'Bayesian Second Price auction'

Two player auction
P_1 , P_2
Let b_1, b_2 denote their respective bids.

If $\left.\begin{array}{l} b_1 \geq b_2 \\ \end{array}\right\}$ P_1 wins

If $\left.\begin{array}{l} b_2 > b_1 \\ \end{array}\right\}$ P_2 wins.

winner with highest bid wins auction.

If $b_1 \geq b_2$, the P_1 wins and pays the second highest bid ie b_2.

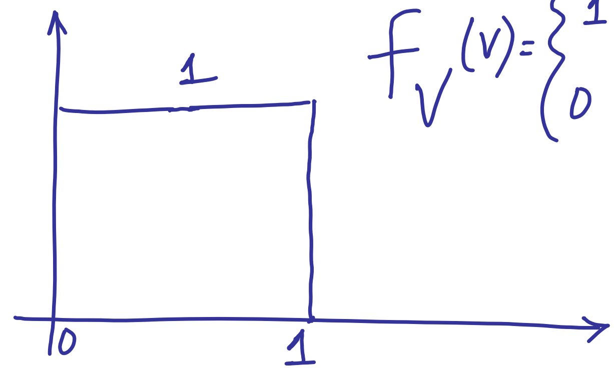
On the other hand, if $b_2 > b_1$, the P_2 wins, and pays the second highest bid b_1.

Each player has a private valuation.

Let the valuations of P_1, P_2 be denoted by V_1, V_2 respectively.

V_1, V_2 are independent.

V_1, V_2 are distributed as uniform random variables in $[0, 1]$.



$$f_V(v) = \begin{cases} 1 & 0 \le v \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The Nash equilibrium of this second price auction

$$b_1 = V_1$$
$$b_2 = V_2$$

Let us start with the assumption that

$$b_2 = V_2 .$$

Let us now find the best response bid b of Player 1.

Case 1: let $V_1 \geq V_2$.

Player 2 is bidding $b_2 = V_2$

If $b \geq V_2$, P_1 wins the auction

and pays the second highest bid i.e. $b_2 = V_2$.

Net payoff $= V_1 - V_2 \geq 0$

On the other hand if he bids $b \leq b_2 = V_2$ then player P_1 loses the auction and his net payoff is 0.

Therefore, we see that any bid $b \geq V_2$ is a best response.

In particular $b = V_1$ is a best response.

Let us now consider $V_1 < V_2$

Player P_2 is bidding $b_2 = V_2$

If P_1 bids $b \geq V_2 = b_2$, then he wins the auction and pays second highest bid i.e $b_2 = V_2$. Net payoff $= V_1 - V_2 < 0$

On the other hand, if he bids any $b < b_2 = V_2$ then he loses the auction and his payoff is 0.

Therefore any bid $b < b_2 = V_2$ is a best response.

In particular $b = V_1$ is a best response.

If player P_2 is bidding $b_2 = V_2$, then $b_1 = V_1$ is a best response for player P_1.

Similarly, it can be shown that if Player P_1 is bidding $b_1 = V_1$ then $b = V_2$ is a best response for P_2.

Hence, the Nash equilibrium of this second price auction is

$$b_1 = V_1$$
$$b_2 = V_2$$

Therefore, each player bidding his true valuation is the Nash equilibrium for the second price auction.

Expected revenue of second price auction:

$b_1 = V_1$

$b_2 = V_2$

Observe if $V_1 \geq V_2$ then $b_1 \geq b_2$

Therefore P_1 wins and pays second highest bid ie $b_2 = V_2$

if $V_1 \geq V_2$, revenue $= V_2$

If $V_2 > V_1$

Then $b_2 > b_1$

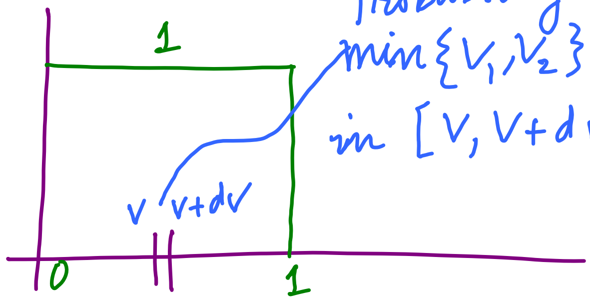
Player P_2 wins and pays the second highest valuation ie V_1.

Therefore revenue : V_1

Therefore, one can conclude that the revenue to the auctioneer in this Bayesian second price auction is

$$\min\{V_1, V_2\}.$$

V_1, V_2 are independent



Probability that $\min\{V_1, V_2\}$ lies in $[v, v+dv]$?

Case 1: $V_1 \leq V_2$

V_1 lies in $[v, v+dv]$

V_2 lies in $[v+dv, 1]$

$$= \Pr(V_1 \in [v, v+dv]) \times \Pr(V_2 \in [v+dv, 1])$$

$$= dv \times (1 - v - dv)$$

$$= dv(1 - v - dv)$$

$$= dv(1 - v)$$

$$\approx (1 - v)\, dv.$$

Similarly, we can consider the other scenario

$$v_2 < v_1$$

$$v_2 \in [v, v + dv]$$

$$v_1 \in [v + dv, 1]$$

$$= \Pr(v_2 \in [v, v+dv]) \times \Pr(v_1 \in [v+dv, 1])$$

$$= dv \times (1 - v - dv)$$

$$= dv(1 - v).$$

Net probability that the $\min\{v_1, v_2\}$ lies in the interval $[v, v+dv]$ is $2(1-v)\,dv$

Revenue to auctioneer

$= \min\{V_1, V_2\}.$

Since minimum lies in $[V, V+dV]$, revenue $= V$

Expected revenue $= \Pr \times V$

$= 2(1-V)\,dV \cdot V$

Expected revenue

$$= \int_0^1 2(1-V)\,V\,dV$$

$$= \int_0^1 2(V-V^2)\,dV$$

Expected revenue

$$= \int_0^1 2(v - v^2)\, dv$$

$$= 2\left\{ \frac{v^2}{2}\Big|_0^1 - \frac{v^3}{3}\Big|_0^1 \right\}$$

$$= 2\left\{ \frac{1}{2} - \frac{1}{3} \right\} = 2 \times \frac{1}{6} = \frac{1}{3}$$

Expected revenue

$$= \frac{1}{3}$$

'Revenue equivalence principle'