5. Levels of reliability methods

There are different levels of reliability analysis, which can be used in any design methodology depending on the importance of the structure. The term **'level'** is characterized by the extent of information about the problem that is used and provided. The methods of safety analysis proposed currently for the attainment of a given limit state can be grouped under four basic "levels" (namely levels IV, III, II, and I) depending upon the degree of sophistication applied to the treatment of the various problems.

1. In **level I** methods, the probabilistic aspect of the problem is taken into account by introducing into the safety analysis suitable "characteristic values" of the random variables, conceived as fractile of a predefined order of the statistical distributions concerned. These characteristic values are associated with partial safety factors that should be deduced from probabilistic considerations so as to ensure appropriate levels of reliability in the design. In this method, the reliability of the design deviate from the target value, and the objective is to minimize such an error. Load and Resistance Factor Design (LRFD) method comes under this category.

2. Reliability methods, which employ two values of each uncertain parameter (i.e., mean and variance), supplemented with a measure of the correlation between parameters, are classified as **level II** methods.

3. **Level III** methods encompass complete analysis of the problem and also involve integration of the multidimensional joint probability density function of the

random variables extended over the safety domain. Reliability is expressed in terms of suitable safety indices, viz., reliability index, β and failure probabilities.

4. **Level IV** methods are appropriate for structures that are of major economic importance, involve the principles of engineering economic analysis under uncertainty, and consider costs and benefits of construction, maintenance, repair, consequences of failure, and interest on capital, etc. Foundations for sensitive projects like nuclear power projects, transmission towers, highway bridges, are suitable objects of level IV design.

5. 1. Space of State Variables

For analysis, we need to define the state variables of the problem. The *state variables* are the basic load and resistance parameters used to formulate the performance function. For 'n' state variables, the limit state function is a function of 'n' parameters .

If all loads (or load effects) are represented by the variable Q and total resistance (or capacity) by R, then the space of state variables is a two-dimensional space as shown in Figure 1. Within this space, we can separate the "safe domain" from the "failure domain"; the boundary between the two domains is described by the limit state function $g(R,Q)=0$.

Since both R and Q are random variables, we can define a joint density function f_{RQ} (r, q). A general joint density function is plotted in Figure 2. Again, the limit state function separates the safe and failure domains. The probability of failure is calculated by integration of the joint density function over the failure domain [i.e., the region in which $g(R, Q) <0$]. As noted earlier, this probability is often very difficult to evaluate, so the concept of a reliability index is used to quantify structural reliability.
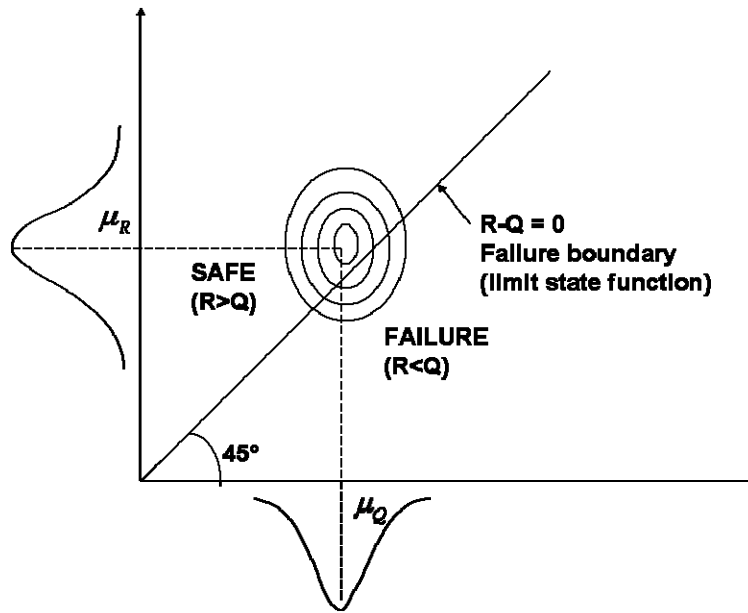
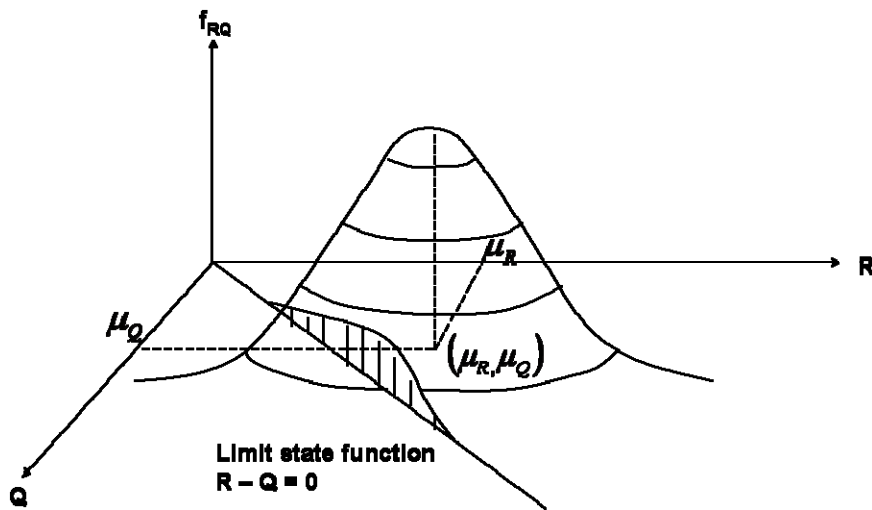**Figure 1 - Safe domain and failure domain in two dimensional state spaces.**

Figure 2 - Three-dimensional representation of a possible joint density function f_{RQ}

5.3. RELIABILITY INDEX

Reduced Variables

It is convenient to convert all random variables to their "standard form;' which is a non dimensional form of the variables. For the basic variables R and Q, the standard forms can be expressed as

$$Z_R = \frac{R - \mu_R}{\sigma_R}$$

$$Z_Q = \frac{R - \mu_Q}{\sigma_Q} \quad \text{----------------------------- (1)}$$

The variables Z_R and Z_Q, are sometimes called *reduced variables*. By rearranging Equation no.1, the resistance R and the load Q can be expressed in terms of the reduced variables as follows:

$$R = \mu_R + Z_R \sigma_R$$

$$R = \mu_Q + Z_Q \sigma_Q \text{----------------------------- (2)}$$

The limit state function g(R, Q) = R-Q can be expressed in terms of the reduced variables by using Eqs. 2. The result is

$$g(Z_R, Z_Q) = \mu_R + Z_R \sigma_R - \mu_Q - Z_Q \sigma_Q = (\mu_R - \mu_Q) + Z_R \sigma_R - Z_Q \sigma_Q \text{------------(3)}$$

For any specific value of g(Z_R, Z_Q), Equation no.3 represents a straight line in the space reduced variables Z_R and Z_Q. The line corresponding to g(Z_R, Z_Q) =0 separates the safe and failure domain in the space of reduced variables. The loads Q and resistances R are some times indicated in terms of capacity C and demand D as well in literature.

Design for a given reliability index

$$P_s = R = 1 - \phi \left(\frac{C - D}{\sqrt{\sigma_C^2 + \sigma_D^2}} \right) \qquad if \ variable \ are \ normal \qquad \text{------(1)}$$

$$P_s = 1 - P_f = 1 - \phi \left[\frac{\ln \left\{ \frac{\overline{C}}{D} \sqrt{\frac{1 + V_D^2}{V_C^2}} \right\}}{\sqrt{\ln(1 + V_D^2)(1 + V_C^2)}} \right] \quad if \ C \ and \ D \ are \ \log normal \quad \text{--------(2)}$$

we define $\text{CFS} = \dfrac{\overline{C}}{D}$ and write (1) and (2) in terms of CFS

$$\beta = \frac{CFS - 1}{\sqrt{(CFS)^2 V_C^2 + V_D^2}} \quad for \ C \ and \ D \ are \ normal$$

$$\beta = \frac{\ln\left(CFS\sqrt{\frac{1 + V_D^2}{1 + V_C^2}}\right)}{\sqrt{\ln\left(1 + V_D^2\right)\left(1 + V_C^2\right)}}$$

5.3.1. General Definition of the Reliability Index

A version of the reliability index was defined as the inverse of the coefficient of variation The reliability index is the *shortest* distance from the origin of reduced variables to the is illustrated in Figure 3, line g(Z_R, Z_Q) = 0 .This definition, which was introduced by Hasofer and Lind (1974) following formula:

Using geometry we can calculate the reliability index (shortest distance) from the following formula:

$$\beta = \frac{\mu_R - \mu_Q}{\sqrt{\sigma_R^2 + \sigma_Q^2}} \quad \text{------------------------------- (4)}$$

where β is the inverse of the coefficient of variation of the function g(R, Q) = R-Q When R and Q are uncorrelated for normally distributed random variables R and Q, it can be shown that the reliability index is related to the probability of failure by

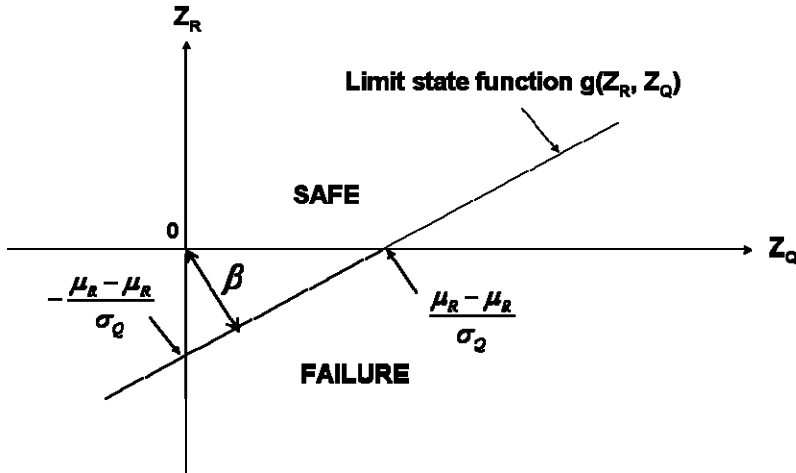$$\beta = -\Phi^{-1}(P_f) \quad or \quad P_f = \Phi(-\beta)$$

Figure 3 - Reliability index defined as the shortest distance in the space of reduced variables.

Table 1 provides an indication of how β varies with P_f.

Table 1- Reliability index β and probability of failure P_f

P_f	β
10^{-1}	1.28
10^{-2}	2.33
10^{-3}	3.09
10^{-4}	3.71
10^{-5}	4.26
10^{-6}	4.75
10^{-7}	5.19
10^{-8}	5.62
10^{-9}	5.99

The definition for a two variable case can be generalized for n variables as follows. Consider a limit state function $g(X_1, X_2 \ldots\ldots X_n)$, Where the X_i variables are all uncorrelated. The Hasofer-Lind reliability index is defined as follows:

1. Define the set of reduced variables $\{Z_1, Z_2, \ldots, Z_n\}$ using

$$Z_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}}$$

------------------------------(5)

2. Redefine the limit state function by expressing it in terms of the reduced variables $(Z_1, Z_2, .., Z_n)$.

3. The reliability index is the shortest distance from the origin in the n-dimensional space of reduced variables to the curve described by $g(Z_1, Z_2, \ldots, Z_n) = 0$.

5.4. First-order second moment method (FOSM)

This method is also referred to as *mean value first-order second moment (MVFOSM) method*, and it is based on the first order Taylor series approximation of the performance function linearized at the mean values of the random variables. It uses only second-moment statistics (mean and variance) of the random variables. Originally, Cornell (1969) used the simple two variable approaches. On the basic assumption that the resulting probability of Z is a normal distribution, by some relevant virtue of the central limit theorem, Cornell (1969) defined the reliability index as the ratio of the expected value of Z over its standard deviation. The Cornell reliability index (β_c) is the absolute value of the ordinate of the point corresponding to $Z = 0$ on the standardized normal probability plot as given in Figure 4 and equation.
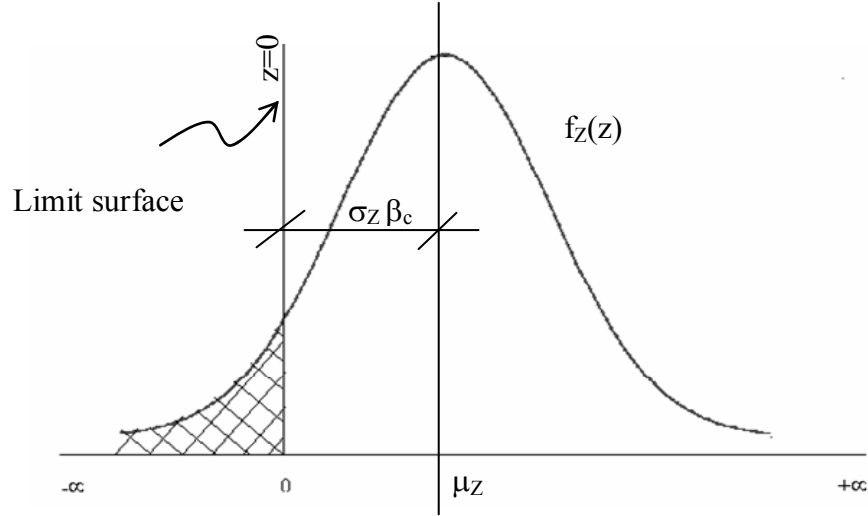
Figure 4- Definition of limit state and reliability index

$$\beta_c = \frac{\mu_z}{\sigma_z} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \text{---------------------------------(6)}$$

On the other hand, if the joint probability density function $f_X(x)$ is known for the multi variable case, then the probability of failure p_f is given by

$$p_f = \int_L f_X(x)\,dX \text{---------------------------------(7)}$$

where L is the domain of X where g(X)<0.

In general, the above integral cannot be solved analytically, and an approximation is obtained by the FORM approach. In this approach, the general case is approximated to an ideal situation where X is a vector of independent Gaussian variables with zero mean and unit standard deviation, and where g(X) is a linear function. The probability of failure p_f is then:

$$p_f = P(g(X) < 0) = P(\sum_{i=1}^{n} \alpha_i X_i - \beta < 0) = \Phi(-\beta) \text{---------------------------------(8)}$$

where α_i is the direction cosine of random variable X_i, β is the distance between the origin and the hyper plane $g(X)=0$, n is the number of basic random variables X, and Φ is the standard normal distribution function.

The above formulations can be generalized for many random variables denoted by the vector X. Let the performance function is in the form given as

$$Z = g(X) = g(X_1, X_2....X_n) \text{ -----------------------------------(9)}$$

A Taylor series expansion of the performance function about the mean value is given by equation.

$$Z = g(\mu_X) + \sum_{i=1}^{n} \frac{\partial g}{\partial X_i}(X_i - \mu_{X_i}) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 g}{\partial X_i \partial X_j}(X_i - \mu_{X_i})(X_j - \mu_{X_j}) + ... \text{ ------(10)}$$

where derivatives are evaluated at the mean values of the random variables $(X_1, X_2... X_n)$ and μ_{X_i} is the mean value of X_i. Truncating the series in linear terms, the first order mean and variance of Z can be obtained as

$$\mu_Z \approx g(\mu_{X_1}, \mu_{X_2},.....\mu_{X_n})$$

and, $\sigma_Z^2 \approx \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial g}{\partial X_i}\frac{\partial g}{\partial X_j}\text{var}(X_i, X_j)$ --------------------------------(11)

where var (X_i, X_j) is the covariance of X_i and X_j. If the variances are uncorrelated, then the variance for z is given as

$$\sigma_Z^2 \approx \sum_{i=1}^{n} \left(\frac{\partial g}{\partial X_i} \right)^2 \text{var}(X_i) \text{-----------------------------------} (12)$$

The reliability index can be calculated by taking the ratio of mean $\left(\mu_z\right)$ and standard deviation of Z $\left(\sigma_z\right)$ as in Equation 2.66.

$$\beta = \frac{\mu_z}{\sigma_z}$$

5.4.1. Linear limit state functions

Consider *a linear* limit state function of the form

$$g(X_1, X_2, X_3..........X_n) = a_0 + a_1 X_1 + a_2 X_2 + a_3 X_3............a_n X_n = a_0 + \sum_{i=1}^{n} a_i X_i$$

where the a_i terms (i = 0, 1, 2……. n) are constants and the X_i terms are *uncorrelated* random variables. If we apply the three-step procedure outlined above for determining the Hasofer-Lind reliability index, we would obtain the following expression for β:

$$\beta = \frac{a_0 + \sum_{i=1}^{n} a_i \mu_{X_i}}{\sqrt{\sum_{i=1}^{n} (a_i \sigma_{X_i})^2}} \text{------------------------------} (13)$$

Observe that the reliability index ,β , in Eq. 5.18 depends only on the deviations of the random variables Therefore this is called a *second moment measure* of structural safety because only the first two moments (mean and variance) to calculate . There is no explicit relationship between β and the distributions of the type of probability to calculate β variables. If the random variables are all normally distributed

and uncorrelated then this formula is *exact* in the sense that β and P_f are related by Eq**.** 5.15. Otherwise, Eq. 5.15 Provides only an approximate means of Probability of failure. The method discussed above has some limitations and deficiencies. It does not use the distribution information about the variable and function g( ) is linearized at the mean values of the X_i variables. If g( ) is non-linear, neglecting of higher order term in Taylor series expansion introduces significant error in the calculation of reliability index. The more important observation is that the Equations 2.58 and 2.64 do not give constant value of reliability index for mechanically equivalent formulations of the same performance function. For example, safety margin R-S<0 and R/S <1 are mechanically equivalent yet these safety margins will not lead to same value of probability of failure. Moreover, MVFOSM approach does not use the distribution information about the variables when it is available.

Nonlinear limit state functions

Now consider the case of a *nonlinear* limit state function. When the function is nonlinear, we can obtain an approximate answer by linearizing the nonlinear function using a Taylor series expansion. The result is

$$g(X_1, X_2, X_3........X_n) \approx g(x_1^*, x_2^*.....x_n^*) + \sum_{i=1}^{n}(X_i - x_i^*)\frac{\partial g}{\partial X_i}\Big|_{\text{evaluated at }(x_1^*, x_2^*.....x_n^*)}$$

where $(x_1^*, x_2^*.....x_n^*)$ is the point about which the expansion is performed**.** One choice for this linearization point is the point corresponding to the mean values of the random variables.   Thus   Eq.   5.19   becomes

$$g(X_1, X_2, X_3.......X_n) \approx g(\mu x_1, \mu x_2,...........\mu x_n) + \sum_{i=1}^{n}(X_i - \mu x_i)\frac{\partial g}{\partial X_i}\Big|_{\text{evaluatedat }(x_1^*, x_2^*.....x_n^*)}$$

Since Eq. 5.20 is a linear function of the X_i variables, it can be rewritten to be exactly like Eq. 5.17. Thus Eq. 5.18 can be used as an approximate solution for ti reliability index . After some algebraic manipulations, the following expression for results:

$$\beta = \frac{g(\mu_{X_1}, \mu_{X_2}, \ldots \ldots \mu_{X_n})}{\sqrt{\sum_{i=1}^{n}(a_i \sigma_{Xi})^2}} \qquad where \quad a_i = \frac{\partial g}{\partial X_i}\Big|_{evaluated\ at\ (x_1^*, x_2^* \ldots x_n^*)}$$

The reliability index defined in Eq. 5.21 is called *a first-order second-moment mean value reliability index.* It is a long name, but the underlying meaning of each part of the name is very important:

First order because we use first-order terms in the Taylor series expansion , *Second moment* because only means and variances are needed. *Mean value* because the Taylor series expansion is about the mean values.

5.4.2. Comments on the First-Order Second-Moment Mean Value Index

The first-order second-moment mean value method is based on approximating non normal CDFs of the state variables by normal variables, as shown in Figure 5.15 for the
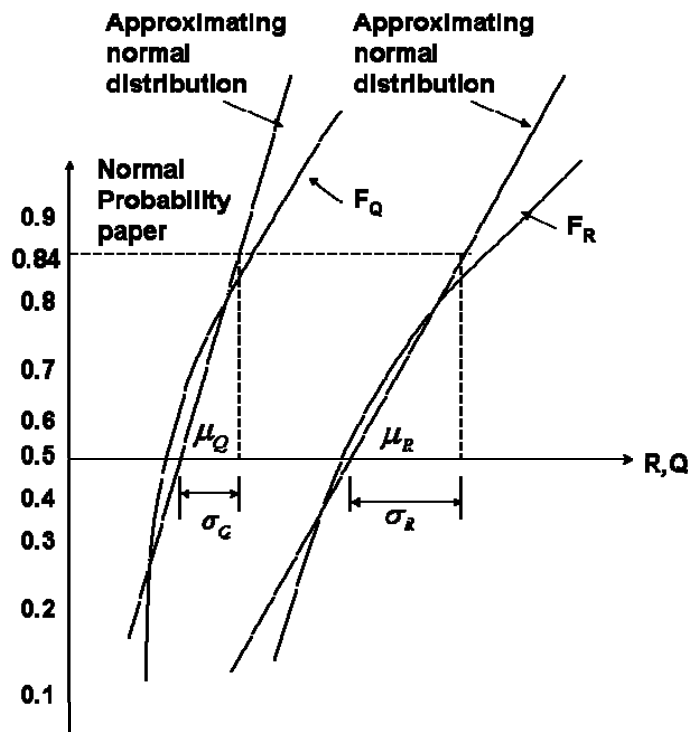
Figure 5 - Mean value second - moment formulation.

simple case in which $g(R, Q) = R - Q$. The method has both advantages and disadvantages in structural reliability analysis.

Among its advantages:

1. It is easy to use.
2. It does not require knowledge of the distributions of the random variables.

Among its disadvantages:

1. Results are inaccurate if the tails of the distribution functions cannot be approximate by a normal distribution.
2. 2. There is an invariance problem: the value of the reliability index depends on the specific form of the limit state function.

The invariance problem is best clarified by an example.

5.5. Advanced first-order second moment method (AFOSM)

It is essential that irrespective of method of evaluation of reliability of a limit state, all the mechanically equivalent performance functions must produce same safety indices. However the MVFOSM method fails to satisfy the above condition in some cases, such as in case of correlated variables and nonlinear limit state formulations. Hence, a new approach, called Hasofer-Lind reliability index (Hasofer and Lind 1974) was developed to tackle the problem of variant reliability indices produced using Cornell index. In this method the reduced variables are defined as given in Equation 2.67.

$$X_i' = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}}, \text{ i=1, 2 ...n} \quad \text{----------------------------------(14)}$$

where X_i' is a random variable with zero mean and unit standard deviation. The above equation is used to transform the original limit state $g(X) = 0$ to reduced limit state $g'(X) = 0$. X is referred to as the original co-ordinate system and X′ reduced co-ordinate system. Note that if X_i is normal in original co-ordinate system it will be standard normal in reduced co-ordinate system.

The Hasofer-Lind reliability index (β_{HL}) can be defined as the minimum distance from the origin of the axes in the reduced co-ordinate system to the limit state surface. The minimum distance point on the limit state surface is called the design point or checking point. Considering the limit state function in two variables as given in Equation 2.68, wherein R and S should be normal variables, the reduced variables can be written as given in equations.

$$Z = R - S = 0$$

$$R' = \frac{R - \mu_R}{\sigma_R}$$

$$S' = \frac{S - \mu_S}{\sigma_S}$$

Substituting values of R' and S' in the above equation, the limit state equation in the reduced co-ordinate system can be written as

$$g() = \sigma_R R' - \sigma_S S' + \mu_R - \mu_S = 0 \text{----------------------------------(15)}$$

The position of the limit state surface relative to the origin in the reduced coordinate system is a measure of the reliability of the system. By simple trigonometry, the distance of the limit state line from the origin can be calculated and it will give the reliability index value.

$$\beta_{HL} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \text{-----------------------------------(16)}$$

This is same as the reliability index defined by the MVFOSM method, if both R and S are normal. In this definition the reliability index is invariant, because regardless of the form in which the limit state equation is written, its geometric shape and the distance from the origin remains constant.

To be specific, β is the First-order second moment reliability index, defined as the minimum distance from the origin of the standard, independent normal variable space to the failure surface as discussed in detail by Hasofer and Lind (1974). Figure 2.5 shows the plot depicting the functional relationship between probability of failure (p_f) and

reliability index (β), and classifies the performance of designs based on these two values. As seen from the figure, the performance is high if the reliability index is equal to 5, which corresponds to a probability of failure of approximately 3×10^{-7}.

5.5.1. Hasofer-Lind Reliability Index

Hasofer and Lind proposed a modified reliability index that did not exhibit the invariance problem illustrated in Example *5.3*. The "correction" is to evaluate the limit state function at a point known as the "design point" instead of the mean values. The design point is a point on the failure surface g = 0. Since this design point is generally not known a priori, an iteration technique must be used (in general) to solve for the reliability index.

5.5.2. AFOSM Method for Normal Variables

The *Hasofer-Lind* (H-L) method is applicable for normal random variables. It first defines the reduced variables as

$$X_i' = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}} (i = 1,2,.......,n) \text{ --------------------------------(17)}$$

where X_i' is a random variable with zero mean and unit standard deviation. Above equation is used to transform the original limit state g(X) = 0 to the reduced limit state g(X`)= 0. The X coordinate system is referred to as the *original coordinate syatem*. The X` coordinate system is referred to as the *transformed* or *reduced coordinate* system. Note that if X_i is normal, X_i' is standard normal.. The safety index β_{Hi} is defined as the minimum distance from the origin of the axes in the reduced coordinate system to the limit state surface (failure surface). It can he expressed as

$$\beta_{Hi} = \sqrt{(x^{`*})^I (x^{`*})} \text{ --------------------------------(18)}$$

The minimum distance point on the limit state surface is called the *design point* or *checking point.* It is denoted by vector x* in the original coordinate system and by vector x`* in the reduced coordinate system. These vectors represent the values of all the random variables, that is, X_1, X_2. ..., X_n, at the design point corresponding to the coordinate system being used.

This method can be explained with the help of figure shown Consider the linear limit state equation in two variables.

Z=R—S=0 This equation is similar to above equation. Note that *R* and S need not be normal variables. A set of reduced variables is introduced as

$$R` = \frac{R - \mu_R}{\sigma_R} \quad \text{-----------------------------------(19)}$$

and

$$S` = \frac{S - \mu_S}{\sigma_S} \quad \text{----------------------------------(20)}$$
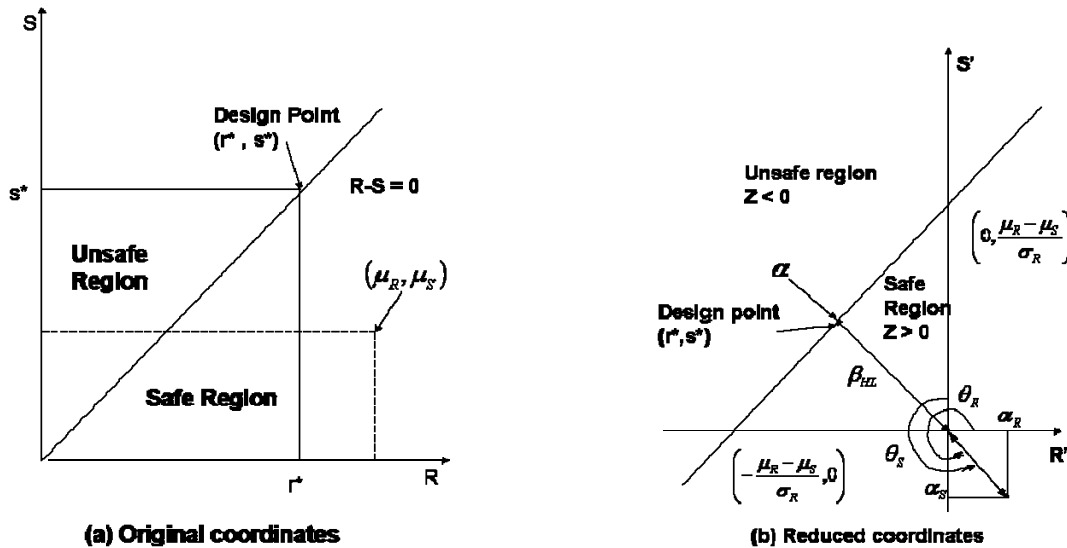

Figure 6 - Hasofer -Lind Reliability Index: Linear Performance Function

If we substitute these into above equation the limit state equation in the reduced coordinate system becomes

$$g(\ )= \sigma_R R' - \sigma_S S' + \mu_R - \mu_S = 0 \text{ -----------------------------------(21)}$$

The transformation of the limit state equation from the original to the reduced coordinate system is shown in above figure. The safe and failure regions are also shown. From the figure, it is apparent that if the failure line (limit state line) is closer to the origin in the reduced coordinate system the failure region is larger and if it is farther away from the origin, the failure region is smaller. Thus, the position of the limit state surface relative to the origin in the reduced coordinate system is a measure of the reliability of the system. The coordinates of the intercepts of above equation on the R` and S` axes can be shown to be $\left[-(\mu_R - \mu_S)/\sigma_R, 0\right] and \left[0,(\mu_R - \mu_S)/\sigma_S\right]$, respectively .Using simple trigonometry, we can calculate the distance of the limit state line from the origin as

$$\beta_{HL} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \text{ -----------------------------------(22)}$$

This distance is referred to as the reliability index or safety index. It is the same as the reliability index defined by the MVFOSM method in above equation if both R and S are normal variables. However, it is obtained in a completely different way based on geometry. It indicates that if the limit state is linear and if the random variables R and S arc normal, both methods will give an identical reliability or safety index.

In general, for many random variables represented by the vector $X = (x_1, x_2, \ldots \ldots x_n)$ in the originates the safe state and $g(X') < 0$ denotes the failure state, Again, the Hasofer-Lind reliability index is defined coordinated system and $X^` = X_1^`, X_2^`, X_3^` \ldots \ldots X_n^`$ in the reduced coordinate system the limit state $g(X`) = 0$ is a nonlinear function as shown in the reduced coordinates for two variables in figure .At this stage, $X_i^{`}$'s are assumed to be uncorrelated. Here $g(X`) > 0$ denoted as the minimum distance from the origin to the design point on the limit state in the reduced coordinates and can be expressed by above equation , where x` - represents the coordinates of the design point or the point of minimum distance from the origin to the limit state. In this definition the reliability index is invariant, because regardless of the form in which the limit state equation is written, its

geometric shape and the distance from the origin remain constant. For the limit state surface where the failure region is away from the origin,
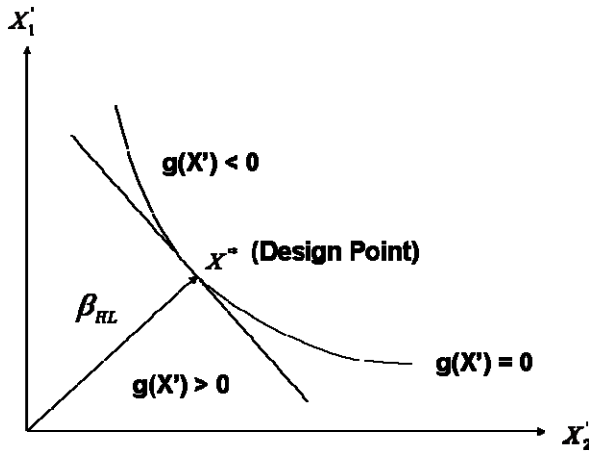

Figure 7 - **Hasofer - Lind Reliability Index: Nonlinear Performance Function**

it is easy to see from figure that x` is the most probable failure point. The Hasofer-Lind reliability index can be used to calculate a first-order approximation of the failure probability as $p_f = \Phi(-\beta_{HL})$. This is the integral of the standard normal density function along the ray joining the origin and x`* it is obvious that the nearer x`* is to the origin, the larger is the failure probability. Thus the minimum distance point on the limit state surface is also the most probable failure point. The point of minimum distance from the origin to the limit state surface, x`*, represents the worst combination of the stochastic variables and is appropriately named the design point or the *most probable point* (MPP) of failure.

For nonlinear limit states, the computation of the minimum distance becomes an optimization problem:

Minimized $D = \sqrt{x'^T x'}$

*Subjected to the constra*int $g(x) = g(x') = 0$

where x` represents the coordinates of the checking point on the limit state equation in the reduced coordinates to be estimated. Using the method of Lagrange multipliers, we can obtain the minimum distance as

$$\beta_{HL} = -\frac{\sum_{i=1}^{n} x_i^{'*}\left(\frac{\partial g}{\partial X_i^{'}}\right)^*}{\sqrt{\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i^{'}}\right)^{2*}}} \quad \text{------------------------------------(23)}$$

Where $\left(\partial g / \partial X_i^{'}\right)^*$ is the ith partial derivative evaluated at the design point with coordinates $\left(x_1^{'*}, x_2^{'*}.....x_n^{'*}\right)$. The asterisk after the derivative indicates that it is evaluated at $\left(x_1^{'*}, x_2^{'*}.....x_n^{'*}\right)$. The design point in the reduced coordinates is given by:

$$x_i^{'*} = -\alpha_i \beta_{HL} \quad \left(i = 1, 2, ...n\right)$$

Where

$$\alpha_i = \frac{\left(\frac{\partial g}{\partial X_i^{'}}\right)^*}{\sqrt{\sum_{i=1}^{n}\left(\frac{\partial g}{\partial X_i^{'}}\right)^{2*}}} \quad \text{- -------------------------------(24)}$$

are the direction cosines along the coordinate axes $X_i^{'}$ in the space of the original coordinates and using equation, we find the design point to be

$$x_i^* = \mu_{X_i} - \alpha_i \sigma_{X_i} \beta_{HL}$$

An algorithm was formulated by Rackwitz (1976) to compute β_{HL} and $x_i^{'*}$ as follows:

• Step 1. Define the appropriate limit state equation.

• Step 2. Assume initial values of the design point $x_i^{'*}$, $i = 1, 2,...........n$. Typically, the initial design point may be assumed to be at the mean values of the random variables. Obtain the reduced variates $x_i^{'*} = \left(x_i - \mu_{X_i}\right)/\sigma_{X_i}$.

• Step 3. Evaluate $\left(\partial g / \partial X_i^{'*}\right)$ and α_i at $x_i^{'*}$

• Step 4. Obtain the new design point $x_i^{'*}$ in terms of β_{HL} as in equation.

• Step 5. Substitute the new $x_i^{'*}$ in the limit state equation g(x`*) = 0 and solve for β_{HL}.

• Step 6. Using the value β_{HL} obtained in Steps 5. Re-evaluate $x_i^{'*} = -\alpha_i \beta_{HL}$

• Step 7. Repeat Steps 3 through 6 until converges.

This algorithm is shown geometrically in figure. The algorithm constructs a linear approximation to the limit state at every search point and finds the distance from the origin to the limit state. In figure, Point B represents the initial design point. usually assumed to be at the mean values of the random variables, as noted in Step 2. Note that B is not on the limit state equation $g(X') = 0$ the tangent to the limit state at B is represented by the line BC. Then AD will give an estimate of β_{HL} in the first iteration, as noted in Step 5. As the iteration continues, β_{HL}, value converges.

Ditlevsen (1979a) showed that for a nonlinear limit state surface lacks comparability; the ordering of β_{HL} values may not be consistent with the ordering of actual reliabilities. An example of this is shown in figure with two limit state surfaces: one flat and the other curved. The shaded region to the right of each limit state represents the corresponding failure region. Clearly, the structure with the flat limit state surface has a different reliability than the one with the curved limit state surface; however, the β_{HL} values are identical for both surfaces and suggest equal reliability. To overcome this inconsistency, Ditlevsen (1979a) introduced the generalized reliability index, β_g defined as

$$\beta_g = \Phi^{-1}\left[\int\limits_{g(X')>0}.........\int \phi(x_1')\phi(x_2').......\phi(x_n')dx_1'dx_2'.......x_n'\right] \text{----------------------------------(25)}$$

where Φ and φ are the cumulative distribution function and the probability density function of a standard normal variable. Respectively, Because the reliability index in this definition includes the entire sale region, it provides a consistent ordering of second-moment reliability. The integral in the equation looks similar to that in equation and is difficult to compute directly. Hence, Ditlevsen (1979a) proposed approximating the nonlinear limit state by a polyhedral surface consisting of tangent hyper-planes at selected points on the surface.
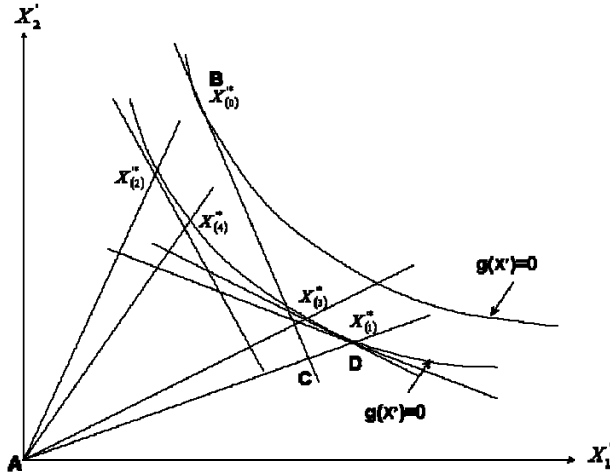
Figure 8 - **Algorithm for finding β_HL**

Note: A number in parentheses indicates iteration numbers

Consider a limit state function $g(X_1, X_2, \ldots, X_n)$ where the random variables X_i are all *uncorrelated.* (If the variables are correlated, then a transformation can be used to obtain uncorrelated variables. See Example 5.15.) The limit state function is rewritten in terms of the standard form of the variables (reduced variables) using

$$Z_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}}$$

As before, the Hasofer-Lind reliablity index is defined as the shortest distance from the origin of the reduced variable space to the limit state function g = 0. Thus far nothing has changed from the previous presentation of the reliability index. In fact, if the limit state function is linear, then the reliability index is still calculated as in Eq 5.18

$$\beta = \frac{a_0 + \sum_{i=1}^{n} a_i \mu_{X_i}}{\sqrt{\sum_{i=1}^{n} (a_i \sigma_{X_i})^2}} \qquad \text{-------------------------------} \quad (26)$$

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If the limit state function is *nonlinear* , however, iteration is required to find the design point $\{Z_1^*, Z_2^* ...... Z_n^*\}$ in reduced variable space such that still corresponds to the Shortest distance. This concept is illustrated in Figures 5.17 through 5.19 for the case of two random variables.
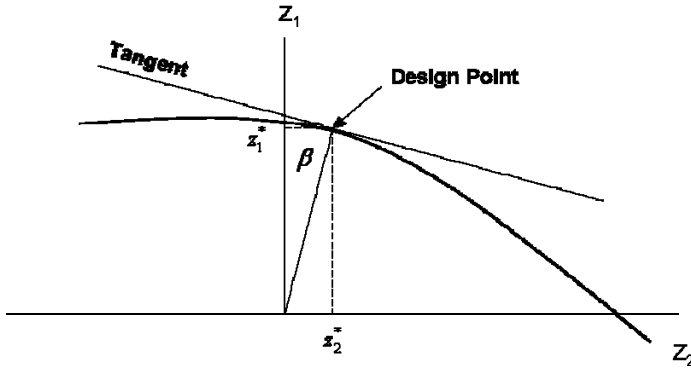

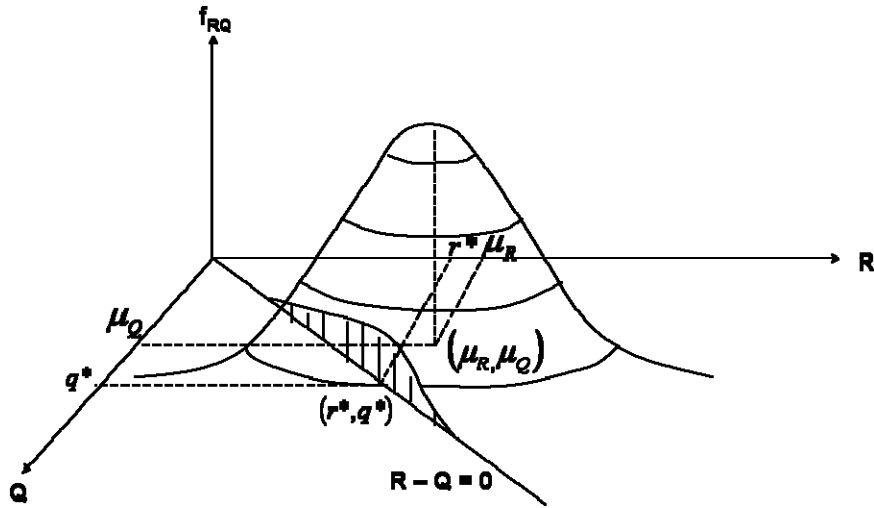**Figure 9 - Hasofer - Lind reliability index**

Figure 10 - Design point on the failure boundary for the linear limit state function g = R – Q
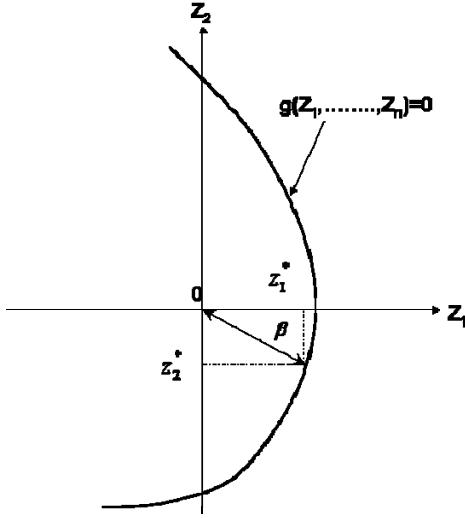
Figure 11 - Design point and reliability index for highly nonlinear limit state function.

The iterative procedure requires us to solve a set of $(2n + 1)$ simultaneous equation with $(2n + I)$ unknowns: $\beta, \alpha_1, \alpha_2 \ldots, \alpha_n$, $Z_1^*, Z_2^* \ldots Z_n^*$ where

$$\alpha_i = \frac{-\left.\dfrac{\partial g}{\partial Z_i}\right|_{evaluated \ at \ design \ po\operatorname{int}}}{\sqrt{\displaystyle\sum_{k=1}^{n}\left(\left.\dfrac{\partial g}{\partial Z_k}\right|_{evaluated \ at \ design \ po\operatorname{int}}\right)^2}}$$

$$\frac{\partial g}{\partial Z_i} = \frac{\partial g}{\partial X_i}\frac{\partial X_i}{\partial Z_i} = \frac{\partial g}{\partial X_i}\sigma_{X_i}$$

$$\sum_{i=1}^{n}(\alpha_i)^2$$

$$z_i^* = \beta\alpha_i$$

$$g\left(z_1^*, z_2^* \ldots z_n^*\right) = 0$$

Equation 5.23b is just an application of the chain rule of differentiation Equation 5.23c is a requirement on the values of the α_i variables, which can be confirmed by looking at

Eq.5.23a. Equation 5.25 is a mathematical statement of the require that the design point must be on the failure boundary .

There are two alternative procedures *for* Performing the iterative analysis: the simultaneous equation procedure and the matrix procedure The steps in the *simultaneous equation procedure are* as follows:

1. Formulate the limit state function and appropriate parameters for all random variables involved

2. Express the limit state function in terms of reduced variates z_i.

3. Use Eq. 5.24 to express the limit state function in terms of β and α_i.

4 Calculation then a values. Use Eq. 5.24 here also to express each α_i as a function of all β and α_i.

5. Conduct the initial cycle: Assume numerical values of β and α_i , noting that the α_i values must satisfy Eq. 5.23c.

6. Use the numerical values of β and α_i on the right-hand sides of the equations formed in Steps 3 and 4 above.

7. Solve the $n + 1$ simultaneous equations in Step 6 for β and α_i.

8. Go back to Step 6 and repeat. Iterate until the β and α_i values converge.

The *matrix procedure* consists of the following steps:

1. Formulate the limit state function and appropriate parameters for all random variables $X_i(i = 1,2, \ldots , n)$ involved.

2. Obtain an initial design point $\{x_i^*\}$ by assuming values for n-1 of the random variables X_i. (Mean values are often a reasonable initial choice.) Solve the limit state equation $g = 0$ for the remaining random variable. This ensures that the design point is on the failure boundary.

3. Determine the reduced variates $\{Z_i^*\}$ corresponding to the design point $\{x_i^*\}$ using

$$Z_i^* = \frac{x_i^* - \mu_{X_i}}{\sigma_{X_i}}$$

4. Determine the partial derivatives of the limit state function with respect to the reduced variates using Eq. 5.23b. For convenience, define a column vector $\{G\}$ as the vector whose elements are these partial derivatives multiplied by -1:

$$\{G\} = \begin{Bmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{Bmatrix} \quad where \ G_i = -\frac{\partial g}{\partial Z_i}\bigg|_{evaluated \ at \ design \ point} \qquad \text{-------------------} \quad (27)$$

5. calculate an estimate of β using the following formula:

$$\beta = \frac{\{G\}^T \{z*\}}{\sqrt{\{G\}^T \{G\}}} \quad where \ \{z*\} = \begin{Bmatrix} z_1^* \\ z_2^* \\ \vdots \\ z_n^* \end{Bmatrix} \qquad \text{-------------------------}(28)$$

The superscript T denotes transpose. If the limit state equation is *linear,* then Eq 5.28 reduces to Eq. 5.18.

6. Calculate a column vector containing the sensitivity factors using

$$\{\alpha\} = \frac{\{G\}}{\sqrt{\{G\}^T \{G\}}} \qquad \text{------------------------------} \quad (29)$$

7. Determine a new design point in reduced variates for n-1 of the variables using

$$Z_i^* = \alpha_i \beta$$

8. Determine the corresponding design point values in original coordinates for the n-1 values in Step 7 using

$$X_i^* = \mu_{X_i} + Z_i^* \sigma_{X_i}$$

9. Determine the value of the remaining random variable (i.e., the one not found in Steps 7 and 8) by solving the limit state function $g = 0$.

10. Repeat Steps 3 to 9 until and the design point $\{x_i^*\}$ converge.