max\{V_1, V_2\}$ lies in the infinitesimal interval $[v, v+dv]$?

Scenario 1: V_1 is the maximum

V_1 lies in $[v, v+dv]$

V_2 lies in $[0, v]$.

$$Pr = Pr(V_1 \in [v, v+dv]) \times Pr(V_2 \in [0, v])$$

$$= dv \times v = v\,dv$$

Scenario 2: V_2 is maximum

V_2 lies in $[v, v+dv]$

V_1 lies in $[0, v]$

$$Pr = Pr\left(V_1 \in [0,v]\right) \times Pr\left(V_2 \in [v, v+dv]\right)$$

$$= v \times dv = v\,dv$$

Probability that $\max\{V_1, V_2\}$ lies in $[v, v+dv]$

$$= v\,dv + v\,dv$$

$$= 2v\,dv.$$

Average revenue corresponding to $\max\{v_1, v_2\} \in [v, v+dv]$

$$= \frac{1}{2} v \times 2v\,dv$$

$$= v^2\,dv$$

Net average revenue to the auctioneer

$$= \int_0^1 v^2\,dv$$

$$= \left.\frac{v^3}{3}\right|_0^1 = \frac{1}{3}.$$

The expected revenue of the auctioneer $= \frac{1}{3}$.

Sealed bid first price auction