Single parity check code (SPC code)

$(n, n-1)$ linear binary code

$n-1 = k$

$$\underline{m} = [m_1 \; m_2 \; \dots \; m_{n-1}] \xrightarrow{\text{Encoder}} \underline{c} = [m_1 \; m_2 \; \dots \; m_{n-1} \; p]$$

parity

$p = m_1 \oplus m_2 \oplus \dots \oplus m_{n-1}$ XOR of all message bits

$(3,2)$ SPC code:

$\underline{m}$	$\underline{c}$
0 0	0 0 0
0 1	0 1 1
1 0	1 0 1
1 1	1 1 0

# of 1s in $\underline{c}$: even

SPC code: consists of all even-weight vectors of length n

$$G = \begin{array}{c} \\ n-1 \end{array} \overset{n}{\left[\begin{array}{c|c} I_{n-1} & \begin{matrix} 1 \\ 1 \\ \vdots \\ 1 \end{matrix} \end{array} \right]}$$

all 1's

$$H = \begin{array}{c} 1 \end{array} \overset{n}{\left[1 \; 1 \; 1 \; \dots \; 1 \right]}$$

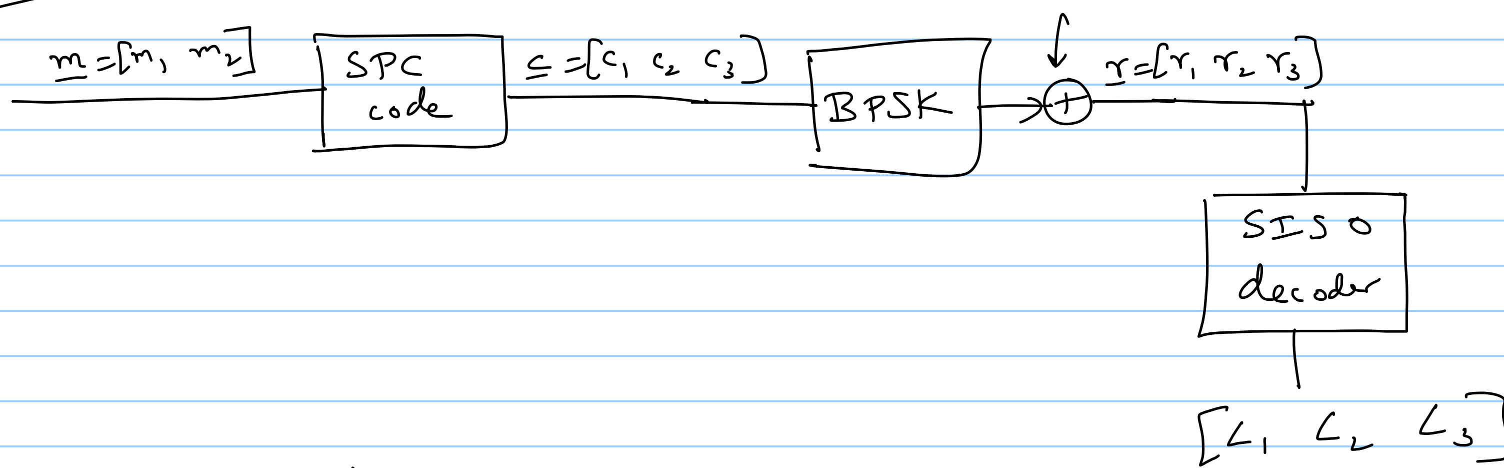
all 1's in a row

$\updownarrow$

$c_1 \oplus c_2 \oplus c_3 \oplus \dots \oplus c_n = 0$

SISO decoder for SPC code

(3,2) code



$L_1 \leftarrow$ intrinsic $\propto r_1$

extrinsic : what do r_2 and r_3 say about c_1 ?

intrinsic

$c_2 \quad c_3$ $\qquad c_1 = c_2 \oplus c_3$

parity check condition

$c_1 = c_2 \oplus c_3$

$$l_2 = \log \frac{\overbrace{Pr(c_2=0 \mid r_2)}^{p_2}}{\underbrace{Pr(c_2=1 \mid r_2)}_{1-p_2}} \qquad l_3 = \log \frac{\overbrace{Pr(c_3=0 \mid r_3)}^{p_3}}{\underbrace{Pr(c_3=1 \mid r_3)}_{1-p_3}}$$

Given p_2 & p_3: what is $p_1 = Pr(c_1=0 \mid r_2, r_3)$?

$\log \frac{p_1}{1-p_1}$: extrinsic LLR

c_1	c_2	c_3
0	0	0
0	1	1
1	0	1
1	1	0

$$p_1 = p_2\, p_3 + (1-p_2)(1-p_3)$$

$$1-p_1 = p_2(1-p_3) + (1-p_2)\, p_3$$

$$l_{ext,1} = \log \frac{p_1}{1-p_1}$$

$$p_1 - (1-p_1) = p_2\big(p_3 - (1-p_3)\big) + (1-p_2)\big((1-p_3) - p_3\big)$$

$$\frac{p_1 - (1-p_1)}{p_1 + (1-p_1)} = \frac{\big(p_2 - (1-p_2)\big)}{p_2 + (1-p_2)} \frac{\big(p_3 - (1-p_3)\big)}{p_3 + (1-p_3)}$$

$$\tanh(x) = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

odd function: $x < 0 \quad \tanh(x) < 0$

$x > 0 \quad \tanh(x) > 0$

$$\frac{1 - \left(\frac{1-p_1}{p_1}\right)}{1 + \left(\frac{1-p_1}{p_1}\right)} = \frac{1 - \frac{1-p_2}{p_2}}{1 + \frac{1-p_2}{p_2}} \cdot \frac{1 - \frac{1-p_3}{p_3}}{1 + \frac{1-p_3}{p_3}}$$

$$\frac{1 - e^{-l_{ext,1}}}{1 + e^{-l_{ext,1}}} = \frac{1 - e^{-l_2}}{1 + e^{-l_2}} \cdot \frac{1 - e^{-l_3}}{1 + e^{-l_3}}$$

$$c_1 = c_2 \oplus c_3 \xRightarrow{\text{tanh rule}} \boxed{\tanh\left(\frac{l_{ext,1}}{2}\right) = \tanh\left(\frac{l_2}{2}\right) \cdot \tanh\left(\frac{l_3}{2}\right)}$$

Signs : $\operatorname{sgn}(l_{ext,1}) = \operatorname{sgn}(l_2) \cdot \operatorname{sgn}(l_3)$

Absolute value : $\tanh\left(\frac{|l_{ext,1}|}{2}\right) = \tanh\left(\frac{|l_2|}{2}\right) \tanh\left(\frac{|l_3|}{2}\right)$

$x > 0$

$f(x) = \left|\log \tanh \frac{|x|}{2}\right|$

$\boxed{f^{-1}(x) = f(x)}$

$$\log \tanh\left(\frac{|l_{ext,1}|}{2}\right) = \log \tanh \frac{|l_2|}{2} + \log \tanh \frac{|l_3|}{2}$$

$$f(|l_{ext,1}|) = f(|l_2|) + f(|l_3|)$$

$$\boxed{\begin{array}{c} |l_{ext,1}| = f\big(f(|l_2|) + f(|l_3|)\big) \\ \hline \operatorname{sgn}(l_{ext,1}) = \operatorname{sgn}(l_2)\operatorname{sgn}(l_3) \end{array}}$$

$r = [r_1 \; r_2 \; r_3] \longrightarrow$ SISO decoder $\longrightarrow [L_1 \quad L_2 \quad L_3]$ $\qquad L_i = l_i + l_{ext,i}$

SPC code

intrinsic: $[l_1 \quad l_2 \quad l_3]$

extrinsic: $[l_{ext,1} \quad l_{ext,2} \quad l_{ext,3}]$

$$\boxed{f(x) = \left|\log \tanh \frac{|x|}{2}\right|}$$

$c_2 = c_1 \oplus c_3$

$$|l_{ext,2}| = f\big(f(|l_1|) + f(|l_3|)\big)$$

$$\operatorname{sgn}(l_{ext,2}) = \operatorname{sgn}(l_1)\operatorname{sgn}(l_3)$$

$$c_3 = c_1 \oplus c_2 \qquad |l_{ext,3}| = f\big(f(|l_1|) + f(|l_2|)\big)$$

$$\operatorname{sgn}(l_{ext,3}) = \operatorname{sgn}(l_1)\,\operatorname{sgn}(l_2)$$

$$S = f(|l_1|) + f(|l_2|) + f(|l_3|) \qquad P = \operatorname{sgn}(l_1)\,\operatorname{sgn}(l_2)\,\operatorname{sgn}(l_3)$$

$$l_{ext,1} = \underbrace{f\big(S - f(|l_1|)\big)}_{\text{absolute value}} \cdot \underbrace{P \cdot \operatorname{sgn}(l_1)}_{\text{sign}}$$

$$l_{ext,2} = f\big(S - f(|l_2|)\big) \cdot P \cdot \operatorname{sgn}(l_2)$$

$$l_{ext,3} = f\big(S - f(|l_3|)\big) \cdot P \cdot \operatorname{sgn}(l_3)$$