Introduction:

In this lecture we are going to develop the laminate theory with thermal and hygral effects. Then we will develop the relations for effective coefficients of thermal and hygral expansion for laminate. Further, we will develop governing differential equation for laminate. We will conclude this lecture with some sample numerical examples based on this.

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Laminate Theory with Thermal Effects:

In the classical laminate theory we make the assumption that the total strain is superposition of mechanical strain and thermal strains. Thus,

$$\begin{aligned}\{\epsilon\} &= \{\epsilon^{(\sigma)}\} + \{\epsilon^{(T)}\}\\ \{\epsilon\} &= \{\epsilon^{(0)}\} + z\{\kappa\} + \{\epsilon^{(T)}\}\end{aligned} \qquad (5.96)$$

where, $\{\epsilon^{(T)}\}$ is the thermal strain in laminate. We can write the stresses in global direction for laminate as

$$\{\sigma\}_{xy} = [\overline{Q}]\,\{\epsilon\}_{xy}$$

Thus, using Equation (5.95) in above equation, we get

$$\{\sigma\}_{xy} = [\overline{Q}]\,\left(\{\epsilon^{(0)}\}_{xy} + z\,\{\kappa\}_{xy} - \{\epsilon^{(T)}\}_{xy}\right) \qquad (5.97)$$

Now we find the resultant in-plane forces as

$$\{N\}_{xy} = \int_{-H}^{H}\{\sigma\}_{xy}\,dz = \int_{-H}^{H}[\overline{Q}]^k\,\{\epsilon^{(0)}\}_{xy}\,dz + \int_{-H}^{H}[\overline{Q}]^k\,\{\kappa\}_{xy}\,z\,dz - \int_{-H}^{H}[\overline{Q}]^k\,\{\epsilon^{(T)}\}_{xy}dz$$

Recalling the development of classical laminate theory with the use of Equation (5.21), we write

$$\{N\}_{xy} = [A]\{\epsilon^{(0)}\}_{xy} + [B]\{\kappa\}_{xy} - \int_{-H}^{H}[\overline{Q}]^k\,\{\epsilon^{(T)}\}_{xy}dz \qquad (5.98)$$

Let us define

$$\begin{aligned}\{N^{(T)}\}_{xy} &= \int_{-H}^{H}[\overline{Q}]^k\,\{\epsilon^{(T)}\}_{xy}dz\\ &= \sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}[\overline{Q}]^k\,\{\epsilon^{(T)}\}_{xy}^k\,dz\\ &= \sum_{k=1}^{N}[\overline{Q}]^k\,\{\epsilon^{(T)}\}_{xy}^k(z_k - z_{k-1})\end{aligned} \qquad (5.99)$$

as the effective laminate thermal forces per unit length. Thus, with this definition we can write Equation (5.97) as

$$\{N\}_{xy} + \{N^{(T)}\}_{xy} = [A]\{\epsilon^{(0)}\}_{xy} + [B]\{\kappa\}_{xy} \qquad (5.100)$$

Now let us define the resultant moments using Equation (5.96) as

$$\{M\}_{xy} = \int_{-H}^{H} \{\sigma\}_{xy}\, z\, dz$$

$$= \int_{-H}^{H} \left[\overline{Q}\right]^{k} \left\{\epsilon^{(0)}\right\}_{xy} z\, dz + \int_{-H}^{H} \left[\overline{Q}\right]^{k} \{\kappa\}_{xy}\, z^{2}\, dz - \int_{-H}^{H} \left[\overline{Q}\right]^{k} \left\{\epsilon^{(T)}\right\}_{xy} z\, dz$$

With the use of Equation (5.21) and Equation (5.27), we write

$$\{M\}_{xy} = [B]\left\{\epsilon^{(0)}\right\}_{xy} + [D]\{\kappa\}_{xy} - \int_{-H}^{H} \left[\overline{Q}\right]^{k} \left\{\epsilon^{(T)}\right\}_{xy} z\, dz \quad (5.101)$$

Let us define

$$\left\{M^{(T)}\right\}_{xy} = \int_{-H}^{H} \left[\overline{Q}\right]^{k} \left\{\epsilon^{(T)}\right\}_{xy} z\, dz$$

$$= \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \left[\overline{Q}\right]^{k} \left\{\epsilon^{(T)}\right\}_{xy}^{k} z\, dz \qquad (5.102)$$

$$= \frac{1}{2} \sum_{k=1}^{N} \left[\overline{Q}\right]^{k} \left\{\epsilon^{(T)}\right\}_{xy}^{k} \left(z_k^2 - z_{k-1}^2\right)$$

as the effective laminate thermal moments per unit length. Thus, with this definition we can write Equation (5.101) as

$$\{M\}_{xy} + \left\{M^{(T)}\right\}_{xy} = [B]\left\{\epsilon^{(0)}\right\}_{xy} + [D]\{\kappa\}_{xy} \qquad (5.103)$$

Combining Equation (5.99) and Equation (5.102), we write

$$\begin{Bmatrix} N + N^{(T)} \\ M + M^{(T)} \end{Bmatrix}_{xy} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \epsilon^{(0)} \\ \kappa \end{Bmatrix}_{xy} \qquad (5.104)$$

The inverse of this equation is written as

$$\begin{Bmatrix} \epsilon^{(0)} \\ \kappa \end{Bmatrix}_{xy} = \begin{bmatrix} A' & B' \\ B'^{T} & D' \end{bmatrix} \begin{Bmatrix} N + N^{(T)} \\ M + M^{(T)} \end{Bmatrix}_{xy} \qquad (5.105)$$

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Laminate Coefficient of Thermal Expansion

The derivation of laminate coefficients of thermal expansion is dealt here for symmetric laminates. This is because there is no extension-bending coupling due to the fact that for symmetric laminates $[B]=0$.

Let us derive the expression for the coefficient of thermal expansion for laminate. Let us define the coefficient of thermal expansion for laminate, $\{\alpha^*\}$, as the laminate mid-plane strain, $\{\epsilon^{(0)}\}$, to the per unit uniform change in temperature, ΔT. Thus,

$$\{\alpha^*\}=\frac{\{\epsilon^{(0)}\}}{\Delta T} \quad (5.106)$$

Now for symmetric laminates with pure thermal loading, we have

$$\{\epsilon^{(0)}\}=[A]^{-1}\{N^{(T)}\} \quad (5.107)$$

Combining Equation (5.105) and Equation (5.106), we get

$$\{\alpha^*\}=\frac{[A]^{-1}\{N^{(T)}\}}{\Delta T} \quad (5.108)$$

For a uniform temperature change the equivalent thermal force is given as

$$\{N^{(T)}\}=\Delta T\int_{-H}^{H}[\overline{Q}]^k\{\alpha\}_{xy}^k\,dz \quad (5.109)$$

Thus, the laminate coefficient of thermal expansion becomes

$$\{\alpha^*\}=[A]^{-1}\int_{-H}^{H}[\overline{Q}]^k\{\alpha\}_{xy}^k\,dz \quad (5.110)$$

It is known that the $[\overline{Q}]^k$ and $\{\alpha\}_{xy}^k$ are constant in each lamina in thickness direction. Thus, the integration over thickness can be simplified as the summation over laminae thicknesses as

$$\{\alpha^*\}=[A]^{-1}\sum_{k=1}^{N}[\overline{Q}]^k\{\alpha\}_{xy}^k\,t_k \quad (5.111)$$

where, $t_k=z_k-z_{k-1}$ is the thickness of *k*th lamina.

Figure 5.12 shows the variation of α_{xx}^* for $[\pm\theta]_S$ laminate along with α_{xx} of layer for $+\theta$ for AS4/3501-6 Epoxy material from Soden et al [4]. Similarly, Figure 5.13 and Figure 5.14 show the variation of α_{yy}^* and α_{xy}^*.

From these figures it is seen that these coefficients vary from positive to negative values. Further, it is observed that the coefficient of thermal expansion depends upon stacking sequence. This fact is very important from laminate designing point of view where it is used in an environment with large thermal gradient. One can choose a laminate sequence for which a coefficient of thermal expansion is zero.

From Equation (5.109) and Equation (5.110) for a uniform temperature change, the equivalent thermal force is written as

$$\{N^{(T)}\} = [A]\,\{\alpha^*\}\,\Delta T \qquad (5.112)$$

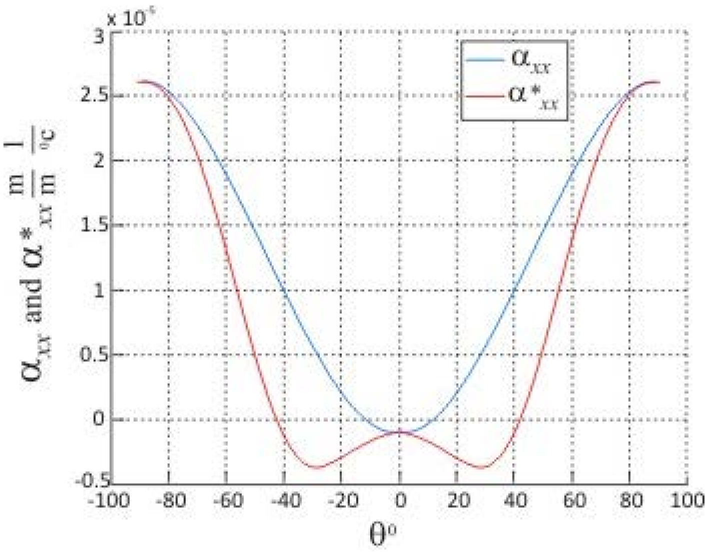

Figure 5.12: Variation of α_{xx} for lamina and of α^*_{xx} for $[\pm\theta]_S$ laminate

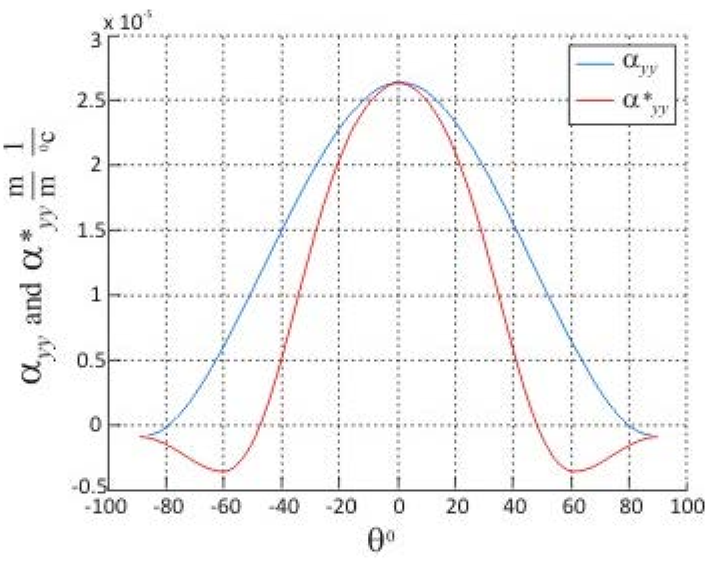

Figure 5.13: Variation of α_{yy} for lamina and of α^*_{yy} for $[\pm\theta]_S$ laminate

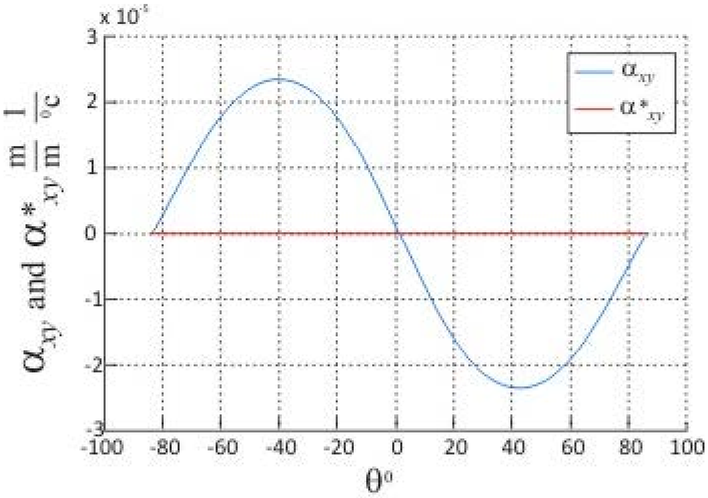

Figure 5.14: Variation of α_{xy} for lamina and of α^*_{xy} for $[\pm\theta]_S$ laminate

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Laminate Theory with Hygral Effects:

In this study, the total strain is assumed to be superimposition of mechanical and hygral strains.

$$\begin{aligned}\{\epsilon\} &= \{\epsilon^{(\sigma)}\} + \{\epsilon^{(H)}\}\\ \{\epsilon\} &= \{\epsilon^{(0)}\} + z\{\kappa\} + \{\epsilon^{(H)}\}\end{aligned} \tag{5.113}$$

where, $\{\epsilon^{(H)}\}$ is the hygral strain in laminate. The stress in global direction is given as

$$\{\sigma\}_{xy} = [\bar{Q}]\left(\{\epsilon^{(0)}\}_{xy} + z\,\{\kappa\}_{xy} - \{\epsilon^{(H)}\}_{xy}\right) \tag{5.114}$$

The resultant in-plane forces are given as

$$\{N\}_{xy} + \{N^{(H)}\}_{xy} = [A]\{\epsilon^{(0)}\}_{xy} + [B]\{\kappa\}_{xy} \tag{5.115}$$

where,

$$\{N^{(H)}\}_{xy} = \sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}[\bar{Q}]^k\,\{\epsilon^{(H)}\}_{xy}^k\,dz \tag{5.116}$$

is defined as the resultant in-plane forces per unit length due to hygral strains.
In a similar way, we can give the resultant moments as

$$\{M\}_{xy} + \{M^{(H)}\}_{xy} = [B]\{\epsilon^{(0)}\}_{xy} + [D]\{\kappa\}_{xy} \tag{5.117}$$

where,

$$\{M^{(H)}\}_{xy} = \sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}[\bar{Q}]^k\,\{\epsilon^{(H)}\}_{xy}^k\,z\,dz \tag{5.118}$$

is defined as resultant moments per unit length due to hygral strains. Combining Equation (5.116) and Equation (5.118), we write

$$\begin{Bmatrix} N + N^{(H)} \\ M + M^{(H)} \end{Bmatrix}_{xy} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}\begin{Bmatrix} \epsilon^{(0)} \\ \kappa \end{Bmatrix}_{xy} \tag{5.119}$$

The inverse of this equation is written as

$$\begin{Bmatrix} \epsilon^{(0)} \\ \kappa \end{Bmatrix}_{xy} = \begin{bmatrix} A' & B' \\ B'^T & D' \end{bmatrix}\begin{Bmatrix} N + N^{(H)} \\ M + M^{(H)} \end{Bmatrix}_{xy} \tag{5.120}$$

Laminate Coefficients of Hygral Expansion

Here we derive the laminate coefficients hygral expansion, $\{\beta^*\}$, for symmetric laminates. Let us define the coefficient of hygral expansion for laminate as laminate mid-plane strains to the per percentage change in moisture absorption, ΔM. Thus,

$$\{\beta^*\} = \frac{\{\epsilon^{(0)}\}}{\Delta M} \tag{5.121}$$

For symmetric laminates with hygral loads alone, we can write

$$\{\epsilon^{(0)}\} = [A]^{-1}\{N^{(H)}\} \tag{5.122}$$

Combining Equation (5.121) and Equation (5.122), we get

$$\{\beta^*\} = \frac{[A]^{-1}\{N^{(H)}\}}{\Delta M} \tag{5.123}$$

For a uniform moisture absorption the equivalent hygral force is given as

$$\{N^{(H)}\} = \Delta M \int_{-H}^{H}[\overline{Q}]^k \{\beta\}_{xy}^k \, dz \tag{5.124}$$

Thus, the laminate coefficient of hygral expansion becomes

$$\{\beta^*\} = [A]^{-1}\int_{-H}^{H}[\overline{Q}]^k \{\beta\}_{xy}^k \, dz \tag{5.125}$$

It is known that the $[\overline{Q}]^k$ and $\{\beta\}_{xy}^k$ are constant in each lamina in thickness direction. Thus, the integration over thickness can be simplified as summation over laminae thicknesses as

$$\{\beta^*\} = [A]^{-1}\sum_{k=1}^{N}[\overline{Q}]^k \{\beta\}_{xy}^k \, t_k \tag{5.126}$$

Combining Equation (5.124) and Equation (5.125) we can get the effective hygral force as

$$\{N^{(H)}\} = [A]\{\beta^*\}\Delta M \tag{5.127}$$

Figure 5.15 shows the variation of β_{xx}^* for $[\pm\theta]_S$ laminate along with β_{xx} of a layer of $+\theta$ for T300/5208 material from Pipes et al [5]. Similarly, Figure 5.16 and Figure 5.16 show the variation of β_{yy}^* and β_{xy}^*.

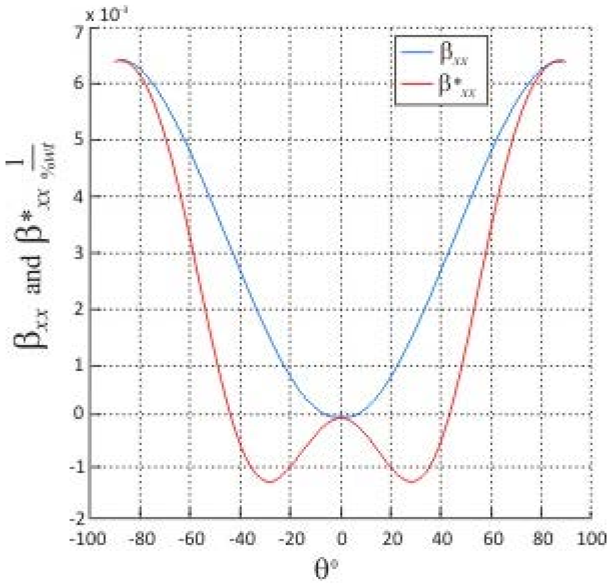

Figure 5.15: Variation of β_{xx} for lamina and of β^*_{xx} for $[\pm\theta]_S$ laminate

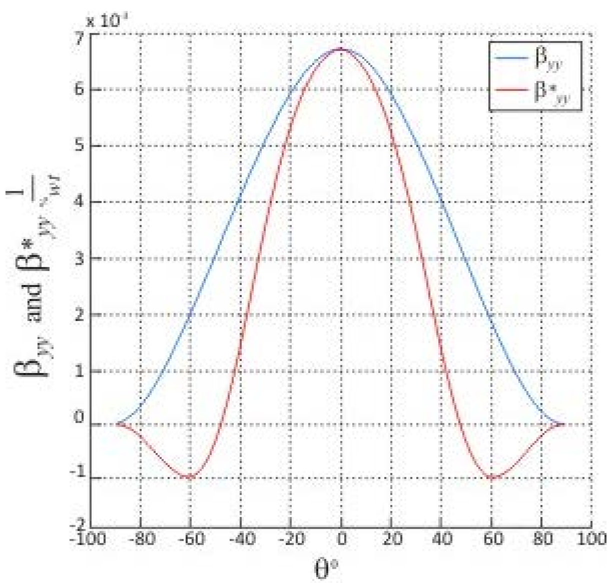

Figure 5.16: Variation of $\boldsymbol{\beta}_{yy}$ for lamina and of β^*_{yy} for $[\pm\theta]_S$ laminate

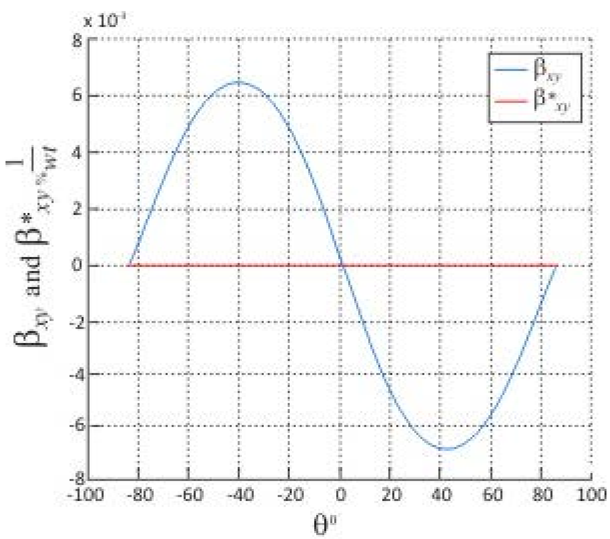

Figure 5.17: Variation of $\boldsymbol{\beta}_{xy}$ for lamina and of β^*_{xy} for $[\pm\theta]_S$ laminate

Hygro-thermal Effects on Laminate:

When thermal as well as hygral strains are present in total strain, then

$$\begin{aligned}\{\epsilon\} &= \left\{\epsilon^{(\sigma)}\right\} + \left\{\epsilon^{(T)}\right\} + \left\{\epsilon^{(H)}\right\} \\ \{\epsilon\} &= \left\{\epsilon^{(0)}\right\} + z\{\kappa\} + \left\{\epsilon^{(T)}\right\} + \left\{\epsilon^{(H)}\right\}\end{aligned} \tag{5.128}$$

The resultant in-plane forces are given as

$$\{N\}_{xy} + \left\{N^{(T)}\right\}_{xy} + \left\{N^{(H)}\right\}_{xy} = [A]\left\{\epsilon^{(0)}\right\}_{xy} + [B]\{\kappa\}_{xy} \tag{5.129}$$

and the resultant moments are given as

$$\{M\}_{xy} + \left\{M^{(T)}\right\}_{xy} + \left\{M^{(H)}\right\}_{xy} = [B]\left\{\epsilon^{(0)}\right\}_{xy} + [D]\{\kappa\}_{xy} \tag{5.130}$$

Equation (5.115) and (5.116) are combined as

$$\begin{Bmatrix} N + N^{(T)} + N^{(H)} \\ M + M^{(T)} + M^{(H)} \end{Bmatrix}_{xy} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \epsilon^{(0)} \\ \kappa \end{Bmatrix}_{xy} \tag{5.131}$$

The inverse of this equation is written as

$$\begin{Bmatrix} \epsilon^{(0)} \\ \kappa \end{Bmatrix}_{xy} = \begin{bmatrix} A' & B' \\ B'^{T} & D' \end{bmatrix} \begin{Bmatrix} N + N^{(T)} + N^{(H)} \\ M + M^{(T)} + M^{(H)} \end{Bmatrix}_{xy} \tag{5.132}$$

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Governing Differential Equations for Classical Laminate Theory

The equilibrium equations for a laminate are

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}=0$$

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}=0 \qquad (5.133)$$

$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}=0$$

In the laminate, in general, we consider that the transverse shear stresses are vanishing at the top and bottom of the laminate, that is $\tau_{xz}=\tau_{yz}=0$ at $z=+H$ and $z=-H$. Now, integrate Equation (5.133) with respect to *z*. The first two of the above equation give us

$$\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0$$

$$\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_{yy}}{\partial y}=0 \qquad (5.134)$$

The third of the Equation (5.133) gives

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q=0 \qquad (5.135)$$

where,

$$\left(Q_x, Q_y\right)=\int_{-H}^{H}\left(\tau_{xz}, \tau_{yz}\right) dz \text{ and } q=\sigma_{zz}|_{z=+H}-\sigma_{zz}|_{z=-H}$$

Now, multiply the first of Equation (5.133) with *z* and integrate with respect to *z* to get

$$\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{xy}}{\partial y}+\int_{-H}^{H}\frac{\partial \tau_{xz}}{\partial z} z\, dz=0 \qquad (5.136)$$

Now, let us write

$$z\frac{\partial \tau_{xz}}{\partial z}=\frac{\partial}{\partial z}(z\,\tau_{xz})-\tau_{xz}$$

Now recalling that $\tau_{xz}=0$ at $z=+H$ and $z=-H$ we can write for the third term in Equation (5.136) as

$$\int_{-H}^{H}\frac{\partial}{\partial z}(z\,\tau_{xz})\, dz=\ [z\,\tau_{xz}]_{-H}^{+H}=0 \text{ and } -\int_{-H}^{H}\tau_{xz}\, dz=-\,Q_x$$

Thus, Equation (5.136) becomes

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x \quad (5.137)$$

Similarly, we can write

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} = Q_y \quad (5.138)$$

Now putting Equation (5.137) and Equation (5.138) in Equation (5.135) we get

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + q = 0 \quad (5.139)$$

Note that this equation is identical with the homogeneous plate theory. However, in these equations the definition of the resultants is different.

One can express the moment resultants in terms of *A*, *B* and *D* matrices and the derivatives of mid-plane displacements as given below.

Equation (5.137) can be written as

$$\begin{aligned} &A_{11}\, u_{0,xx} + 2A_{16}\, u_{0,xy} + A_{66}\, u_{0,yy} + A_{16}\, v_{0,xx} + (A_{12} + A_{66}) v_{0,xy} + A_{66}\, v_{0,yy} \\ &- B_{11} w_{0,xxx} - 3B_{16} w_{0,xxy} - (B_{12} + 2B_{66}) w_{0,xyy} - B_{26} w_{0,yyy} = 0 \end{aligned} \quad (5.140)$$

Equation (5.138) becomes

$$\begin{aligned} &A_{16}\, u_{0,xx} + 2A_{26}\, v_{0,yy} + A_{66}\, u_{0,yy} + A_{26}\, u_{0,yy} + (A_{12} + A_{66}) u_{0,xy} + A_{66}\, v_{0,xx} \\ &- B_{16} w_{0,xxx} - 3B_{26} w_{0,xyy} - (B_{12} + 2B_{66}) w_{0,xxy} - B_{22} w_{0,yyy} = 0 \end{aligned} \quad (5.141)$$

And Equation (5.139) becomes

$$\begin{aligned} &-B_{11}\, u_{0,xxx} - 3\, B_{16}\, u_{0,xxy} - (B_{12} + 2B_{66}) u_{0,xyy} - B_{26}\, u_{0,yyy} - B_{16}\, v_{0,xxx} \\ &-(B_{12} + 2B_{66})\, v_{0,xyy} - B_{22}\, v_{0,yyy} + D_{11} w_{0,xxxx} + 4D_{16} w_{0,xxxy} \\ &+(D_{12} + 2D_{66}) w_{0,xxyy} + 4D_{26} w_{0,xyyy} + D_{22} w_{0,yyyy} = q \end{aligned} \quad (5.142)$$

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Examples:

***Note*:** In the following examples, each lamina has a thickness of 1 *mm* and material is AS4/3501-6 Epoxy from Soden et al [4].

***Example 5.9*:** For the laminate in Example 5.2 (that is, $[0/90/0]$), calculate the laminate coefficients of thermal expansions.

***Solution*:** This is a symmetric matrix. Hence, $[B]=0$. *A* matrix is given as:

$$[A]=\begin{bmatrix}264.43 & 9.30 & 0\\ 9.30 & 148.82 & 0\\ 0 & 0 & 19.80\end{bmatrix}GPa-mm$$

Now we have

$$[A^{*}]=[A]^{-1}=\begin{bmatrix}0.00378 & -0.00023 & 0\\ -0.00023 & 0.00672 & 0\\ 0 & 0 & 0.05050\end{bmatrix}\frac{1}{GPa-mm}$$

Now we calculate effective thermal forces for fictitious thermal change of $\Delta T=1°C$.

$$\begin{Bmatrix}N_{xx}^{(T)}\\ N_{yy}^{(T)}\\ N_{xy}^{(T)}\end{Bmatrix}=\Delta T\sum_{k=1}^{3}[\bar{Q}]^{k}\begin{Bmatrix}\alpha_{xx}\\ \alpha_{yy}\\ \alpha_{xy}\end{Bmatrix}^{k}t_k$$

$$\begin{Bmatrix}N_{xx}^{(T)}\\ N_{yy}^{(T)}\\ N_{xy}^{(T)}\end{Bmatrix}=(1)\begin{bmatrix}126.86 & 3.10 & 0\\ 3.10 & 11.07 & 0\\ 0 & 0 & 6.60\end{bmatrix}\begin{Bmatrix}-0.01\\ 0.26\\ 0\end{Bmatrix}\times 10^{-4}\times(1)$$

$$+(1)\begin{bmatrix}11.07 & 3.10 & 0\\ 3.10 & 126.86 & 0\\ 0 & 0 & 6.60\end{bmatrix}\begin{Bmatrix}0.26\\ -0.01\\ 0\end{Bmatrix}\times 10^{-4}\times(1)$$

$$+(1)\begin{bmatrix}126.86 & 3.10 & 0\\ 3.10 & 11.07 & 0\\ 0 & 0 & 6.60\end{bmatrix}\begin{Bmatrix}-0.01\\ 0.26\\ 0\end{Bmatrix}\times 10^{-4}\times(1)$$

which gives,

$$\begin{Bmatrix}N_{xx}^{(T)}\\ N_{yy}^{(T)}\\ N_{xy}^{(T)}\end{Bmatrix}=\begin{Bmatrix}1.922\\ 5.232\\ 0\end{Bmatrix}\times 10^{-4}\ GPa-mm$$

The laminate coefficients of thermal expansion are calculated as

$$\begin{Bmatrix} \alpha^*_{xx} \\ \alpha^*_{yy} \\ \alpha^*_{xy} \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} N^{(T)}_{xx} \\ N^{(T)}_{yy} \\ N^{(T)}_{xy} \end{Bmatrix}$$

$$= \begin{bmatrix} 0.00378 & -0.00023 & 0 \\ -0.00023 & 0.00672 & 0 \\ 0 & 0 & 0.05050 \end{bmatrix} \begin{Bmatrix} 1.922 \\ 5.232 \\ 0 \end{Bmatrix} \times 10^{-4}$$

$$= \begin{Bmatrix} 0.6062 \\ 3.4716 \\ 0 \end{Bmatrix} \times 10^{-6} \ \frac{m}{m}\frac{1}{{}^\circ C}$$

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Example 5.10: For the laminate in Example 5.3 (that is, $[\pm 45]_S$), calculate the laminate coefficients of thermal expansions.

Solution: This is a symmetric matrix. Hence, $[B]=0$. *A* matrix is given as:

$$[A]=\begin{bmatrix}170.56 & 117.76 & 0\\ 117.76 & 170.56 & 0\\ 0 & 0 & 131.76\end{bmatrix} GPa-mm$$

And

$$[A]^{-1}=\begin{bmatrix}0.01120 & -0.00773 & 0\\ -0.00773 & 0.01120 & 0\\ 0 & 0 & 0.00759\end{bmatrix}\frac{1}{GPa-mm}$$

Now we calculate effective thermal forces for fictitious thermal change of $\Delta T=1°C$. In the thermal force calculation we have assumed that the middle two layers are combined to form one layer of 2 *mm* thickness.

$$\begin{Bmatrix}N_{xx}^{(T)}\\ N_{yy}^{(T)}\\ N_{xy}^{(T)}\end{Bmatrix}=\Delta T\sum_{k=1}^{3}[\bar{Q}]^k\begin{Bmatrix}\alpha_{xx}\\ \alpha_{yy}\\ \alpha_{xy}\end{Bmatrix}^k t_k$$

$$\begin{Bmatrix}N_{xx}^{(T)}\\ N_{yy}^{(T)}\\ N_{xy}^{(T)}\end{Bmatrix}=(1)\begin{bmatrix}42.64 & 29.44 & 28.95\\ 29.44 & 42.64 & 28.95\\ 28.95 & 28.95 & 32.94\end{bmatrix}\begin{Bmatrix}0.125\\ 0.125\\ -0.270\end{Bmatrix}\times 10^{-4}\times(1)$$

$$+(1)\begin{bmatrix}42.64 & 29.44 & -28.95\\ 29.44 & 42.64 & -28.95\\ -28.95 & -28.95 & 32.94\end{bmatrix}\begin{Bmatrix}0.125\\ 0.125\\ 0.270\end{Bmatrix}\times 10^{-4}\times(2)$$

$$+(1)\begin{bmatrix}42.64 & 29.44 & 28.95\\ 29.44 & 42.64 & 28.95\\ 28.95 & 28.95 & 32.94\end{bmatrix}\begin{Bmatrix}0.125\\ 0.125\\ -0.270\end{Bmatrix}\times 10^{-4}\times(1)$$

which gives,

$$\begin{Bmatrix}N_{xx}^{(T)}\\ N_{yy}^{(T)}\\ N_{xy}^{(T)}\end{Bmatrix}=\begin{Bmatrix}4.774\\ 4.774\\ 0\end{Bmatrix}\times 10^{-4}\ GPa-mm$$

The laminate coefficients of thermal expansion are calculated as

$$\begin{Bmatrix} \alpha^*_{xx} \\ \alpha^*_{yy} \\ \alpha^*_{xy} \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} N^{(T)}_{xx} \\ N^{(T)}_{yy} \\ N^{(T)}_{xy} \end{Bmatrix}$$

$$= \begin{bmatrix} 0.01120 & -0.00773 & 0 \\ -0.00773 & 0.01120 & 0 \\ 0 & 0 & 0.00759 \end{bmatrix} \begin{Bmatrix} 0.4774 \\ 0.4774 \\ 0 \end{Bmatrix} \times 10^{-3}$$

$$= \begin{Bmatrix} 1.6565 \\ 1.6565 \\ 0 \end{Bmatrix} \times 10^{-6} \frac{m}{m} \frac{1}{°C}$$

Example 5.11: Material properties for T300/5208 material are given here [**Error! Reference source not found.**]. Calculate the laminate coefficients of hygral expansion for $[\pm 45]_S$ laminate with each layer of 1 *mm* thickness.

$$E_1 = 143\ GPa, E_2 = 10.1\ GPa, G_{12} = 4.14\ GPa, \nu_{12} = 0.31, \beta_1 = 0, \beta_2 = 6.67 \times 10^{-3} / \%\ wt$$

Solution: This is a symmetric matrix. Hence, $[B] = 0$. *A* matrix is given as:

$$[A] = \begin{bmatrix} 177.01 & 143.89 & 0 \\ 143.89 & 177.01 & 0 \\ 0 & 0 & 147.87 \end{bmatrix} GPa - mm$$

And

$$[A]^{-1} = \begin{bmatrix} 0.01665 & -0.01353 & 0 \\ -0.01353 & 0.01665 & 0 \\ 0 & 0 & 0.00676 \end{bmatrix} \frac{1}{GPa - mm}$$

Now we calculate effective thermal forces for fictitious hygral change of $\Delta M = \frac{1}{\%\ wt}$. In the hygral force calculation we have assumed that the middle two layers are combined to form one layer of 2 *mm* thickness

$$\begin{Bmatrix} N^{(H)}_{xx} \\ N^{(H)}_{yy} \\ N^{(H)}_{xy} \end{Bmatrix} = \Delta M \sum_{k=1}^{3} [\bar{Q}]^k \begin{Bmatrix} \beta_{xx} \\ \beta_{yy} \\ \beta_{xy} \end{Bmatrix}^k t_k$$

$$\begin{Bmatrix} N^{(H)}_{xx} \\ N^{(H)}_{yy} \\ N^{(H)}_{xy} \end{Bmatrix} = (1) \begin{bmatrix} 44.25 & 35.97 & 33.45 \\ 35.97 & 44.25 & 33.45 \\ 33.45 & 33.45 & 36.96 \end{bmatrix} \begin{Bmatrix} 3.335 \\ 3.335 \\ -6.670 \end{Bmatrix} \times 10^{-3} \times (1)$$

$$+ (1) \begin{bmatrix} 44.25 & 35.97 & -33.45 \\ 35.97 & 44.25 & -33.45 \\ -33.45 & -33.45 & 36.96 \end{bmatrix} \begin{Bmatrix} 3.335 \\ 3.335 \\ 6.670 \end{Bmatrix} \times 10^{-3} \times (2)$$

$$+ (1) \begin{bmatrix} 44.25 & 35.97 & 33.45 \\ 35.97 & 44.25 & 33.45 \\ 33.45 & 33.45 & 36.96 \end{bmatrix} \begin{Bmatrix} 3.335 \\ 3.335 \\ -6.670 \end{Bmatrix} \times 10^{-3} \times (1)$$

which gives,

$$\begin{Bmatrix} N_{xx}^{(H)} \\ N_{yy}^{(H)} \\ N_{xy}^{(H)} \end{Bmatrix} = \begin{Bmatrix} 0.1777 \\ 0.1777 \\ 0 \end{Bmatrix} GPa - mm$$

The laminate coefficients of hygral expansion are calculated as

$$\begin{Bmatrix} \beta_{xx}^{*} \\ \beta_{yy}^{*} \\ \beta_{xy}^{*} \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} N_{xx}^{(H)} \\ N_{yy}^{(H)} \\ N_{xy}^{(H)} \end{Bmatrix}$$

$$= \begin{bmatrix} 0.01665 & -0.01353 & 0 \\ -0.01353 & 0.01665 & 0 \\ 0 & 0 & 0.00676 \end{bmatrix} \begin{Bmatrix} 0.1777 \\ 0.1777 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.5544 \\ 0.5544 \\ 0 \end{Bmatrix} \times 10^{-3} \ /\% \ wt$$

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Example 5.12: The laminate in Example 5.1 is subjected to a thermal gradient of $-50°C$. Calculate the thermal residual stress at the top of the laminate.

Solution: The Q matrix for this material is calculated as

$$[Q]=\begin{bmatrix}126.68 & 3.10 & 0\\ 3.10 & 11.07 & 0\\ 0 & 0 & 6.60\end{bmatrix} GPa$$

Now, $\overline{Q}$ matrix for $0°$ and $90°$ is calculated as

$$[\overline{Q}(0)]=\begin{bmatrix}126.68 & 3.10 & 0\\ 3.10 & 11.07 & 0\\ 0 & 0 & 6.60\end{bmatrix} GPa, \quad [\overline{Q}(90)]=\begin{bmatrix}11.07 & 3.10 & 0\\ 3.10 & 126.68 & 0\\ 0 & 0 & 6.60\end{bmatrix} GPa$$

The coefficients of thermal expansion in global directions for $0°$ and $90°$ layers are

$$\begin{Bmatrix}\alpha_{xx}\\ \alpha_{yy}\\ \alpha_{xy}\end{Bmatrix}_{0°}=\begin{Bmatrix}-0.01\\ 0.26\\ 0\end{Bmatrix}\times 10^{-4}\frac{m}{m}\frac{1}{°C} \text{ and } \begin{Bmatrix}\alpha_{xx}\\ \alpha_{yy}\\ \alpha_{xy}\end{Bmatrix}_{90°}=\begin{Bmatrix}0.26\\ -0.01\\ 0\end{Bmatrix}\times 10^{-4}\frac{m}{m}\frac{1}{°C}$$

And the *A*, *B* and *D* matrices are calculated as

$$[A]=\begin{bmatrix}137.75 & 6.20 & 0\\ 6.20 & 137.75 & 0\\ 0 & 0 & 13.2\end{bmatrix} GPa-mm$$

$$[B]=\begin{bmatrix}-57.805 & 0 & 0\\ 0 & 57.805 & 0\\ 0 & 0 & 0\end{bmatrix} GPa-mm^2$$

$$[D]=\begin{bmatrix}45.916 & 2.066 & 0\\ 2.066 & 45.916 & 0\\ 0 & 0 & 4.4\end{bmatrix} GPa-mm^3$$

The thermal forces are

$$\begin{Bmatrix} N_{xx}^{(T)} \\ N_{yy}^{(T)} \\ N_{xy}^{(T)} \end{Bmatrix} = (-50)\begin{bmatrix} 126.86 & 3.10 & 0 \\ 3.10 & 11.07 & 0 \\ 0 & 0 & 6.60 \end{bmatrix}\begin{Bmatrix} -0.01 \\ 0.26 \\ 0 \end{Bmatrix} \times 10^{-4} \times (1)$$

$$+ (-50)\begin{bmatrix} 11.07 & 3.10 & 0 \\ 3.10 & 126.86 & 0 \\ 0 & 0 & 6.60 \end{bmatrix}\begin{Bmatrix} 0.26 \\ -0.01 \\ 0 \end{Bmatrix} \times 10^{-4} \times (1)$$

$$= \begin{Bmatrix} -11.93 \\ -11.93 \\ 0 \end{Bmatrix} \times 10^{-3}\ GPa - mm$$

Now we calculate the thermal moments as

$$\begin{Bmatrix} M_{xx}^{(T)} \\ M_{yy}^{(T)} \\ M_{xy}^{(T)} \end{Bmatrix} = \frac{1}{2}(-50)\begin{bmatrix} 126.86 & 3.10 & 0 \\ 3.10 & 11.07 & 0 \\ 0 & 0 & 6.60 \end{bmatrix}\begin{Bmatrix} -0.01 \\ 0.26 \\ 0 \end{Bmatrix} \times 10^{-4} \times \{(0)^2 - (-1)^2\}$$

$$+\frac{1}{2}(-50)\begin{bmatrix} 11.07 & 3.10 & 0 \\ 3.10 & 126.86 & 0 \\ 0 & 0 & 6.60 \end{bmatrix}\begin{Bmatrix} 0.26 \\ -0.01 \\ 0 \end{Bmatrix} \times 10^{-4} \times \{(-1)^2 - (0)^2\}$$

$$= \begin{Bmatrix} -8.2745 \\ 8.2745 \\ 0 \end{Bmatrix} \times 10^{-3}\ GPa - mm$$

We can write for the thermal forces and moments as

$$\begin{Bmatrix} N^{(T)} \\ M^{(T)} \end{Bmatrix}_{xy} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}\begin{Bmatrix} \epsilon^{(0)} \\ \kappa \end{Bmatrix}_{xy}$$

which becomes,

$$\begin{Bmatrix} -11.93 \\ -11.93 \\ 0 \\ -8.2745 \\ 8.2745 \\ 0 \end{Bmatrix} \times 10^{-3} = \begin{bmatrix} 137.75 & 6.20 & 0 & -57.805 & 0 & 0 \\ 6.20 & 137.75 & 0 & 0 & 57.805 & 0 \\ 0 & 0 & 13.2 & 0 & 0 & 0 \\ -57.805 & 0 & 0 & 45.916 & 2.066 & 0 \\ 0 & 57.805 & 0 & 2.066 & 45.916 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.4 \end{bmatrix}\begin{Bmatrix} \epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix}$$

Solving for the mid-plane strains and curvatures

$$\begin{Bmatrix} \epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = \begin{Bmatrix} -0.33686 \\ -0.33686 \\ 0 \end{Bmatrix} \times 10^{-3}$$

$$\begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} -0.63262 \\ 0.63262 \\ 0 \end{Bmatrix} \times 10^{-3}$$

The total strains at the top of the laminate, that is, at $z = -1.0\ mm$ in $0°$ lamina is

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix}_0 = \begin{Bmatrix} -0.33686 \\ -0.33686 \\ 0 \end{Bmatrix} \times 10^{-3} - 1.0 \begin{Bmatrix} -0.63262 \\ 0.63262 \\ 0 \end{Bmatrix} \times 10^{-3}$$

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix}_0 = \begin{Bmatrix} 0.29576 \\ -0.96948 \\ 0 \end{Bmatrix} \times 10^{-3}$$

We calculate the mechanical strains by subtracting the thermal strains from total strains.

Now the thermal strain in top lamina is

$$\begin{Bmatrix} \epsilon_{xx}^{(T)} \\ \epsilon_{yy}^{(T)} \\ \gamma_{xy}^{(T)} \end{Bmatrix}_0 = \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ \alpha_{xy} \end{Bmatrix}_0 \Delta T = \begin{Bmatrix} -0.01 \\ 0.26 \\ 0 \end{Bmatrix}_0 \times 10^{-4} (-50) = \begin{Bmatrix} 0.05 \\ -1.30 \\ 0 \end{Bmatrix} \times 10^{-3}$$

Thus, the mechanical strains become

$$\begin{Bmatrix} \epsilon_{xx}^{(\sigma)} \\ \epsilon_{yy}^{(\sigma)} \\ \gamma_{xy}^{(\sigma)} \end{Bmatrix}_0 = \begin{Bmatrix} 0.29576 \\ -0.96948 \\ 0 \end{Bmatrix} \times 10^{-3} - \begin{Bmatrix} 0.05 \\ -1.30 \\ 0 \end{Bmatrix} \times 10^{-3} = \begin{Bmatrix} 0.24576 \\ 0.33052 \\ 0 \end{Bmatrix} \times 10^{-3}$$

And the stresses at the top of the laminate are given as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}_0 = \begin{bmatrix} 126.86 & 3.10 & 0 \\ 3.10 & 11.07 & 0 \\ 0 & 0 & 6.60 \end{bmatrix} \begin{Bmatrix} 0.24576 \\ 0.33052 \\ 0 \end{Bmatrix} \times 10^{-3} = \begin{Bmatrix} 32.20 \\ 4.42 \\ 0 \end{Bmatrix} \times 10^{-3} GPa$$

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Homework:

1. Derive the resultant in-plane forces and moments for a laminate with thermal effects.
2. Derive the resultant in-plane forces and moments for a laminate with hygral effects.
3. Derive an expression for laminate coefficient of thermal expansion under the uniform temperature condition.
4. Derive an expression for laminate coefficient of hygral expansion under the condition of uniform moisture absorption.
5. Derive the expressions for resultant in-plane forces and moments for a laminate with hygro-thermal effects.
6. Derive the governing differential equations for classical laminate theory.
7. Calculate the laminate coefficients of thermal expansion for the following laminates of AS4/3501-6 Epoxy from Soden et al [4]. Take thickness of each layer as *1* mm.
    a. $[\mp 45]_S$
    b. $[0/45/0]$
    c. $[0/\pm 45]_S$
    d. $[0/90/0]_S$
8. Calculate the thermal residual stresses for a temperature change of $-75°C$ at the top and bottom of the following laminates of AS4/3501-6 Epoxy. (Write a computer code for this problem. Repeat the Example 5.12).
    a. $[90/0]$
    b. $[0/45]$
9. Calculate the laminate coefficients of hygral expansion for laminate sequences in exercise example 7 with T300/5208 material as in Example 5.11. Take thickness of each layer as *1* mm.

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Module 5: Laminate Theory

Lecture 19: Hygro -thermal Laminate Theory

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- *RB Pipes, JR Vinson, TW Chou. On the hygrothermal response of laminated composite systems. Journal of Composite Materials, 1976, Vol. 10, pp. 129-148.*

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