Module 7: Micromechanics

Lecture 30: CCA Model: Effective Axial Modulus and Poisson's Ratio

Introduction

In this lecture we are going to determine the effective properties of a composite cylinder in terms of properties of fibre and matrix materials using CCA model. We will use the relations and concepts developed in the previous lecture for concentric cylinders.

In this approach the fibre is considered to be transversely isotropic and matrix is isotropic in nature. However, as a special case and whenever possible, we will derive expressions considering both fibre and matrix as isotropic materials.

In the present lecture we will derive the expressions for effective axial modulus and Poisson's ratio.

The Lecture Contains

Previous Next

Effective Axial Modulus

The effective axial modulus is determined from the basic definition of axial modulus. The axial load P is applied to the composite cylinder. The axial stress is uniform across the cross section. Further, this stress can be given as the axial load divided by the cross sectional area, that is,

$$\sigma_{xx} = \frac{P}{\pi b^2} \tag{7.186}$$

If the effective axial modulus is E_1^* is known then the axial strain can be given as

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E_1^*} = \frac{P}{\pi b^2 E_1^*} \tag{7.187}$$

For the axial load applied, the radial stress on the outer boundary, that is, at $r = b$ is zero. Thus,

$$\sigma_{rr}(b) = \sigma_{rr}^{(m)}(b) = 0 \tag{7.188}$$

Thus, from the last of Equation (7.185), this becomes

$$C_{12}^m \varepsilon_{xx} + (C_{11}^m + C_{12}^m)A^m + (C_{12}^m - C_{11}^m)\frac{B^m}{b^2} = 0 \tag{7.189}$$

The effective axial force can also be obtained by integrating the axial stresses in fibre and matrix over the cross sectional area as

$$P = \int_A \sigma_{xx}\, dA = \int_0^{2\pi} \sigma_{xx}\, r dr\, d\theta = 2\pi\left[\int_0^a \sigma_{xx}^{(f)}\, r dr + \int_0^a \sigma_{xx}^{(m)}\, r dr\right] \tag{7.190}$$

Putting the expressions for $\sigma_{xx}^{(f)}$ and $\sigma_{xx}^{(m)}$ from Equation (7.185) and carrying out the integration we get

$$P = \pi a^2\left(C_{11}^f \varepsilon_{xx} + 2C_{12}^f\, A^f\right) + \pi(b^2 - a^2)(C_{11}^m \varepsilon_{xx} + 2C_{12}^m\, A^m) \tag{7.191}$$

The unknown constants A^f, A^m, B^m and ε_{xx} can be determined by solving Equations (7.180), (7.183), (7.189) and (7.191). The unknowns A^f and A^m then can be used in Equation. (7.191) to calculate the axial force P.

The unknown constants A^f, A^m, B^m and ε_{xx} are given as

$$\begin{aligned} A^f &= \frac{a_1}{d}, \quad A^m = \frac{a_2}{d} \\ B^m &= \frac{a_3}{d}, \quad \varepsilon_{xx} = \frac{a_4}{d} \end{aligned} \tag{7.192}$$

where,

$$\begin{aligned} a_1 &= P\left\{a^2\left(C_{11}^m - C_{12}^m\right)\left(C_{12}^f - C_{12}^m\right) + b^2\left[C_{12}^m\left(C_{12}^f - C_{12}^m\right) + C_{11}^m\left(C_{12}^f + C_{12}^m\right)\right]\right\} \\ a_2 &= -PC_{12}^m\left[a^2\left(C_{11}^f - C_{12}^m\right)\left(C_{12}^f - C_{12}^m\right) + b^2 C_{12}^m\left(C_{11}^m - C_{12}^m + C_{22}^f + C_{23}^f\right)\right] \\ a_3 &= Pa^2 b^2 C_{12}^m\left[-C_{11}^m C_{12}^f + C_{12}^m\left(-C_{12}^f + C_{22}^f + C_{23}^f\right)\right] \\ a_4 &= P\left\{a^2\left(C_{11}^m - C_{12}^m\right)\left(C_{11}^m + C_{12}^m - C_{22}^f - C_{23}^f\right) - b^2\left(C_{11}^m + C_{12}^m\right)\left(C_{11}^m - C_{12}^m + C_{22}^f + C_{23}^f\right)\right\} \end{aligned} \quad (7.193)$$

Previous Next

The denominator d is given as

$$d = \pi(b_1 + b_2 + b_3) \tag{7.194}$$

with

$$\begin{aligned}
b_1 &= -b^4[(C_{11}^m)^2 + C_{11}^m C_{12}^m - 2(C_{12}^m)^2](C_{11}^m - C_{12}^m + C_{22}^f + C_{23}^f) \\
b_2 &= a^4(C_{11}^m - C_{12}^m)\begin{bmatrix} -(C_{11}^m)^2 + 2(C_{12}^f - C_{11}^m)^2 + C_{11}^f(C_{11}^m + C_{12}^m - C_{22}^f - C_{23}^f) \\ +C_{11}^m(-C_{12}^m + C_{22}^f + C_{23}^f) \end{bmatrix} \\
b_3 &= -a^2b^2\begin{bmatrix} C_{11}^f(C_{11}^m + C_{12}^m)(C_{11}^m - C_{12}^m + C_{22}^f + C_{23}^f) \\ -2\begin{Bmatrix} (C_{11}^m)^3 + C_{12}^m\left[(C_{12}^f)^2 - 2C_{12}^f C_{12}^m + C_{12}^m(2C_{12}^m - C_{22}^f - C_{23}^f)\right] \\ +C_{11}^m\left[(C_{12}^f)^2 + 2C_{12}^f C_{12}^m + C_{12}^m(-3C_{12}^m + C_{22}^f + C_{23}^f)\right] \end{Bmatrix} \end{bmatrix}
\end{aligned} \tag{7.195}$$

Thus, the effective axial modulus can be found from Equation (7.187) as

$$E_1^* = \frac{\sigma_{xx}}{\varepsilon_{xx}} = \frac{P}{\pi b^2 \varepsilon_{xx}} \tag{7.196}$$

***Note*:** One can determine the effective axial modulus from Equation (7.191). The unknown A^f, A^m can be expressed from this equation in terms of unknown ε_{xx}. For this, one has to solve Equations (7.180), (7.183) and (7.189) for A^f, A^m and B^m. Then divide both sides of this equation by cross sectional area of the composite cylinder and ε_{xx}. This gives the effective axial modulus. For this case, the unknown constants are given as

$$\begin{aligned}
A^f &= \frac{\varepsilon_{xx}\{a^2(C_{11}^m - C_{12}^m)(C_{12}^f + C_{12}^m) + b^2[C_{12}^m(C_{12}^f - C_{12}^m) + C_{11}^m(C_{12}^f + C_{12}^m)]\}}{a^2(C_{11}^m - C_{12}^m)(C_{11}^m + C_{12}^m - C_{22}^f - C_{23}^f) - b^2(C_{11}^m + C_{12}^m)(C_{11}^m - C_{12}^m + C_{22}^f + C_{23}^f)} \\
A^m &= \frac{\varepsilon_{xx}[a^2(C_{11}^m - C_{12}^m)(C_{12}^f - C_{12}^m) + b^2C_{12}^m(C_{11}^m - C_{12}^m + C_{22}^f + C_{23}^f)]}{a^2(C_{11}^m - C_{12}^m)(C_{11}^m + C_{12}^m - C_{22}^f - C_{23}^f) - b^2(C_{11}^m + C_{12}^m)(C_{11}^m - C_{12}^m + C_{22}^f + C_{23}^f)} \\
B^m &= \frac{\varepsilon_{xx}a^2b^2[C_{11}^m C_{12}^f + C_{12}^m(C_{12}^f - C_{22}^f - C_{23}^f)]}{a^2(C_{11}^m - C_{12}^m)(C_{11}^m + C_{12}^m - C_{22}^f - C_{23}^f) - b^2(C_{11}^m + C_{12}^m)(C_{11}^m - C_{12}^m + C_{22}^f + C_{23}^f)}
\end{aligned} \tag{7.197}$$

Previous Next

Special Case: Fibre and Matrix Materials are Isotropic

The constitutive equations for an isotropic and linear elastic material, we write

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} \tag{7.198}$$

where, μ and λ are the Lamé constants. Further, μ is shear modulus. To be consistent with the notations for shear modulus one can use the symbol G in the following derivations. However, we will use the symbol μ in this case. The axial stresses in fibre and matrix are given as

$$\begin{aligned} \sigma_{xx}^{(f)} &= \left(2\mu_f + \lambda_f\right)\varepsilon_{xx} + 2\lambda_f A^f \\ \sigma_{xx}^{(m)} &= \left(2\mu_m + \lambda_m\right)\varepsilon_{xx} + 2\lambda_m A^m \end{aligned} \tag{7.199}$$

Here, $k = \mu + \lambda$ is defined as bulk modulus. The axial stress in fibre and matrix is spatially uniform. The radial stresses in fibre and matrix are given as

$$\begin{aligned} \sigma_{rr}^{(f)} &= \lambda_f\,\varepsilon_{xx} + 2k_f A^f \\ \sigma_{rr}^{(m)} &= \lambda_m\,\varepsilon_{xx} + 2k_m A^m - \frac{2\mu_m}{r^2}B^m \end{aligned} \tag{7.200}$$

The hoop stresses in fibre and matrix are

$$\begin{aligned} \sigma_{\theta\theta}^{(f)} &= \lambda_f\,\varepsilon_{xx} + 2k_f A^f \\ \sigma_{\theta\theta}^{(m)} &= \lambda_m\,\varepsilon_{xx} + 2k_m A^m - \frac{2\mu_m}{r^2}B^m \end{aligned} \tag{7.201}$$

These are the same as radial stresses. Further, as mentioned earlier all shear stresses are zero.

Previous Next

The first continuity condition is the first of Equation (7.180). The continuity of radial stresses at the fibre and matrix interface from Equation (7.182) gives

$$2k_f A^f a^2 - 2k_m A^m a^2 + 2\mu_m B^m = a^2 \varepsilon_{xx}\left(\lambda_m - \lambda_f\right) \tag{7.202}$$

The radial stress free condition on outer surface as in Equation (7.188) gives the relation

$$2k_m b^2 A^m - 2\mu_m B^m = -b^2 \lambda_m \varepsilon_{xx} \tag{7.203}$$

The effective axial force as given in Equation (7.190) is written as

$$P = \pi a^2\left[\left(2\mu_f + \lambda_f\right)\varepsilon_{xx} + 2\lambda_f A^f\right] + \pi\left(b^2 - a^2\right)\left[\left(2\mu_m + \lambda_m\right)\varepsilon_{xx} + 2\lambda_m A^m\right] \tag{7.204}$$

Here, A^f, A^m, B^m and ε_{xx} are the unknowns. These can be obtained by solving the Equations (7.180) and Equations (7.202) through (7.204). These are given as

$$\begin{aligned} A^f &= \frac{b_1}{c}, \quad A^m = \frac{b_2}{c} \\ B^m &= \frac{b_3}{c}, \quad \varepsilon_{xx} = \frac{b_4}{c} \end{aligned} \tag{7.205}$$

where,

$$\begin{aligned} b_1 &= -P\left[a^2\left(\lambda_f - \lambda_m\right)\mu_m + b^2\left(k_m\lambda_f + \lambda_m\mu_m\right)\right] \\ b_2 &= -P\left[a^2\left(\lambda_f - \lambda_m\right)\mu_m + b^2\lambda_m\left(k_f + \mu_m\right)\right] \\ b_3 &= -Pa^2 b^2\left(k_m\lambda_f - k_f\lambda_m\right) \\ b_4 &= P\left[a^2\left(k_f - k_m\right)\mu_m + b^2 k_m\left(k_f + \mu_m\right)\right] \end{aligned} \tag{7.206}$$

and

$$c = 2\pi\left(c_1 + c_2 + c_3\right) \tag{7.207}$$

with

(7.208)

$$c_1 = b^4\left(k_f + \mu_m\right)[-\lambda_m^2 + k_m(\lambda_m + 2\mu_m)]$$

$$c_2 = -a^4\mu_m\left[\left(\lambda_f - \lambda_m\right)^2 + k_m\left(\lambda_f - \lambda_m + 2\mu_f - \mu_m\right) + k_f\left(-\lambda_f + \lambda_m - 2\mu_f + 2\mu_m\right)\right]$$

$$c_3 = a^2b^2\begin{bmatrix} 2\lambda_m\left(-\lambda_f + \lambda_m\right)\mu_m + k_f\left[\lambda_m^2 + k_m\left(\lambda_f - \lambda_m + 2\mu_f - 2\mu_m\right) + \lambda_m\mu_m + 2\mu_m^2\right] \\ -k_m\left[\lambda_f^2 - \lambda_f\mu_m + 2\mu_m\left(\lambda_m - \mu_f + 2\mu_m\right)\right] \end{bmatrix}$$

Then the effective axial modulus E_1^* can be determined from Equation (7.196) with appropriate substitutions of equations for isotropic assumption of fibre and matrix materials.

Previous Next

As an alternate approach, one can determine the effective axial modulus E_1^* from Equation (7.204) if A^f and A^m are expressed in terms of unknown ε_{xx} and divide both sides of this equation by ε_{xx} and cross sectional area of πb^2. A^f and A^m are obtained along with B^m by solving Equations (7.180), (7.202) and (7.203). These unknown constants are given in Equation (7.209).

$$\begin{aligned}
A^f &= -\frac{1}{2}\varepsilon_{xx}\frac{\left[(\lambda_m-\lambda_f)\mu_m a^2-(k_m\lambda_f+\mu_m\lambda_m)b^2\right]}{\left[(k_m-k_f)\mu_m a^2-(\mu_m+k_f)k_m b^2\right]}\\
A^m &= -\frac{1}{2}\varepsilon_{xx}\frac{\left[(\lambda_m-\lambda_f)\mu_m a^2-(k_m\lambda_m+\mu_m\lambda_m)b^2\right]}{\left[(k_m-k_f)\mu_m a^2-(\mu_m+k_f)k_m b^2\right]}\\
B^m &= -\frac{1}{2}\varepsilon_{xx}a^2b^2\frac{-k_m\lambda_f+\lambda_m k_f}{\left[(k_m-\lambda_f)\mu_m a^2-(\mu_m+k_f)k_m b^2\right]}
\end{aligned} \qquad (7.209)$$

The effective axial modulus for this special case is then given as

$$E_1^* = V_f E^{(f)} + (1-V_f)E^{(m)} + \frac{4V_f(1-V_f)(v_f-v_m)^2\mu_m}{\dfrac{(1-V_f)\mu_m}{\left(k_f+\dfrac{\mu_f}{2}\right)}+\dfrac{V_f\mu_m}{\left(k_m+\dfrac{\mu_m}{3}\right)}+1} \qquad (7.210)$$

In this equation the fibre volume fraction is given as

$$V_f = \frac{\pi a^2}{\pi b^2} = \frac{a^2}{b^2}$$

Previous Next

Equivalence of Strain Energy Approach

In this approach the strain energies of the concentric cylinders and equivalent homogeneous cylinder are equated. The strain energy of the equivalent homogeneous cylinder is written as

$$U = \frac{1}{2}\int_V \sigma_{xx}\,\varepsilon_{xx}\,dV = \frac{1}{2}\frac{P^2}{\pi b^2 E_1^*}\Delta L = \frac{1}{2}\frac{P^2}{\pi b^2 E_1^*} \qquad (7.211)$$

In this equation a unit length of the cylinders has been assumed. Further, use of Equation (7.191) can be made in this equation. Now, the strain energy of the concentric cylinders is given as

$$\begin{aligned} U &= \frac{1}{2}\int_V (\sigma_{xx}\varepsilon_{xx} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + \sigma_{rr}\varepsilon_{rr})dV \\ &= \frac{1}{2}\int_{x=0}^{1}\int_{\theta=0}^{2\pi}\int_{r=0}^{b} (\sigma_{xx}\varepsilon_{xx} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + \sigma_{rr}\varepsilon_{rr})dx\,d\theta\,r dr \\ &= \pi\left[\int_{r=0}^{a}\left(\sigma_{xx}^{(f)}\varepsilon_{xx}^{(f)} + \sigma_{\theta\theta}^{(f)}\varepsilon_{\theta\theta}^{(f)} + \sigma_{rr}^{(f)}\varepsilon_{rr}^{(f)}\right)r dr + \int_{r=a}^{b}\left(\sigma_{xx}^{(m)}\varepsilon_{xx}^{(m)} + \sigma_{\theta\theta}^{(m)}\varepsilon_{\theta\theta}^{(m)} + \sigma_{rr}^{(m)}\varepsilon_{rr}^{(m)}\right)r dr\right] \end{aligned} \qquad (7.212)$$

Recall that the stresses are functions of r alone. They are independent of θ and x. Further, for this cylinder system also a unit length has been assumed. Equation (7.184) and Equation (7.185) are substituted in the above equation and integration over r is carried out. It gives us

$$U = \frac{\pi}{2}\left\{\begin{array}{c} a^2\left[C_{11}^f\varepsilon_{xx}^2 + 4C_{12}^f A^f\varepsilon_{xx} + 2\left(C_{22}^f + C_{23}^f\right)(A^f)^2\right] \\ +\left(\dfrac{b^2-a^2}{a^2b^2}\right)(B^m)^2(C_{11}^m - C_{12}^m) \\ +(b^2-a^2)\left[2(A^m)^2(C_{11}^m + C_{12}^m) + 4A^m C_{12}^m\varepsilon_{xx} + C_{12}^m\varepsilon_{xx}^2\right] \end{array}\right\} \qquad (7.213)$$

Further, the unknown constants A^f ,A^m B^m and ε_{xx} as given in Equation (7.192) are used. Then comparison of Equation (7.211) and Equation (7.213) gives the effective axial modulus E_1^*.

Special Case: Fibre and Matrix Materials are Isotropic

The strain energy of the equivalent homogeneous cylinder obtained using Equation (7.211) and Equation (7.204). The strain energy of the concentric cylinders can be obtained using Equation (7.199) – Equation (7.201) and Equation (7.184) in the last of Equation (7.212). The strain energy for concentric cylinders is given as

$$U = \frac{\pi}{2}\left\{\begin{array}{c} a^2\left[4k_f(A^f)^2 + 4\lambda_f\varepsilon_{xx}A^f + \varepsilon_{xx}^2\left(\lambda_f + 2\mu_f\right)\right] \\ +(b^2-a^2)\left[4k_m(A^m)^2 + 4\lambda_m\varepsilon_{xx}A^m + \varepsilon_{xx}^2(\lambda_m + 2\mu_m)\right] \\ +8A^m B^m\mu_m\ln\left(\dfrac{a}{b}\right) \end{array}\right\} \qquad (7.214)$$

In above equation, the unknown constants A^f ,A^m B^m and ε_{xx} and as given Equation (7.205) can be used for further simplifications. Thus, equivalence of strain energies of concentric and equivalent homogeneous cylinder will lead to effective axial modulus E_1^*.

Previous Next

Effective Axial Poisson’s Ratio:

The effective axial Poisson’s ratio can be determined from the preceding problem definition. Here, we define the effective axial Poisson’s ratio as the ratio of associated lateral strain to the axial strain due to applied load in axial direction. Thus,

$$v_{12}^{*} = \frac{w^{(m)}(b)/b}{\varepsilon_{xx}} \tag{7.215}$$

From Equation (7.176) we can write

$$v_{12}^{*} = \frac{-1}{\varepsilon_{xx}}\left(A^{m} + \frac{B^{m}}{b^{2}}\right) \tag{7.216}$$

The constants A^f ,A^m B^m and ε_{xx} and given from Equation (7.192) can be used in the above equation. Further, one can give the constants A^m and B^m in terms of ε_{xx} as in Equation (7.197).

$$v_{12}^{*} = \frac{2V_f C_{11}^{m} C_{12}^{f} + (1-V_f)C_{12}^{m}(C_{11}^{m} - C_{12}^{m} + C_{22}^{f} + C_{23}^{f})}{(-V_f)(C_{11}^{m} - C_{12}^{m})(C_{11}^{m} + C_{12}^{m} - C_{22}^{f} - C_{23}^{f}) + (C_{11}^{m} + C_{12}^{m})(C_{11}^{m} - C_{12}^{m} + C_{22}^{f} + C_{23}^{f})} \tag{7.217}$$

Special Case: Fibre and Matrix Materials are Isotropic

When both fibre and matrix materials are isotropic in nature, then using either Equation (7.205) or Equation. (7.209), the axial Poisson’s ratio is given as

$$v_{12}^{*} = (1-V_f)v^{(m)} + V_f v^{(f)} + \frac{V_f(1-V_f)\left(v^{(f)} - v^{(m)}\right)\left[\frac{\mu_m}{\left(k_m + \frac{\mu_m}{3}\right)} - \frac{\mu_m}{\left(k_f + \frac{\mu_f}{3}\right)}\right]}{\frac{(1-V_f)\mu_m}{\left(k_f + \frac{\mu_f}{3}\right)} + \frac{V_f \mu_m}{\left(k_m + \frac{\mu_m}{3}\right)} + 1} \tag{7.218}$$

Comparing this equation with the corresponding equation in strength of materials approach, we see that the first two terms are exactly same. It should be noted that the third term is not a small term.

Previous Next

Module 7: Micromechanics

Lecture 30: CCA Model: Effective Axial Modulus and Poisson's Ratio

Home Work

1. What are the deformations or load conditions to be imposed on the concentric cylinders to determine the effective axial modulus and Poisson’s ratio?
2. What are the continuity conditions to be imposed on the concentric cylinders to determine the effective axial modulus and Poisson’s ratio?
3. Outline the methodology with key points to determine the effective axial modulus and Poisson’s ratio using CCA model.

Previous Next

References:

- *Z Hashin, BW Rosen. The elastic moduli of fibre-reinforced materials. J. Appl. Mech. 1964, Vol. 31, pp. 223-232.*
- *Z Hashin. Analysis of properties of fiber composites with anisotropic constituents. J. Appl. Mech. 1979, Vol. 46, pp. 543-550.*
- *RM Christensen. Mechanics of Composite Materials. Krieger Publishing Company, Florida, 1991.*
- *Z. Hashin: On elastic behaviour of fibre reinforced materials of arbitrary transverse phase geometry. J. Mech. Phys. Solids, 1965, Vol. 13, pp. 119-134.*
- *Z. Hashin: Analysis of composite materials-a survey. J. Mech. Phys. Solids, 1983, Vol. 50, pp. 481-505.*
- *YC Fung. Foundations of Solid Mechanics. Prentice Hall International, Inc., 1965.*
- *PC Chou, NJ Pagano. Elasticity: Tensor, Dyadic and Engineering Approaches, General Publishing Company, Ltd. Canada, 1967.*
- *CT Herakovich. Mechanics of Fibrous Composites, John Wiley & Sons, Inc. New York, 1998.*
- *MW Hyer, AM Waas. Micromechanics of linear elastic continuous fibre composites, in Comprehensive Composite Materials. Vol. 1: Fiber Reinforcements and General Theory of Composites, Eds. A Kelly, C Zweben. Elsevier, 2000.*
- *Z Hashin. Viscoelastic fiber reinforced materials. AIAA J, 1966, Vol. 4, pp. 1411-1417.*
- *Z Hashin, S Shtrikman. A variational approach to the theory of elastic behaviour of multiphase materials. J. Mech. Phys. Solids, 1963, Vol. 11, pp. 127-140.*

Previous Next