Polar Codes

Introduction and definition

Polar codes

- Invented by Erdal Arikan 2008
- Based on the idea of channel polarization
- First codes to have explicit proof for approaching capacity
- Included as codes for the control channels in the 5G standard
- Sequential in nature
- Defined using generator matrix in a recursive (Kornecker product) definition

Polar transform: G_2

2 bits to 2 bits

$$G_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$ kernel

mod2 (or) XOR

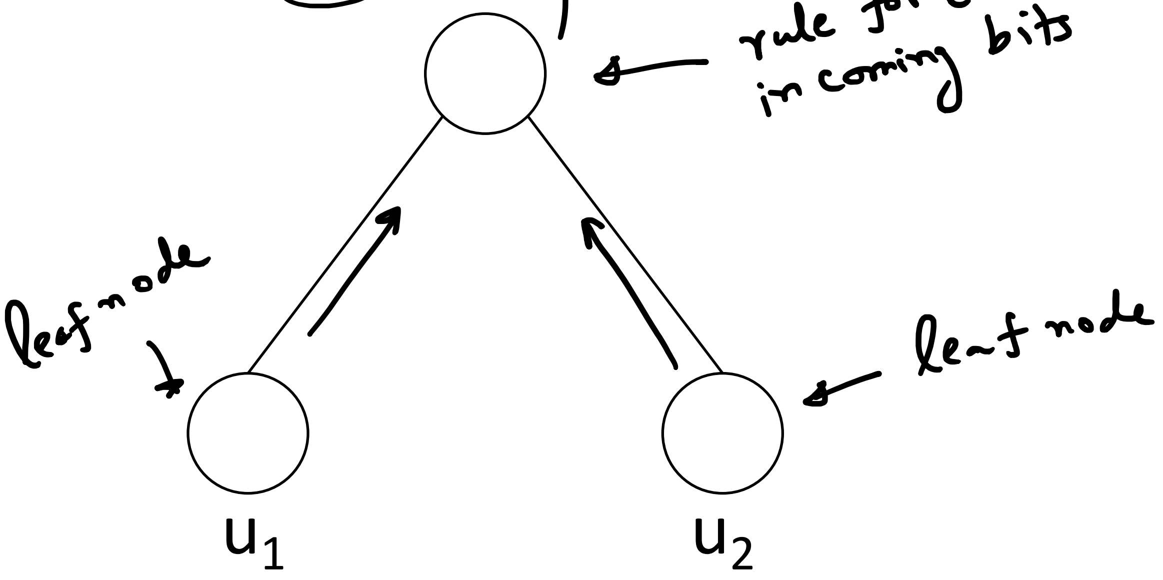
- $[u_1\ u_2]\, G_2 = [u_1{+}u_2\ \ u_2]$

input output

0 0	0 0
0 1	1 1
1 0	1 0
1 1	0 1

Binary tree representation

output $u^{(2)} = [u_1{+}u_2\ \ u_2]$



$u^{(2)}$: length-2 vector

Polar transform: G_4

4 bits to 4 bits

Kronecker product (or) tensor product

$$G_4 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix}} & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \\ \boxed{\begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix}} & \boxed{\begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix}} \end{bmatrix}$$ kernel

+ : mod 2 (or) XOR

- $[u_1\ u_2\ u_3\ u_4]\ G_4 = [\underline{u_1+u_2+u_3+u_4}\ \ \underline{u_2+u_4}\ \ \underline{u_3+u_4}\ \ \underline{u_4}]$

input output

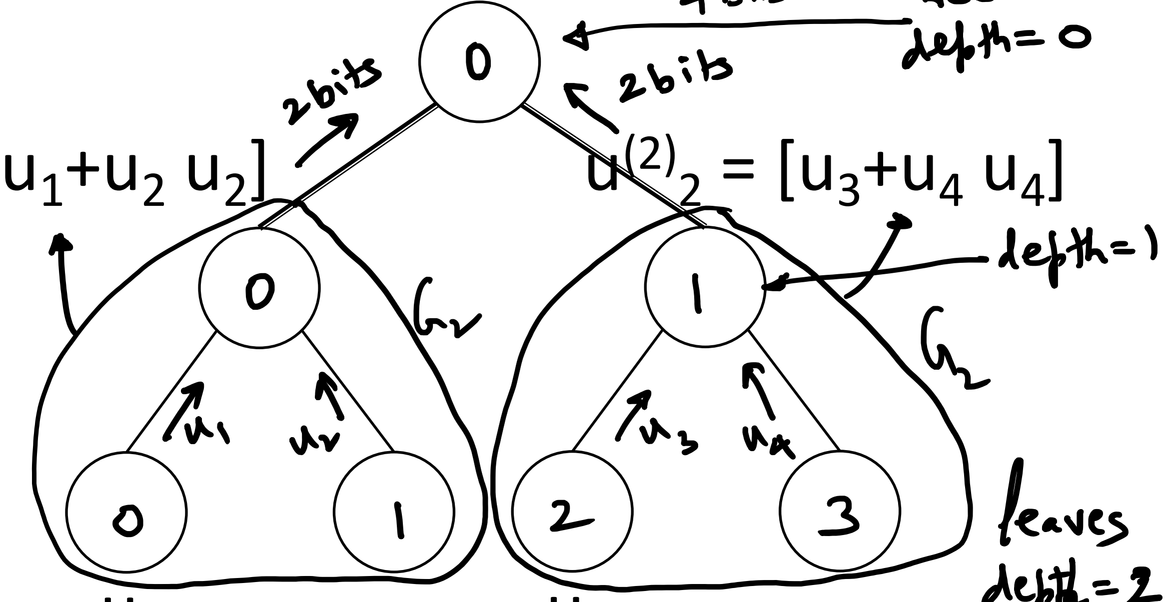
Binary tree representation

depth 2

output $u^{(4)} = [u^{(2)}_1 + u^{(2)}_2 \quad u^{(2)}_2]$

$u^{(2)}_1 = [u_1+u_2\ u_2]$

$u^{(2)}_2 = [u_3+u_4\ u_4]$



u_1 u_2 u_3 u_4

(node, depth) : uniquely determine node

(0, 0) : root

(0, 2) : leaf (u_1) etc....

$$G_8 = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} \otimes G_4 = G_2 \otimes \underbrace{G_2 \otimes G_2}_{G_4}$$

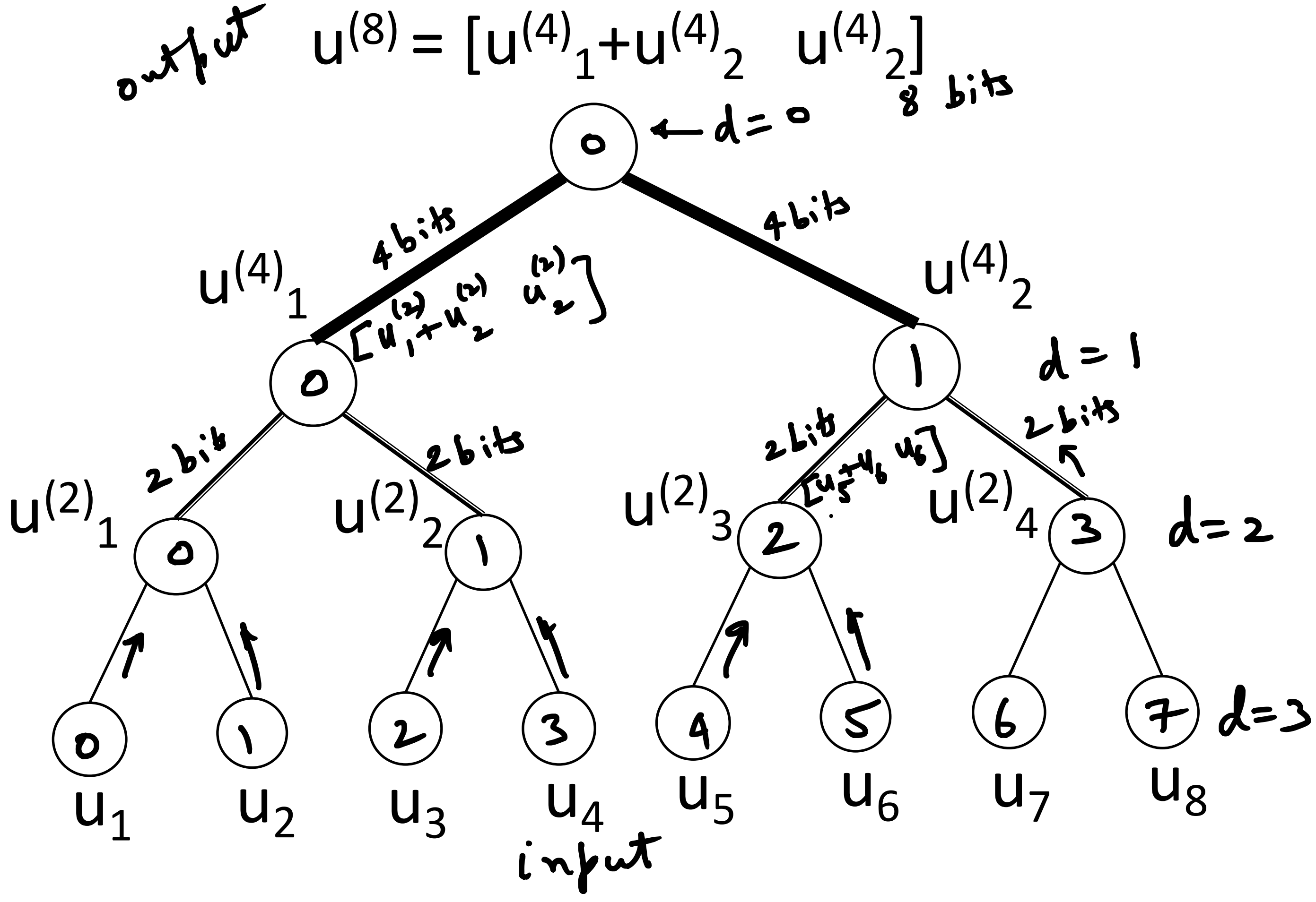
Polar transform: G_8

$$G_8 = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}^{\otimes 3}$$

3 times kronecker product
8 bits to 8 bits

$$= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0\\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}$$

Binary tree representation



- $[u_1\ u_2\ u_3\ u_4\ u_5\ u_6\ u_7\ u_8]\ G_8 = \begin{bmatrix} u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8 & u_2+u_4+u_6+u_8 & u_3+u_4+u_7+u_8 & u_4+u_8 \\ u_5+u_6+u_7+u_8 & u_6+u_8 & u_7+u_8 & u_8 \end{bmatrix}$

input

output

Polar Transform: General

2^n bits to 2^n bits

kernel

$$G_{2^n} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{\otimes n} = G_2 \otimes G_{2^{n-1}} = G_2 \otimes G_2 \otimes G_{2^{n-2}} = \cdots$$

- $N = 2^n$ $n = 1, 2, 3, \ldots$
- G_N : N x N matrix, Kronecker product of 2 x 2 kernel
- Binary tree representation
  - Depth n
  - $u^{(N)} = u\, G_N$: evaluated on tree with u at bottom and $u^{(N)}$ at top
- 5G: uses up to $n = 10$