All pay auction:

Two player auction

P_1, P_2 — bidders

b_1, b_2 — bid of Player 2

bid of Player 1

Player with the highest bid wins.

Both the player pay their bid irrespective of the outcome.

Let us assume that v_1, v_2 denote the valuations of P_1, P_2 respectively —

$\underbrace{V_1, V_2}$ are independent

Distributed as uniform random variables in $[0, 1]$.

$$\left.\begin{aligned} b_1 &= \frac{1}{2} V_1^2 \\ b_2 &= \frac{1}{2} V_2^2 \end{aligned}\right\}$$ Nash Equilibrium for the all pay auction

Let us start by assuming that player P_2 is bidding $b_2 = \frac{1}{2} V_2^2$.

Let player P_1 bid b.

$\pi(b)$ — expected payoff to player 1 as a function of bid b.

$$\pi(b) = Pr(\text{win}) \times (V_1 - b) + Pr(\text{loss}) \times (-b)$$

P_1 wins if $b \geq b_2 = \frac{1}{2} V_2^2$

$$\Rightarrow \frac{1}{2} V_2^2 \leq b$$

$$\Rightarrow V_2 \leq \sqrt{2b}$$

$$Pr(\text{win}) = Pr(V_2 \leq \sqrt{2b})$$

$$Pr(V_2 \leq \sqrt{2b})$$

$$= Pr(V_2 \in [0, \sqrt{2b}])$$

$$= \sqrt{2b} = Pr(\text{win})$$

$$Pr(\text{loss}) = 1 - Pr(\text{win}) = 1 - \sqrt{2b}.$$

$$\pi(b) = \sqrt{2b}\,(v_1 - b) + (1 - \sqrt{2b})(-b)$$

$$= \sqrt{2b}\, v_1 - \cancel{b\sqrt{2b}} - b + \cancel{b\sqrt{2b}}$$

$$= \sqrt{2b}\, v_1 - b$$

$$\pi(b) = \sqrt{2b}\, v_1 - b$$

expected payoff to player 1 as a function of bid b.

$$\frac{\partial \pi(b)}{\partial b} = \sqrt{2} \cdot v_1 \frac{1}{2\sqrt{b}} - 1 = 0$$

$$\sqrt{2}\, V_1 \frac{1}{2\sqrt{b}} - 1 = 0$$

$$\Rightarrow \sqrt{b} = \frac{V_1}{\sqrt{2}}$$

$$\boxed{b = \frac{1}{2} V_1^2}$$

Similarly, it can be shown that if $b_1 = \frac{1}{2} V_1^2$, then $b = \frac{1}{2} V_2^2$ is a best response bid for Player P_2

$$b_1 = \frac{1}{2} v_1^2 \quad b_2 = \frac{1}{2} v_2^2$$

Nash Equilibrium of the Two player all-pay auction game.

Expected revenue of Two player all-pay auction:

$$b_1 = \frac{1}{2} V_1^2$$
$$b_2 = \frac{1}{2} V_2^2$$

Revenue $= b_1 + b_2$

$$= \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$$

Expected Revenue

$$= \frac{1}{2} E\{V_1^2\} + \frac{1}{2} E\{V_2^2\}$$

$$= \frac{1}{2} \int_0^1 V_1^2 f_{V_1}(V_1)\, dV_1 + \frac{1}{2} \int_0^1 V_2^2 f_{V_2}(V_2)\, dV_2$$

$$= \frac{1}{2} \int_0^1 V_1^2\, dV_1 + \frac{1}{2} \int_0^1 V_2^2\, dV_2$$

$$= \frac{1}{2}\int_0^1 v_1^2\,dv_1 + \frac{1}{2}\int_0^1 v_2^2\,dv_2$$

$$= \frac{1}{2}\left.\frac{v_1^3}{3}\right|_0^1 + \frac{1}{2}\left.\frac{v_2^3}{3}\right|_0^1$$

$$= \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{3}$$

Revenue equivalence principle.