3 SPATIAL VARIABILITY USING RANDOM FIELDS

3. 1 Need for spatial variability characterization in design

Many quantities such as properties of materials, concentrations of pollutants, loads etc in civil engineering have spatial variations. Variations are expressed in terms of mean or average values and the coefficients of variation defined in terms of the ratio of standard deviation and mean value expressed as percentage. In addition, the distance over which the variations are well correlated also plays a significant role.

A successful design depends largely on how best the designer selects the basic parameters of the loading/site under consideration from in-situ and/or laboratory test results. Probabilistic methods in civil engineering have received considerable attention in the recent years and the incorporation of soil variability in civil/geotechnical designs has become important. Considerable work was carried out in the area of geotechnical engineering. Guidelines such as those of JCSS (2000) have also been developed in this context. Dasaka (2005) presented a comprehensive compilation on spatial variability of soils. In the following sections, spatial soil variability of soils is addressed and the concepts are applicable to any other property variations as well. Soil has high variability compared to manufactured materials like steel or cement, where variability in material properties is less, as they are produced under high quality control.

3.2 Characterization of variability of design parameters

It is generally agreed that the variability associated with geotechnical properties should be divided in to three main sources, viz., inherent variability, measurement uncertainty, and transformation uncertainty (Baecher and Christian 2003; Ang and Tang 1984).

3.2.1 Inherent variability

The inherent variability of a soil parameter is attributed to the natural geological processes, which are responsible for depositional behaviour and stress history of soil under consideration. The fluctuations of soil property about the mean can be modelled using a zero-mean stationary random field (Vanmarcke 1977). A detailed list of the fluctuations in terms of coefficients of variation for some of the laboratory and in-situ soil parameters, along with the respective scales of fluctuation in horizontal and vertical directions are presented in Baecher and Christian (2003).

3.2.2 Measurement uncertainty

Measurement uncertainty is described in terms of accuracy and is affected by bias (systematic error) and precision (random error). It arises mainly from three sources, viz., equipment errors, procedural-operator errors, and random testing effects, and can be evaluated from data provided by the manufacturer, operator responsible for laboratory tests and/or scaled tests. Nonetheless the recommendations from regulatory authorities regarding the quality of produced data, the measuring equipment and other devices responsible for the measurement of in-situ or laboratory soil properties often show variations in its geometry, however small it may be. There may be many limitations in the formulation of guidelines for testing, and the understanding and implementation of these guidelines vary from operator to operator and contribute to procedural-operator errors in the measurement. The third factor, which contributes to the measurement uncertainty, random testing error, refers to the remaining scatter in the test results that is not assignable to specific testing parameters and is not caused by inherent soil variability.

3.2.3 Transformation uncertainty

Computation models, especially in the geotechnical field contain considerable uncertainties due to various reasons, e.g. simplification of the equilibrium or deformation analysis, ignoring 3-D effects etc. Expected mean values and standard deviations of these factors may be assessed on the basis of empirical or experimental data, on comparison with more advanced computation models. Many design parameters used in geotechnical engineering are obtained from in-situ and laboratory test results. To account for this uncertainty, the model or transformation uncertainty parameter is used.

3.3.4 Evaluation design parameter uncertainty

The total uncertainty of design parameter from the above three sources of uncertainty is combined in a consistent and logical manner using a simple second-moment probabilistic method. The design parameter may be represented as

$$\xi_d = T(\xi_m, \varepsilon) \tag{1}$$

where ξ_m is the measured property of soil parameter obtained from either a laboratory or in-situ test. The measured property can be represented in terms of algebraic sum of non-stationary trend, t, stationary fluctuating component, w, and measurement uncertainty, e. ε is the transformation uncertainty, which arises due to the uncertainty in transforming the in-situ or laboratory measured soil property to the design parameter using a transformation equation of the form shown in Equation 1. Hence, the design property can be represented by Equation 2.

$$\xi_d = T(t + w + e, \varepsilon) \tag{2}$$

Phoon and Kulhawy (1999b) expressed the above equation in terms of Taylor series. Linearizing the Taylor series after terminating the higher order terms at mean values of soil

parameters leads to the Equation 3 for soil design property, subsequently the mean and variance of design property are expressed as given in Equations 4 and 5.

$$\xi_d \approx T(t,0) + w\frac{\partial T}{\partial w}\bigg|_{(t,0)} + e\frac{\partial T}{\partial e}\bigg|_{(t,0)} + \varepsilon\frac{\partial T}{\partial \varepsilon}\bigg|_{(t,0)} \tag{3}$$

$$m_{\xi d} \approx T(t,0) \tag{4}$$

$$SD_{\xi d}^2 = \left(\frac{\partial T}{\partial w}\right)^2 SD_w^2 + \left(\frac{\partial T}{\partial e}\right)^2 SD_e^2 + \left(\frac{\partial T}{\partial \varepsilon}\right)^2 SD_\varepsilon^2 \tag{5}$$

The resulting variance of design parameter after incorporating the spatial average is given by

$$SD_{\xi a}^2 = \left(\frac{\partial T}{\partial w}\right)^2 \Gamma^2(L)SD_w^2 + \left(\frac{\partial T}{\partial e}\right)^2 SD_e^2 + \left(\frac{\partial T}{\partial \varepsilon}\right)^2 SD_\varepsilon^2 \tag{6}$$

Of the above, the treatment and evaluation of inherent soil variability assumes considerable importance as the uncertainties from measurements and transformation process can be handled if proper testing methods are adopted and transformation errors are quantified. Approaches for evaluation of inherent soil variability are developed based on random fields and a brief description of the theory and its relevance to characterisation of soil spatial variability is described in the following sections.

3.4 Random field Theory

Soil properties exhibit an inherent spatial variation, i.e., its value changes from point to point. Vanmarcke (1977a; 1983) provided a major contribution to the study of spatial variability of geotechnical materials using random field theory. In order to describe a soil property stochastically, Vanmarcke (1977a) stated that three parameters are needed to be described: (i) the mean (ii) the standard deviation (or the variance, or the coefficient of variation); and (iii) the scale of fluctuation. He introduced the new parameter, scale of fluctuation, which

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accounts for the distance within which the soil property shows relatively strong correlation from point-to-point.

Figure 3.1(a) shows a typical spatially variable soil profile showing the trend, fluctuating component, and vertical scale of fluctuation. Small values of scale of fluctuation imply rapid fluctuations about the mean, whereas large values suggest a slowly varying property, with respect to the average.

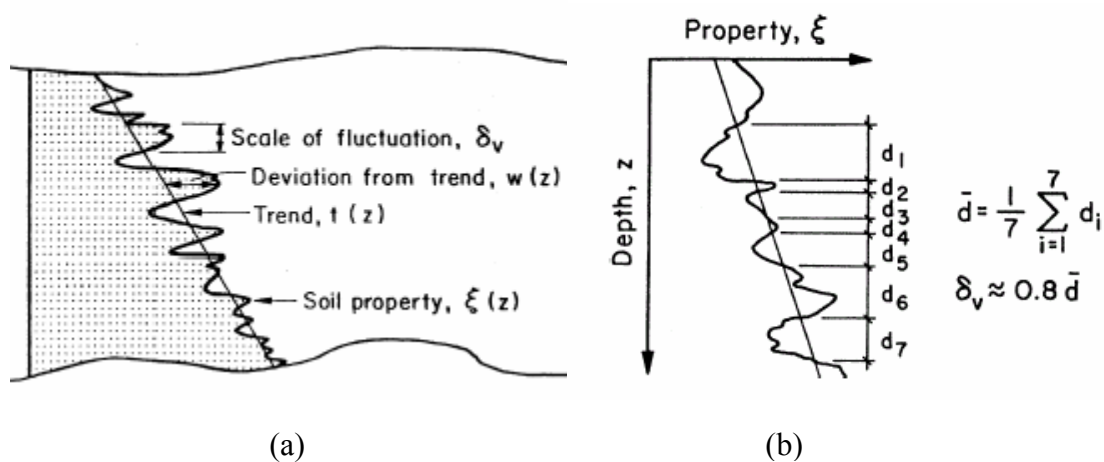(a)   (b)

Figure 3.1(a). Definition of various statistical parameters of a soil property (Phoon and Kulhawy 1999a); (b) approximate definition of the scale of fluctuation (Vanmarcke 1977a)

Vanmarcke (1977a) demonstrated a simple procedure to evaluate an approximate value of the vertical scale of fluctuation, as shown in Figure 3.1(b), which shows that the scale of fluctuation is related to the average distance between intersections, or crossings, of the soil property and the mean.

A random field is a conceivable model to characterize continuous spatial fluctuations of a soil property within a soil unit. In this concept, the actual value of a soil property at each location within the unit is assumed to be a realization of a random variable. Usually, parameters of the random field model have to be determined from only one realization. Therefore the random

field model should satisfy certain ergodicity conditions at least locally. If a time average does not give complete representation of full ensemble, system is non-ergodic. The random field is fully described by the autocovariance function, which can be estimated by fitting empirical autocovariance data using a simple one-parameter theoretical model. This function is commonly normalized by the variance to form the autocorrelation function. Conventionally, the trend function is approximately removed by least square regression analysis. The remaining fluctuating component, $x(z)$, is then assumed to be a zero-mean stationary random field. When the spacing between two sample points exceeds the scale of fluctuation, it can be assumed that little correlation exists between the fluctuations in the measurements. Fenton (1999a & b) observed that the scale of fluctuation often appears to increase with sampling domain.

3.4.1 Statistical homogeneity

Statistical homogeneity in a strict sense means that the entire joint probability density function (joint pdf) of soil property values at an arbitrary number of locations within the soil unit is invariable under an arbitrary common translation of the locations. A more relaxed criterion is that expected mean value and variance of the soil property is constant throughout the soil unit and that the covariance of the soil property values at two locations is a function of the separation distance. Random fields satisfying only the relaxed criteria are called stationary in a weak sense.

Statistical homogeneity (or stationarity) of a data set is an important prerequisite for statistical treatment of geotechnical data and subsequent analysis and design of foundations. In physical sense, stationarity arises in soils, which are formed with similar material type and under similar geological processes. Improper qualification of a soil profile in terms of the statistical homogeneity leads to biased estimate of variance of the mean observation in the

soil data. The entire soil profile within the zone of influence is divided into number of statistically homogeneous or stationary sections, and the data within each layer has to be analysed separately for further statistical analysis. Hence, the partition of the soil profile into stationary sections plays a crucial role in the evaluation of soil statistical parameters such as variance.

3.4.2 Tests for statistical homogeneity

The methods available for statistical homogeneity are broadly categorised as parametric tests and non-parametric tests. The parametric tests require assumptions about the underlying population distribution. These tests give a precise picture about the stationarity (Phoon et al. 2003a).

In geostatistical literature, many classical tests for verification of stationarity have been developed, such as Kendall's τ test, Statistical run test (Phoon et al. 2003a). Invariably, all these classical tests are based on the important assumption that the data are independent. When these tests are used to verify the spatially correlated data, a large amount of bias appears in the evaluation of statistical parameters, and misleads the results of the analysis. To overcome this deficiency, Kulathilake and Ghosh (1988), Kulathilake and Um (2003), and Phoon et al. (2003a) proposed advanced methods to evaluate the statistical homogeneous layers in a given soil profile. The method proposed by Kulathilake and Ghosh (1988), Kulathilake and Um (2003) is semi-empirical window based method, and the method proposed by Phoon et al. (2003a) is an extension of the Bartlett test.

3.4.2.1 Kendall's τ test

The Kendall $\hat{\tau}$ statistic is frequently used to test whether a data set follows a trend. Kendall's $\hat{\tau}$ is based on the ranks of observations. The test statistic, which is also the measure of association in the sample, is given by

$$\hat{\tau} = \frac{S}{n(n-1)/2} \qquad\qquad (7)$$

where n is the number of (X,Y) observations. To obtain S, and consequently $\hat{\tau}$, the following procedure is followed.

1. Arrange the observations (X_i, Y_i) in a column according to the magnitude of the X's, with the smallest X first, the second smallest second, and so on. Then the X's are said to be in natural order.

2. Compare each Y value, one at a time, with each Y value appearing below it. In making these comparisons, it is said that a pair of Y values (a Y being compared and the Y below it) is in natural order if the Y below is larger than the Y above. Conversely, a pair or Y values is in reverse natural order if the Y below is smaller than the Y above.

3. Let P be the number of pairs in natural order and Q the number of pairs in reverse natural order.

4. S is equal to the difference between P and Q;

A total of $\binom{n}{2} = \frac{n(n-1)}{2}$ possible comparisons of Y values can be made in this manner. If all the Y pairs are in natural order, then $P = \frac{n(n-1)}{2}$, Q=0, $S = \frac{n(n-1)}{2} - 0 = \frac{n(n-1)}{2}$, and hence $\tau = \frac{n(n-1)/2}{n(n-1)/2} = 1$, indicating perfect direct correlation between the observations of X and Y.

On the other hand, if all the Y pairs are in reverse natural order, we have $P=0, Q = \frac{n(n-1)}{2}$, $S = 0 - \frac{n(n-1)}{2} = \frac{-n(n-1)}{2}$, and $\tau = \frac{-n(n-1)/2}{n(n-1)/2} = -1$, indicating a perfect inverse correlation between the X and Y observations.

Hence $\hat{\tau}$ cannot be greater than +1 or smaller than -1, thus, $\hat{\tau}$ can be taken as a relative measure of the extent of the disagreement between the observed orders of the Y. The strength of the correlation is indicated by the magnitude of the absolute value of $\hat{\tau}$.

3.4.2.2 Statistical run test

In this procedure, a run is defined as a sequence of identical observations that is followed and preceded by a different observation or no observation at all. The number of runs that occur in a sequence of observations gives an indication as to whether or not results are independent random observations of the same random variable. In this the hypothesis of statistical homogeneity, i.e., trend-free data, is tested at any desired level of significance, α, by comparing the observed runs to the interval between $r_{n;1-\alpha/2}$ *and* $r_{n;\alpha/2}$. Here, n=N/2, N being the total number of data points within a soil record. If the observed number of runs falls outside the interval, the hypothesis would be rejected at the α level of significance. Otherwise, the hypothesis would be accepted.

For testing a soil record with run test, the soil record is first divided into number of sections, and variance of the data in each section is computed separately. The computed variance in each section is compared with the median of the variances in all sections, and the number of runs (r) is obtained. The record is said to be stationary or statistically homogeneous at significance level of α, if the condition given below is satisfied.

$$r_{n;\ 1-\alpha/2} < r \le r_{n;\ \alpha/2} \tag{8}$$

3.4.2.3 Bartlett's approach

The classical Bartlett test is one of the important tests, which examines the equality of two or multiple variances of independent data sets. The following steps are involved in the Bartlett's test.

The sampling window is divided into two equal segments and sample variance ($s_1^2 \ or \ s_2^2$) is calculated from the data within each segment separately. For the case of two sample variances, $s_1^2 \ and \ s_2^2$, the Bartlett test statistic is calculated as

$$B_{stat} = \frac{2.30259(m-1)}{C}\left[2\log s^2 - \left(\log s_1^2 + \log s_2^2\right)\right] \tag{9}$$

where m=number of data points used to evaluate $s_1^2 \ or \ s_2^2$. The total variance, s^2, is defined as

$$s^2 = \frac{s_1^2 + s_2^2}{2} \tag{10}$$

The constant C is given by

$$C = 1 + \frac{1}{2(m-1)} \tag{11}$$

While choosing the segment length, it should be remembered that m≥10 (Lacasse and Nadim 1996). In this technique, the Bartlett statistic profile for the whole data within the zone of influence is generated by moving sampling window over the soil profile under consideration. In the continuous Bartlett statistic profile, the sections between the significant peaks are treated as statistically homogeneous or stationary layers, and each layer is treated separately for further analysis.

3.4.2.4 Modified Bartlett technique

Phoon et al. (2003a, 2004) developed the Modified Bartlett technique to test the condition of null hypothesis of stationarity of variance for correlated profiles suggested by conventional statistical tests such as Bartlett test, Kendall's test etc, and to decide whether to accept or reject the null hypothesis of stationarity for the correlated case. The modified Bartlett test statistic can also be used advantageously to identify the potentially stationary layers within a soil profile. This procedure was formulated using a set of numerically simulated correlated

soil profiles covering all the possible ranges of autocorrelation functions applicable to soil. In this procedure, the test statistic to reject the null hypothesis of stationarity is taken as the peak value of Bartlett statistic profile. The critical value of modified Bartlett statistic is chosen at 5% significance level, which is calculated from simulated soil profiles using multiple regression approach, following five different autocorrelation functions, viz., single exponential, double exponential, triangular, cosine exponential, and second-order Markov.

The data within each layer between the peaks in the Bartlett statistic profile are checked for existence of trend. A particular trend is decided comparing the correlation length obtained by fitting a theoretical function to sample autocorrelation data. If the correlation lengths of two trends of consecutive order are identical, it is not required to go for higher order detrending process. However, it is suggested that no more than quadratic trend is generally required to be removed to transform a non-stationary data set to stationary data set (Jaksa et al. 1999).

The following dimensionless factors are obtained from the data within each layer.

Number of data points in one scale of fluctuation, $k = \dfrac{\delta}{\Delta z}$ $\hspace{2cm}$ (12)

Normalized sampling length, $I_1 = \dfrac{T}{\delta} = \dfrac{n\Delta z}{k\Delta z} = \dfrac{n}{k}$ $\hspace{2cm}$ (13)

Normalized segment length, $I_2 = \dfrac{W}{\delta} = \dfrac{m\Delta z}{k\Delta z} = \dfrac{m}{k}$ $\hspace{2cm}$ (14)

where δ is the scale of fluctuation evaluated, and 'n' is the total of data points in a soil record of T. The Bartlett statistic profile is computed from the sample variances computed in two contiguous windows. Hence, the total soil record length, T, should be greater than 2W. To ensure that $m \geq 10$, the normalized segment length should be chosen as $I_2=1$ for $k \geq 10$ and $I_2=2$ for $5 \leq k < 10$ (Phoon et al. 2003a).

Equations 15 and 16 show the typical results obtained from regression analysis for I_2 equals to 1 and 2 respectively for the single exponential simulated profiles. Similar formulations have also been developed for other commonly encountered autocorrelation functions and reported in Phoon et al. (2003a).

$$B_{crit}=(0.23k+0.71) \ln(I_1)+0.91k+0.23 \qquad \text{for } I_2=1 \qquad\qquad (15)$$

$$B_{crit}=(0.36k+0.66) \ln(I_1)+1.31k-1.77 \qquad \text{for } I_2=2 \qquad\qquad (16)$$

A comparison is made between the peaks of the Bartlett statistic within each layer with B_{crit} obtained from the respective layer. If $B_{max}<B_{crit}$, the layer can be treated as statistically homogeneous and hence, accept the null hypothesis of stationarity. Otherwise, if $B_{max}>B_{crit}$, reject the null hypothesis of stationarity, and treat the sections on either side of the peaks in the Bartlett statistic profile as stationary and repeat the above steps and evaluate whether these sections satisfy the null hypothesis of stationarity. However, while dividing the sections on either side of the peaks in the Bartlett statistic profile, it should be checked for $m \geq 10$, where 'm' is the number of data points in a segment.

3.4.2.5 Dual-window based method

Kulathilake and Ghosh (1988) and Kulathilake and Um (2003) proposed a simple window based method to verify the statistical homogeneity of the soil profile using cone tip resistance data. In this method, a continuous profile of 'BC' distance is generated by moving two contiguous sub-windows throughout the cone tip resistance profile. The distance 'BC', whose units are same as q_c, is the difference of the means at the interface between two contiguous windows. In this method it is verified whether the mean of the soil property is constant with depth, which is a prerequisite to satisfy the weak stationarity. At first, the elevation of the window is taken at a level that coincides with the level of first data point in the q_c profile. After evaluating the BC distance, the whole window is moved down at a shift each time. The

computed distance 'BC' is noted each time at the elevation coinciding the centre of the window (i.e., the intersection of two contiguous sub-windows). This length of sub-window is selected based on the premise that at least 10 data points are available within the sub-window.

The data within the two sub-windows is treated separately, and checked for linear trend in the data of 10 points. The reason behind verifying the data with only linear trend is that within 0.2 m profile, higher-order trends are rarely encountered. In addition, in normally consolidated soils, the overburden stress follows a linear trend with depth. Kulathilake and Um (2003) suggested that the demarcation between existence and non-existence of a linear trend in the data be assumed at a determination coefficient (R^2) of 0.9. It means that if the R^2 value of theoretical linear fit is greater than 0.9, then the data set is said to be having a linearly trend in it, if not the mean value is said to be constant throughout the sub-window. Hence, within a window length (i.e., two contiguous windows) there exist four sets of possibility of trend in the mean values. They are

1. Constant trend in both the contiguous sub-windows

2. Constant trend in upper sub-window and a linear trend in the lower sub-window

3. Linear trend in the upper sub-window and constant trend in the lower sub-window, and

4. Linear trend in both the contiguous sub-windows.

The above four sets possibilities of trend within the contiguous windows are shown in Figure 3.2. As the distance 'BC' increases, the heterogeneity of the q_c at the intersection between two sub-sections increases.
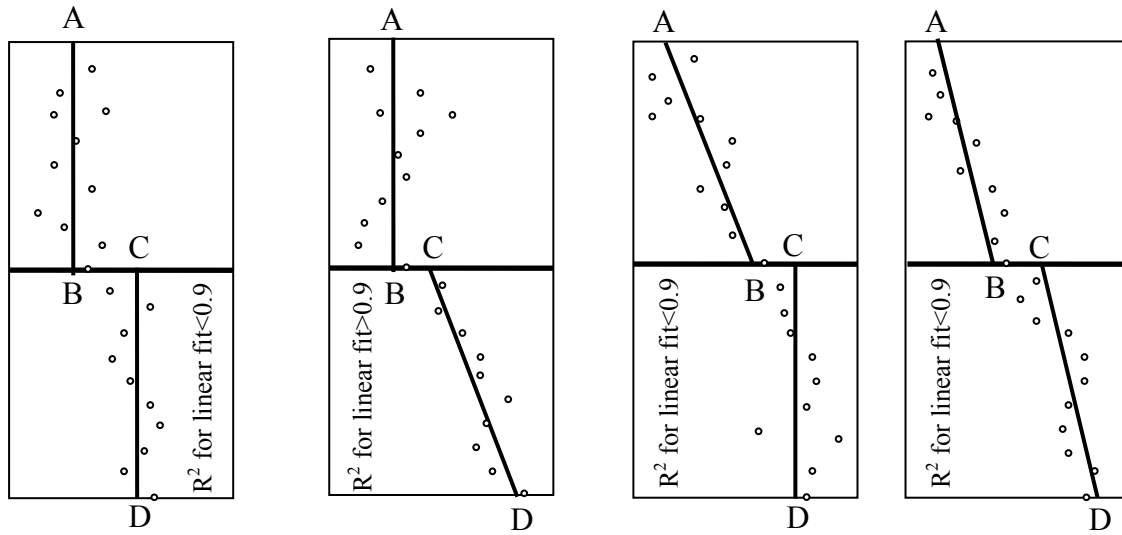
Figure 3.2. Evaluation of 'BC' distance in various possible combinations

3.4.3 Trend removal

Once the statistically homogeneous layers are identified within a soil profile, the individual statistical layers are checked for the existence of trend, and the same is removed before evaluating the variance and autocorrelation characteristics of the data. In general, all soil properties exhibit a trend with depth. The deterministic trend in the vertical soil profile may be attributed to overburden stress, confining pressure and stress history of soil under study. Generally, a smooth curve can be fitted using the Ordinary Least Square (OLS) method, except in special cases such as varved clays, where periodic trends are clearly visible (Phoon et al. 2003a). In most of the studies, the trend line is simply estimated by regression analysis using either linear or polynomial curve fittings.

Other methods have also been applied, such as normalization with respect to some important physical variables, differencing technique, which is routinely used by statisticians for transforming a non-stationary time series to a stationary one. The normalization method of trend removal with respect to a physical quantity accounts for systematic physical effects on the soil profiles. In general, the detrending process is not unique. Different trend removal

procedures will in most cases result in different values of the random fluctuating components and different shapes of the autocorrelation function.

Baecher (1987) commented that the selection of a particular trend function is a decision on how much of the spatial variability in the measurements is treated as a deterministic function of space (i.e., trend) and how much is treated statistically and modelled as random processes. However, the detrending process cannot be entirely arbitrary. After all, the fluctuating components remaining in the detrended soil records must be stationary for meaningful statistical analyses to be undertaken on limited data points. Clearly, the chosen trend function should be reasonable in view of this stationary constraint. The scale of fluctuation or autocorrelation distance evaluated from the non-stationary data is always higher than the corresponding stationary data. In other words, the trend removal invariably reduces the scale of fluctuation of the soil properties. One of the simplest methods to evaluate whether a linear or 2nd order polynomial trend is sufficient to be removed from the experimental data is to calculate the scale of fluctuation for the above both detrended data. If the evaluated scales of fluctuation are closer to each other, a detrending process using the lesser degree polynomial is chosen. In the limit, the scale of fluctuation is zero when the entire profile is treated as a ''trend'' with zero ''random'' variation (Phoon et al. 2003a).

If a trend is evident in the measurements, it should be decided whether or not it should be removed before statistical analysis of a set of raw data. An observed trend that has no physical or geological basis or is not predictable must not be removed prior to statistical analysis, since it is a part of the uncertainty to be characterized (Fenton 1999b). After selecting a proper trend function for the data, the residuals off the trend are calculated. Phoon et al. (2004) pointed out that trend removal is a complex problem, and there is at present no fully satisfactory solution to it. The identified trend in the data is removed by employing any of the following three widely used detrending methods.

3.4.3.1 Decomposition technique

In this method the data set is divided into stationary random field and nonstationary trend, by using the results obtained from either a non-parametric test or a parametric test discussed in the last section. Initially a linear trend is selected and removed from the original data. The linearly detrended data is tested for the weak stationarity. If the residuals off the linear trend do not satisfy the stationarity hypothesis, the above procedure is repeated by choosing a higher order polynomial. However, it is suggested that no more than quadratic trend is normally sufficient to transform a non-stationary data set to stationary data set (Jaksa et al. 1999), and keep them fairly stationary, as complete removal of the trend in the data is rarely achieved.

3.4.3.2 Normalization technique

Normalisation of the data set with respect to a dominant parameter, such as cone tip resistance, q_c, effective overburden pressure, σ'_{vo}, is also used in geotechnical engineering to make the data trend free.

3.4.3.3 Differencing technique

In this method, a nonstationary data set is made stationary by using first, second or higher order differencing technique. This method of testing a time series is suggested by Bowerman and O'Connell (1983), which is suitable for data containing no seasonal variations. According to Bowerman and O'Connell (1983) if the sample autocorrelation function for experimental data dies down fairly quickly, the original data set can be treated as stationary. However, if the sample autocorrelation function dies down extremely slow, then the original data set can be transformed to a stationary set by taking first or second difference of original data set. However, the term "fairly quickly" is rather subjective and extensive judgment is involved in it. Moreover, it is observed that if no seasonal variations exist in the data, no more than

second difference is rarely needed to transform a nonstationary data to stationary data (Jaksa et al. 1999).

2.4.4 Estimation of autocorrelation

Available methods for estimating the sample autocorrelation functions differ in their statistical properties such as the degree of bias, sampling variability, ease of use, computational requirements, etc.. The methods that are commonly used for this purpose are method of moments, Bartlett's approach, method based on maximum likelihood principle, Geostatistics, etc. However, the method of moments is the most common used to estimate sample correlation function of soil properties.

3.4.4.1 Method of moments

A classical way of describing random functions is through the autocorrelation function, $\rho(\Delta z)$. It is the coefficient of correlation between values of a random function at separation of k. The spatial correlation of a soil property can be modelled as the sum of a trend component and a residual term (Vanmarcke 1977a), as shown in Equation 2.17.

x=z+e   17)

where x is the measurement at a given location, z is the trend component, and e is the residual (deviation about the trend). The residuals off the trend tend to exhibit spatial correlation. The degree of spatial correlation among the residuals can be expressed through an auto-covariance function.

$$c(k) = E[P(Z_i) - t(Z_i)][P(Z_j) - t(Z_j)]$$ (18)

where k is the vector of separation distance between point i and j, E[.] is the expectation operator, $P(Z_i)$ is the data taken at location i, and $t(Z_i)$ is the value of the trend at location i.

The normalized form of the autocovariance function given in Equation 19 is known as the autocorrelation function.

$$\rho(k) = c[k]/c[0] \tag{19}$$

where c[0] is the autocovariance function at zero separation distance, which is nothing but variance data.

It is not possible to evaluate 'c_k' nor 'ρ_k' with any certainty, but only to estimate them from samples obtained from a population. As a result, the sample autocovariance at lag k, c_k^*, and sample autocorrelation at lag k, r_k, are generally evaluated. The sample autocorrelation function (ACF) is the graph of r_k for lags k=0,1,2, ...h, where 'h' is the maximum number of lags allowable. Generally, 'h' is taken as a quarter of total number of data points in time series analysis of geotechnical data (Box and Jenkins 1970; Lumb 1975a). Beyond this number, the number of pairs contributing to the autocorrelation function diminishes and produces unreliable results. The sample ACF at lag k, r_k, is generally evaluated using

$$r_k = \frac{\dfrac{1}{(N-k-1)}\sum_{i=1}^{N-k}\left(X_i - \overline{X}\right)\left(X_{i+k} - \overline{X}\right)}{\dfrac{1}{(N-1)}\sum_{i=1}^{N}\left(X_i - \overline{X}\right)^2} \tag{20}$$

If no measurement error or noise is present, r becomes equal to 1 at a lag distance of zero. Statistically homogeneous data are used to evaluate the sample autocorrelation functions. The autocorrelation characteristics of soil properties can be characterized either by autocorrelation distance, or scale of fluctuation, which is theoretically equal to the area under the correlation function. The scale of fluctuation (or correlation radius) for one dimensional real field is defined as shown in Equation 21 (Vanmarcke 1977a).

$$\delta = 2\int_0^\infty \rho(\tau)d\tau \tag{21}$$

More generally, the scale of fluctuation δ is defined as the radius of an equivalent "unit step" correlation function, i.e., $\rho(\tau)=1$ for $\tau \le \delta$ and $=0$ for $\tau > \delta$, τ being the Euclidian lag (JCSS 2000). The autocorrelation distance (or scale of fluctuation) is evaluated from the sample autocorrelation function using method of fitting or based on Bartlett limits, which are described in the following sections.

3.4.4.1.1 Method of fitting

Analytical expressions are fitted to the sample autocorrelation functions using regression analysis based on least square error approach. The least square error is generally characterised by the determination coefficient of the fit. Frequently used single-parameter theoretical auto-correlation functions are exponential, squared exponential, though models such as triangular, second order auto-regressive, spherical, etc. are also not uncommon to fit the sample autocorrelation data in geotechnical engineering. Some of these models are given in Table 3.1.

Table 3.1. Theoretical autocorrelation functions used to determine the autocorrelation distance and scale of fluctuation, δ (Jaksa et al. 1999)

Model No.	Theoretical autocorrelation function	Autocorrelation function	Auto-correlation distance, ρ	Scale of fluctuation, δ
1	Triangular	$\rho_{\Delta z} = \begin{cases} 1 - \dfrac{\|\Delta z\|}{a} & for\ \|\Delta z\| \le a \\ 0 & for\ \|\Delta z\| \ge a \end{cases}$	a	a
2	Single exponential	$\rho_{\Delta z} = \exp\left(-\|\Delta z\|/b\right)$	b	2b
3	Double exponential	$\rho_{\Delta z} = \exp\left(-\left(\|\Delta z\|/c\right)^2\right)$	c	$\sqrt{\pi}\,c$
4	Second-order Markov	$\rho_{\Delta z} = \exp\left(-\|\Delta z\|/d\right)\left(1 + \dfrac{\|\Delta z\|}{d}\right)$	d	4d
5	Cosine exponential	$\rho_{\Delta z} = \exp\left(-\|\Delta z\|/e\right)\cos\left(\dfrac{\Delta z}{e}\right)$	e	e

Table 3.1 shows the autocorrelation distance and corresponding scale of fluctuation for theoretical autocorrelation functions. A small scale of fluctuation (δ) implies rapid fluctuations about the mean and vice versa. and a large reduction in variance over any failure plane; this results in a small "spread" of the performance function. Conversely a large δ means much longer variations about the mean and results in smaller reduction in variance over a failure plane (Mostyn and Soo 1992).

3.4.4.1. 2 Bartlett limits

In the field of time series analysis, the most commonly used method to compute the autocorrelation distance is by Bartlett's approximation. In this method the computed scale of fluctuation corresponds to two standard errors of the estimate, i.e., the lag distance at which the positive Bartlett's limits given by Equation 2.21, superimposed on the autocorrelation plot crosses the autocorrelation function (Jaksa et al. 1999).

$$|r_h| = \pm \frac{1.96}{\sqrt{N}} \tag{22}$$

The scale of fluctuation of cone tip resistance varies from site to site. Moreover, it also varies with type of soil, as Jaksa et al. (2004) reports smaller scales of fluctuation in sands than clays due to their nature of formation. Further, Fenton and Vanmarcke (1998) argue that the scale of fluctuation depends largely on the geological processes of transport of raw materials, layer deposition, and common weathering rather than on the actual property studied. Nonetheless, DeGroot and Baecher (1993) observed that the scale of fluctuation is also function of sampling interval on in-situ measured property.

3.4.5 Effect of anisotropy in correlation scales

Most soils in nature are usually anisotropic due to their mode of sedimentation and consolidation that cause preferred particle orientations. There are generally two types of

anisotropy. Inherent or initial anisotropy manifests itself in the soil deposits as a result of applied stresses at the time of formulation in the form of first-structure on a macroscopic scale or as a fabric orientation on the microscopic scale. Stress or induced anisotropy arises from changes in the effective stress state produced by subsequent loading history. This anisotropy can cause the elastic, strength and compressibility parameters of the soil deposits to vary with direction, and hence cannot be ignored.

The soil properties exhibit large variations and their directional behaviour is observed by many researchers (Vanmarcke 1983; Jaksa et al. 1999; Phoon and Kulhawy 1999a; Griffiths and Fenton 2000; Nobahar and Popescu 2002; Fenton and Griffiths 2003; Jaksa et al. 2004; Sivakumar Babu and Mukesh 2004; and Uzielli et al. 2005; Wei et al. 2005). The autocorrelation distances in vertical and horizontal directions are never the same, but in general, differ by an order of magnitude, with horizontal scale of fluctuation being higher than that in the vertical (Uzielli et al. 2005). Attempts have been made in the literature to formulate autocorrelation models for 1, 2, and 3-dimensional soil space (Vanmarcke 1977a; and Kulathilake and Miller 1987). The effect of anisotropy of soil properties on the bearing capacity in a probabilistic framework has not been studied extensively in the literature. Many times, due to economic feasibility, speed of exploration, availability of equipment and time constraints vertical cone penetration data alone is obtained and used in the evaluation of strength properties (Wei et al. 2005).

The autocovariance structure is called isotropic if the normalized autocovariance depends on the Euclidian distances between field points only, instead of the axis directional distance components, components, i.e.,

$$\rho(\Delta x, \Delta y, \Delta z) = \rho\left(\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}\right) \qquad (23)$$

Isotropy implies that the autocorrelation function is invariant to orthonormal transformation of the field coordinates. Also the autocorrelation structure may be partly isotropic, for example with respect to horizontal field directions:

$$\rho(\Delta x, \Delta y, \Delta z) = \rho\left(\sqrt{\Delta x^2 + \Delta y^2}, \Delta z\right) \tag{24}$$

For complete anisotropy, the exponential correlation function in 3-D space is

$$\rho(\Delta x, \Delta y, \Delta z) = \exp\left(-\frac{|\Delta x|}{D_x} - \frac{|\Delta y|}{D_y} - \frac{|\Delta z|}{D_z}\right) \tag{25}$$

If an isotropy in the horizontal direction is assumed, then the exponential correlation function shown in Equation 2.25 is reduced to

$$\rho(\Delta x, \Delta y, \Delta z) = \exp\left(-\frac{\sqrt{\Delta x^2 + \Delta y^2}}{D_h} - \frac{|\Delta z|}{D_z}\right) \tag{26}$$

Similar theoretical autocorrelation functions in 3-D field for other distributions can also be formulated on the similar lines shown above.

3.4.6 Spatial averaging

Parameters in geotechnical analyses usually refer to averages of a soil property over a sliding surface or a rupture zone in an ultimate failure analysis or significantly strained volumes in a deformation analysis. If the dimensions of such surfaces or volumes exceed the scales of fluctuation of the soil property, spatial averaging of fluctuations is substantial. This implies that the variance of an averaged soil property over a sliding surface or affected volume is likely to be substantially less than the field variance, which is mainly based on small sample tests (e.g. triaxial tests) or small affected volumes in insitu tests (JCSS 2002).

Because of the spatial variability of soil properties, encountering a sufficiently low strength to induce failure in localized areas is more likely than such an encounter over the entire zone of

influence. Both the conventional analyses based on the factor of safety and the simplified probabilistic analyses fail to address this issue of scale of failure. Over the depth interval ΔZ the spatial average soil property is given as

$$u(\Delta Z) = \frac{1}{\Delta Z} \int_{\Delta Z} u(\Delta Z) dz \qquad (27)$$

The spatial average of the soil property u(x,y,z) over a volume V is given in the same way as

$$u_v = \frac{1}{V} \iiint_V u(x, y, z) dxdydz \qquad (28)$$

Averaging distance depends on the nature of the problem in hand. For design of shallow foundations in shear criterion, this distance is equal to the extent of shear failure zone within the soil mass (Cherubini 2000). This distance for shallow foundations in cohesionless soil subjected to vertical loading is approximately taken as 2B below the base of footing in the vertical direction and 3.5B from the centre of footing in the horizontal direction, where B is the width of the footing.

3.4.7 Evaluation of variance reduction function

The combined effect of spatial correlation and spatial averaging of soil properties over the failure domain are beneficially utilized to reduce the variance of the measured data within the zone of interest. The derivation of the variance reduction functions in terms of spatial correlation and spatial average is described in the following section. JCSS (2002) presents the evaluation of variance reduction function by both exact approach and simplified approach.

3.4.7.1 Variance reduction for data in 1-D space

The variability of soil property u_i from point to point is measured by standard deviation σ_i and the standard deviation of the spatial average property $u_{\Delta Z}$ is by $\sigma_{\Delta Z}$. The larger the length

(or the volume) over which the property is averaged, higher is the fluctuation of u_i that tends to cancel out in the process of spatial averaging. This causes reduction in standard deviation as the size of the averaging length or volume increases, which is given by

$$\Gamma_u(\Delta Z) = \frac{\sigma_{\Delta z}}{\sigma_i} \tag{29}$$

A simple relationship of the variance reduction function in terms of scale of fluctuation and averaging distance is given in Equation 2.30 (Vanmarcke 1977a).

$$\left.\begin{aligned}\Gamma^2(\Delta Z) &= \frac{\delta}{\Delta Z} & \frac{L}{\delta} &> 1.0 \\ \Gamma^2(\Delta Z) &= 1.0 & \frac{L}{\delta} &\leq 1.0\end{aligned}\right\} \tag{30}$$

The Equation 2.30 indicates that with decrease in scale of fluctuation and increase in averaging distance, the value of variance reduction function reduces, which in turn reduces standard deviation of the spatially averaged soil property. In other words, the more erratic the variation (i.e., less correlated the soil property) of the soil property with distance and larger the soil domain considered, larger will be the reduction in variability of the average property. This phenomenon is a result of the increasing likelihood that unusually high property values at some point will be balanced by low values at other point (Vanmarcke 1977a). However, Vanmarcke (1983) emphasized that the variance reduction function $\gamma(T)$ is related to the autocorrelation function $\rho(\tau)$ as given by.

$$\gamma(T) = \frac{1}{T^2}\int_0^T\int_0^T \rho(t_1 - t_2)\,dt_1 dt_2 \tag{31}$$

which reduces to

$$\gamma(T) = \frac{2}{T}\int_0^T \left(1 - \frac{\tau}{T}\right)\rho(\tau)\,d\tau \tag{32}$$

From Equation 2.32, the variance reduction functions for triangular, exponential, and squared exponential autocorrelation functions can be worked out as given in Equations 2.33 to 2.35, respectively.

$$\gamma(T) = \begin{cases} 1 - \dfrac{T}{3a} & \text{for } T \leq a \\ \left(\dfrac{a}{T}\right)\left(1 - \dfrac{a}{3T}\right) & \text{for } T \geq a \end{cases} \tag{33}$$

$$\gamma(T) = 2\left(\frac{b}{T}\right)^2\left(\frac{T}{b} - 1 + \exp(-T/b)\right) \tag{34}$$

$$\gamma(T) = \left(\frac{d}{T}\right)^2\left(\sqrt{\pi}\,\frac{T}{d}E\left(\frac{T}{d}\right) + \exp\left(-\left(\frac{T}{d}\right)^2\right) - 1\right) \tag{35}$$

where a, b, d are referred to as the autocorrelation distances, T is the averaging length, the distance over which the geotechnical properties are averaged over a failure surface, and E(·) is the error function, which increases from 0 to 1 as its argument increases from 0 to ∞. In terms of standard Gaussian cumulative distribution function $E(u)=2[F_U(u)-0.5]$.

As the averaging length, T→∞ the variance reduction function, $\gamma(T)$ →0. In other words, the chances associated with failure of huge volume of soil are very rare. In addition, $\gamma(T)$ is inversely proportional to T at very large values of T.

The variance reduction factor for averaging in one, 2 or 3-D random field may be approximated as given in Equations .

$$\begin{aligned} \Gamma^2(L_1 \ldots L_n) &= 1 & \text{for } (L_1 \ldots L_n) \leq \alpha_n \\ &= \frac{\alpha_n}{(L_1 \ldots L_n)} & \text{for } (L_1 \ldots L_n) \geq \alpha_n \end{aligned} \tag{36}$$

where n=1, 2, 3, and L_1, L_2 and L_3 are the lengths over which averaging takes place and α_1, α_2, α_3 are the correlation radii. In case of "separable" autocorrelation functions, i.e. which can

be written as a multiplication of factors for each of the dimensions of a 2- or 3-D surface or volume, the total variance reduction factor can, for the 3-D case be written as:

$$\Gamma^2(L_1 L_2 L_3) = \Gamma^2(L_1)\Gamma^2(L_2)\Gamma^2(L_3) \tag{37}$$

Similar to the above, Vanmarcke (1977a) also proposed an approximate and simplified resultant variance reduction factor in 2-D space as the product of individual variance reduction factors in vertical and horizontal directions in terms of scale of fluctuation (δ) and spatial averaging distance (L) in the respective directions as shown in Equation (38).

$$\Gamma_A^2 = \Gamma_v^2 \times \Gamma_h^2 \tag{29}.$$

The above propositions have been used in the analysis of spatial variability of soils and the influence of spatial variability in foundation design is presented in Sivakumar Babu et al (2005) Dasaka et al (2005) and Sumanta Haldar and Sivakumar Babu (2006).