Introduction to Error-Correction Codes

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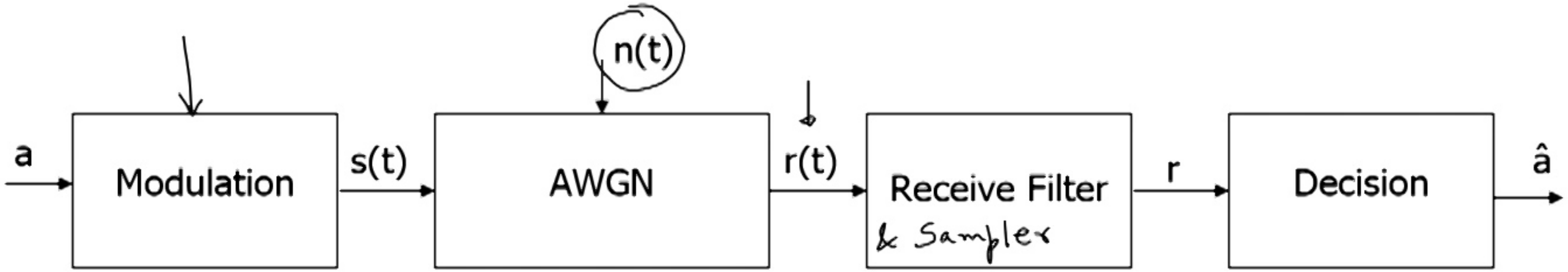
Outline

- Digital Signaling
  - Modulation Schemes: BPSK
  - Bit-Error Rate (BER) vs. Signal-to-Noise Ratio (SNR)
- Error-Correction Coding (ECC)
  - ECC in Digital Communication Systems
  - BER vs. Normalized SNR with ECC
- ECC Operation
  - Encoding and Decoding
  - Simple Codes: Repetition codes etc.

Modulation over AWGN channel



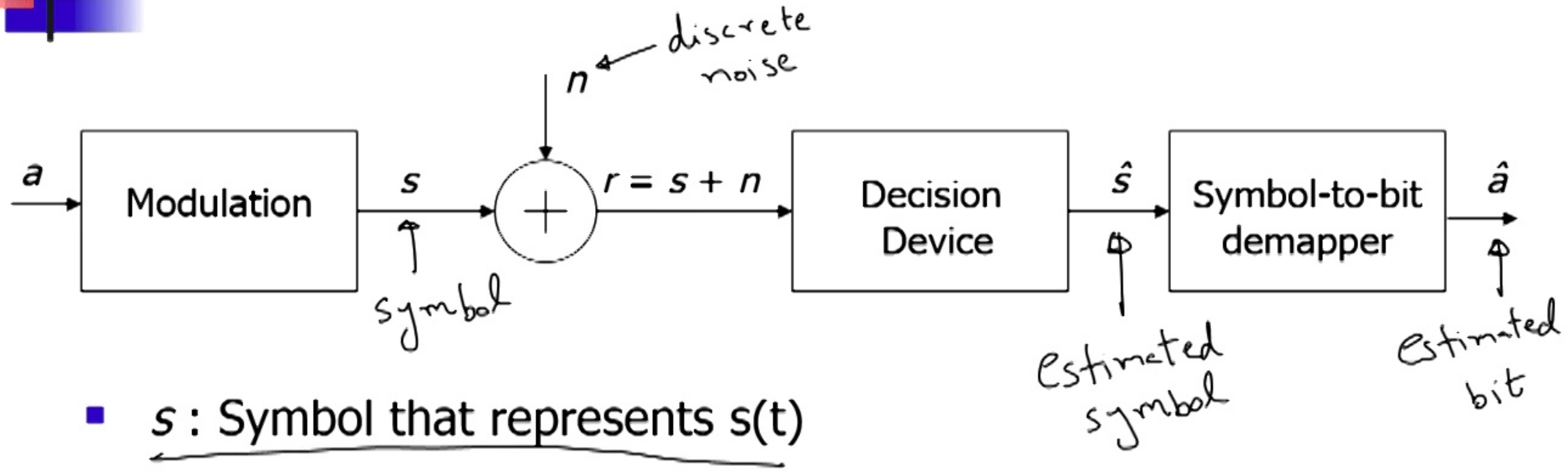
a : Bits to be transmitted
â : Receiver's estimated value of transmitted bit

s(t) : Waveform transmitted into the channel

n(t): White Gaussian Noise
r(t) = s(t) + n(t) : Waveform received after addition of noise

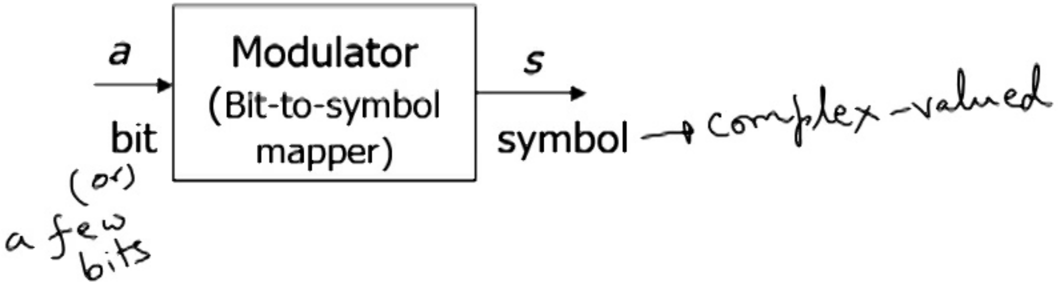
r : Information about waveform after filtering at the receiver

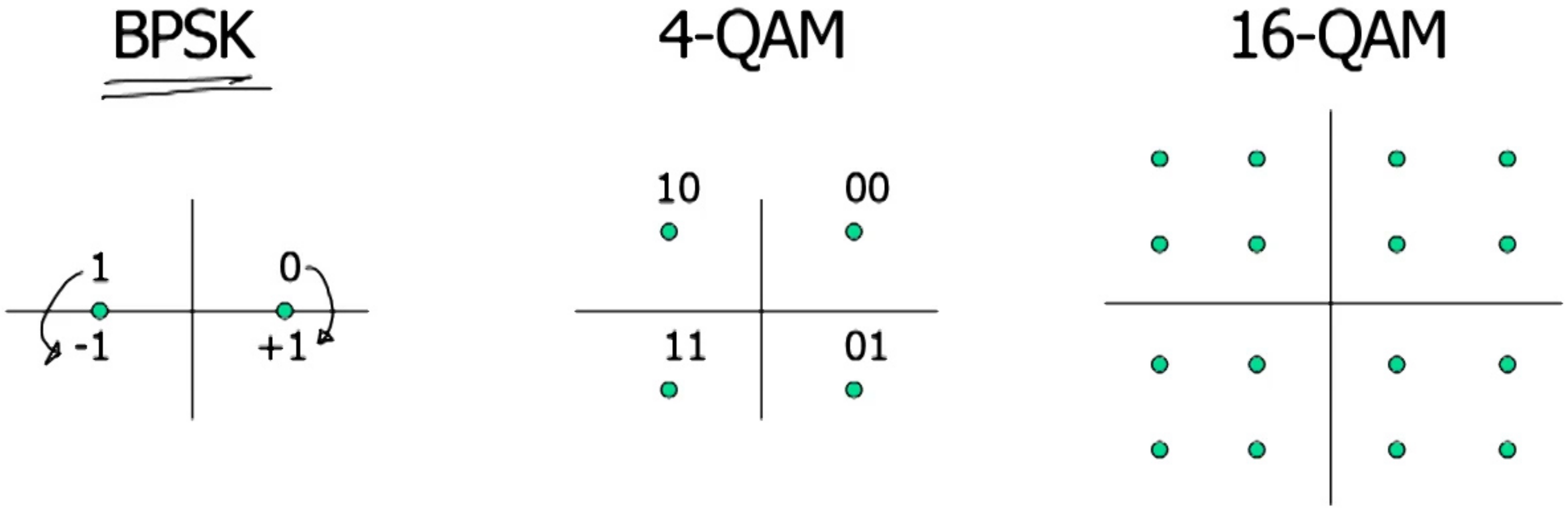
Discrete-time AWGN channel



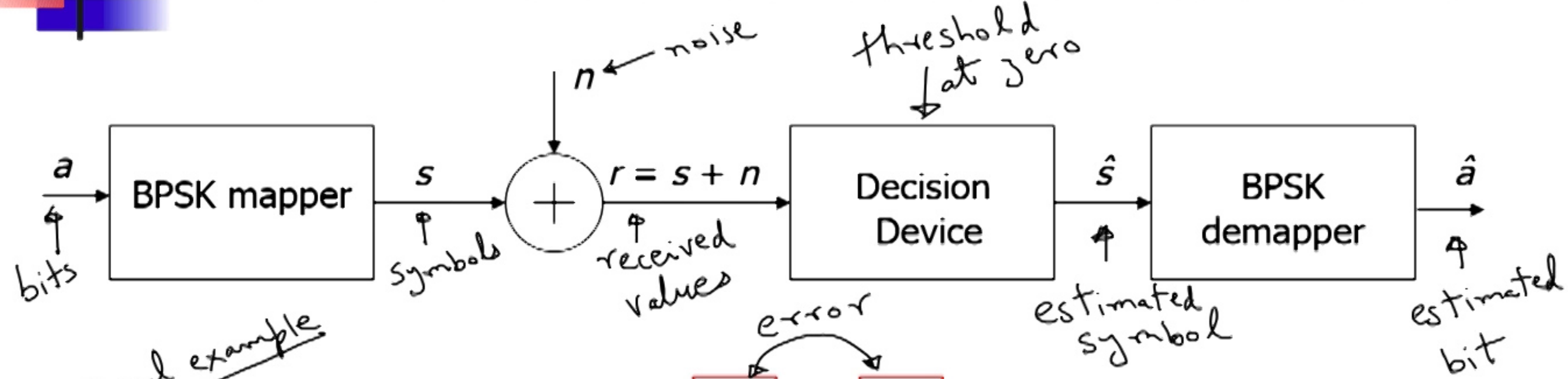
- s : Symbol that represents s(t)
- n : Noise after receive filtering
  - Gaussian-distributed
  - Independent from symbol to symbol
- Assumption: Receive filter produces sufficient statistics

Modulation Schemes





BPSK over AWGN



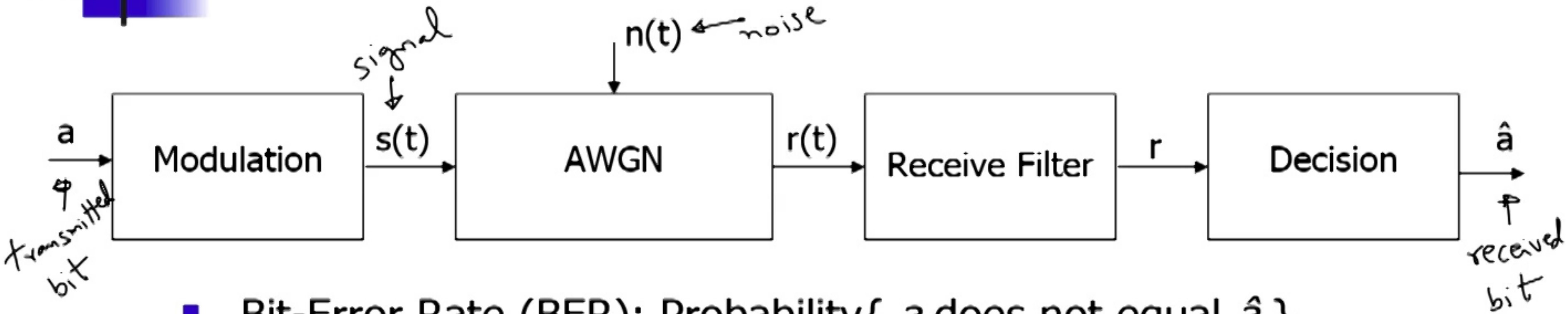
Typical example

error

a :	0	0	1	0	1	$\boxed{1}$	0	$\boxed{1}$	1	0	1
s :	+1	+1	−1	+1	−1	$\boxed{-1}$	+1	$\boxed{-1}$	−1	+1	−1
r :	0.8	0.2	-0.8	1.9	-0.6	$\boxed{0.2}$	1.3	$\boxed{0.1}$	-1.2	0.3	-1.1
$\hat{s}$:	+1	+1	−1	+1	−1	$\boxed{+1}$	+1	$\boxed{+1}$	−1	+1	−1
$\hat{a}$:	0	0	1	0	1	$\boxed{0}$	0	$\boxed{0}$	1	0	1

Note: <u>Optimum Decision</u> => $\hat{s}$ is the symbol closest to r

Error Rates and SNR



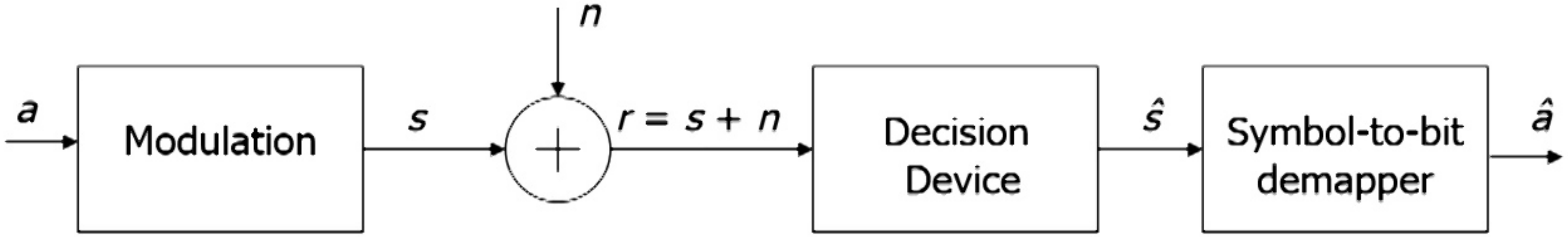
- Bit-Error Rate (BER): Probability{ a does not equal $\hat{a}$ }
- Estimating BER:
  - Transmit N bits (say N = 10000000…)
  - Find number of errors n_e (n_e must be at least 100)
  - BER = n_e/N

Simulation method

SNR	SNR (dB)
1	0 dB
2	≈ 3 dB
⋮	⋮

- Signal-to-Noise Ratio (SNR) = Signal power / Noise power
- SNR (in dB) = $10 \log_{10}(\underbrace{\text{Signal power / Noise power}}_{\text{SNR}})$

SNR in Continuous/Discrete time



Continuous-time AWGN Channel:

Signal Power = P; Noise PSD = $N_0/2$; Bandwidth = 2W

Noise power = $\frac{N_0}{2} \times 2W = N_0 W$

SNR = $P/(N_0 W)$

Time per symbol = T = 1/(2W)

symbol rate = 2W

Discrete-time AWGN Channel:

Energy per symbol, E_s = PT = P/(2W)

energy = power × time

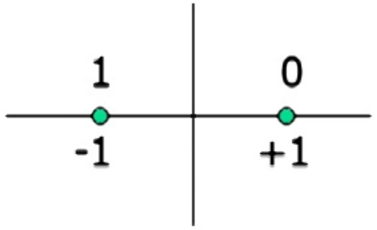
Noise Energy = σ^2 = Variance of noise n = $N_0/2$

"noise" converts from continuous to discrete-time

SNR in discrete time = $\boxed{E_s / \sigma^2 = P/(N_0 W)}$ = SNR in continuous time

Calculating SNR

BPSK



- symbols are equally likely

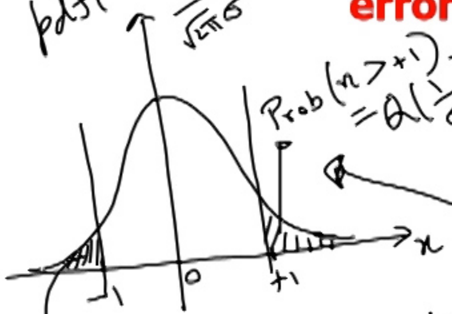
- Signal Energy, E_s
  - Mean of square of symbols = $[(-1)^2 + (+1)^2]/2 = 1$
- Noise Power = σ^2
- SNR = $1 / \sigma^2$

BER vs SNR for BPSK over AWGN

pdf$(n=x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$

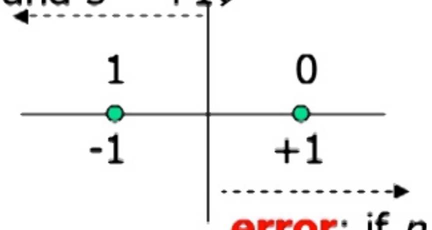
threshold at zero

error: if n < -1 and s = +1



Prob$(n>+1) = Q(\frac{1}{\sigma})$

Prob$(n<-1) = Q(1/\sigma)$



error: if n > 1 and s = -1

$$\mathrm{BER} = \overbrace{\mathrm{Prob}\{s = +1\}}^{1/2}\mathrm{Prob}\{n < -1\} + \overbrace{\mathrm{Prob}\{s = -1\}}^{1/2}\overbrace{\mathrm{Prob}\{n > +1\}}^{Q(1/\sigma)}$$

$\mathrm{SNR} = \frac{1}{\sigma^2}$

$\frac{1}{\sigma} = \sqrt{\mathrm{SNR}}$

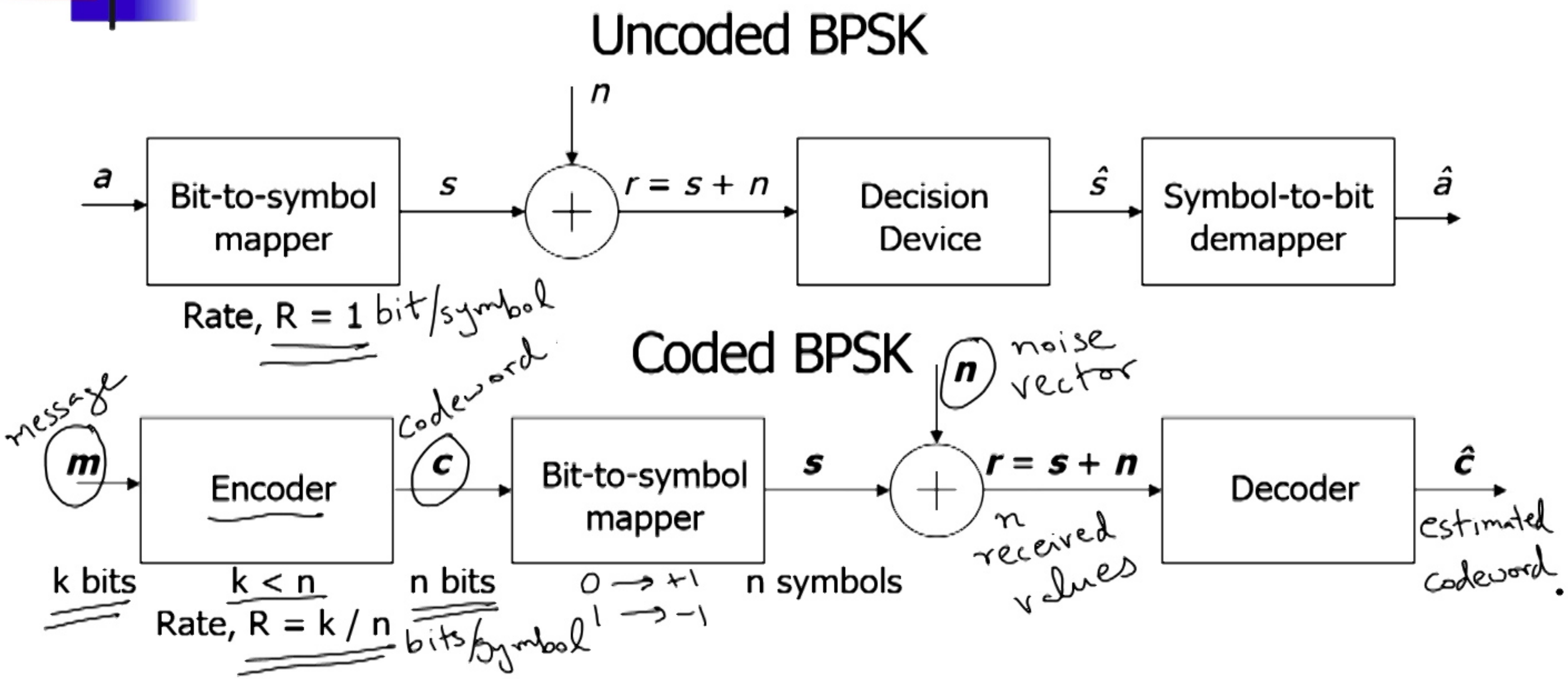
Using pdf(n = x) with variance σ^2,

$$\mathrm{BER} = Q(1/\sigma) = \underbrace{Q(\mathrm{sqrt(SNR)})}_{Q(\sqrt{\mathrm{SNR}})}$$

Q function:

$$Q(x) = 0.5 * \mathrm{erfc}\,(x / \mathrm{sqrt}(2))$$

Error-Correction Coding



Coding enables same BER at lower SNRs!

E_b/N_0 vs. BER

Signal energy per information bit:

$E_b = E_s/R = \frac{n E_s}{k}$ n codeword bits, k → message bits

Noise power:

$N_0/2 = \sigma^2$

$E_s = R E_b$

SNR = $E_s/\sigma^2 = RE_b/(N_0/2)$

Or

SNR = $2R\,(E_b/N_0)$ $\quad \frac{E_b}{N_0} = \frac{SNR}{2R}$

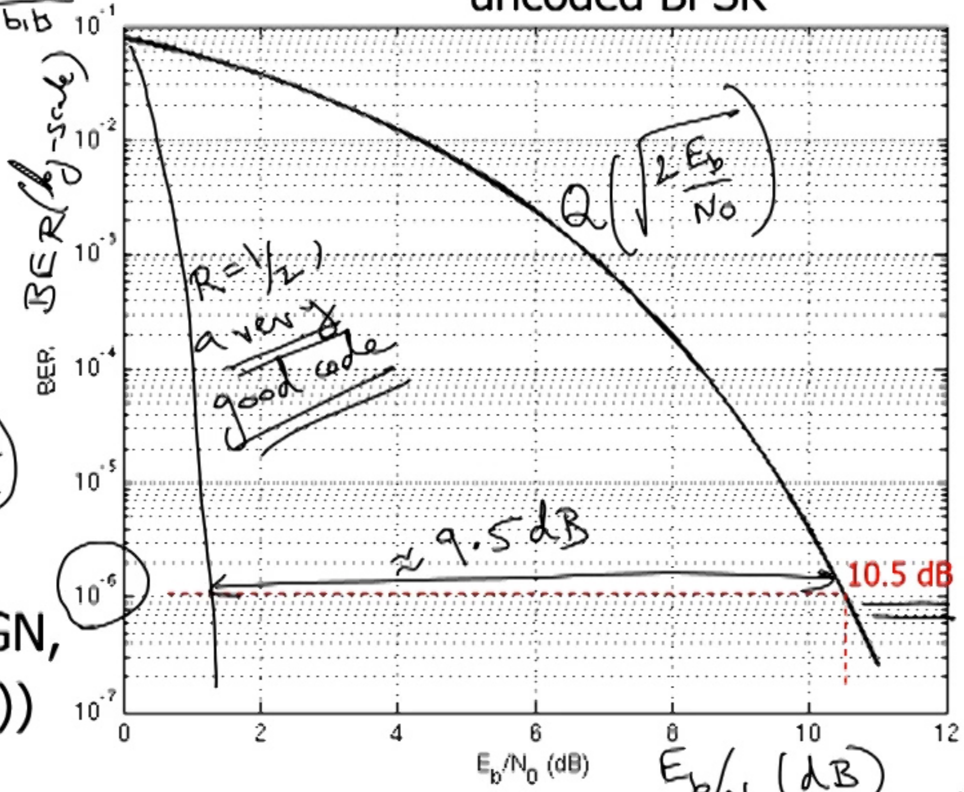
BPSK: $\boxed{\frac{E_b}{N_0} = \frac{1}{2R\sigma^2}}$

For uncoded BPSK over AWGN,

BER = $Q(1/\sigma) = Q(\text{sqrt}(\text{SNR}))$

$= Q(\text{ sqrt}(2(E_b/N_0)) \text{ })$

Plot of BER vs. E_b/N_0 (dB) for uncoded BPSK



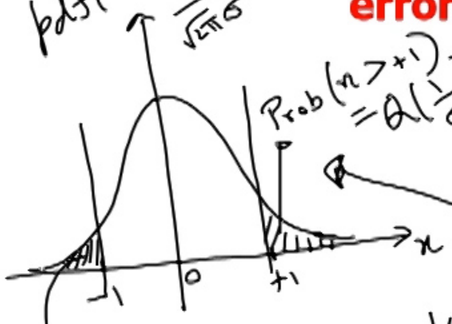
E_b/N_0 (dB) $= 10\log_{10}\left(\frac{1}{2R\sigma^2}\right)$

R=1 (uncoded)

$\boxed{\text{Coding enables same BER at lower } E_b/N_0\text{'s!}}$

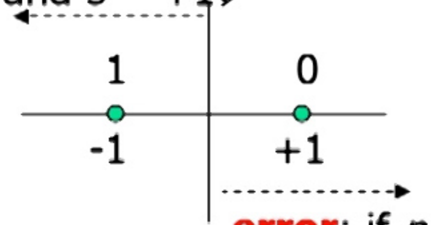
BER vs SNR for BPSK over AWGN

$\text{pdf}(n=x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$



threshold at zero

error: if n < -1 and s = +1



error: if n > 1 and s = -1

$$\text{BER} = \underbrace{\overset{1/2}{\text{Prob}\{s = +1\}}\text{Prob}\{n < -1\}} + \underbrace{\overset{1/2}{\text{Prob}\{s = +1\}}\overset{Q(1/\sigma)}{\text{Prob}\{n < +1\}}}$$

Using pdf(n = x) with variance σ^2,

$$\text{BER} = Q(1/\sigma) = \underbrace{Q(\text{sqrt}(\text{SNR}))}_{Q(\sqrt{SNR})}$$

$SNR = \frac{1}{\sigma^2}$

$\frac{1}{\sigma} = \sqrt{SNR}$

Q function:

$Q(x) = 0.5 * \text{erfc}(x / \text{sqrt}(2))$