Module 7: Micromechanics

Lecture 33: Three Phase CCA Model: Effective Transverse Shear Modulus

Introduction

In the previous lectures we have introduced the concept of CCA model and have derived the expressions for effective axial modulus; Poisson’s ratio; shear modulus and plane strain bulk modulus. Now, only one of the effective properties is left to be determined so that we can get all effective properties of the composite. This property is transverse shear modulus.

In the present lecture, we will derive the expression for effective transverse shear modulus.

The Lecture Contains

Previous Next

Three-Phase Composite Cylinder for Transverse Shear Modulus

The determination of effective transverse shear modulus with concentric cylinders is quite difficult. At present, for this problem no exact solution has been presented. Hence, a different model is presented for the determination of transverse shear modulus. In this model, all cylinders except one are replaced with equivalent homogeneous material. This model is shown in Figure 7.12. In this model, the outer composite cylinder can have infinite radius. The outer cylinder represents the composite and hence, it should not affect the overall effective properties of the three cylinders. This model is called as ***three phase composite cylinder model* or *three phase model***. The closed form expression for this property proposed by Christensen and Lo [10] has been presented here.

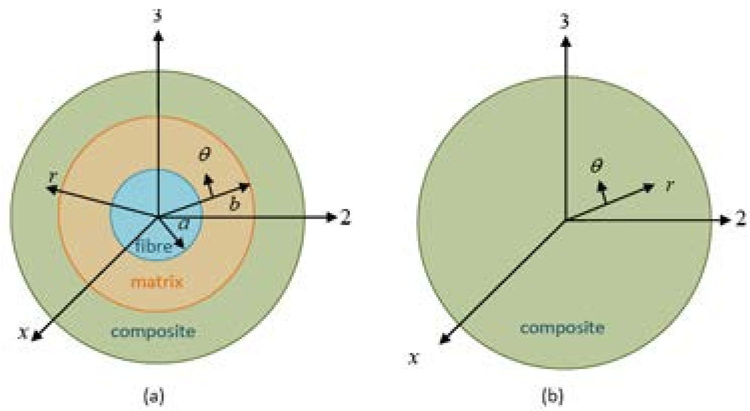

Figure 7.12: (a) Three phase concentric cylinder assemblage model (b) homogeneous single cylinder

Previous Next

The state of deformation imposed is such that the far away from the fibre and matrix a state of pure shear is produced.

The planar stresses for the cylinder can be given in terms of stress function ϕ as

$$\begin{aligned}\sigma_{\theta\theta} &= \frac{\partial^2\phi}{\partial r^2}\\ \sigma_{rr} &= \frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\\ \tau_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\end{aligned} \tag{7.254}$$

where, the stress function ϕ, following the work of [11], is given as

$$\phi = \left(-\frac{r^2}{2}+a_3\frac{b^2}{2}+c_3\frac{b^4}{4r^4}\right)\cos 2\theta \tag{7.255}$$

Here, a_3 and c_3 are the constants that are evaluated using the boundary conditions. Using Equation (7.255) in Equation (7.254), the stresses become

$$\begin{aligned}\sigma_{\theta\theta} &= \left(-1+\frac{3}{2}c_3\frac{b^4}{r^4}\right)\cos 2\theta\\ \sigma_{rr} &= \left(1-2a_3\frac{b^2}{r^2}-\frac{3}{2}c_3\frac{b^4}{r^4}\right)\cos 2\theta\\ \tau_{r\theta} &= -\left(1+a_3\frac{b^2}{r^2}+\frac{3}{2}c_3\frac{b^4}{r^4}\right)\sin 2\theta\end{aligned} \tag{7.256}$$

Now, considering the plane strain condition with $\varepsilon_{xx}=0$ and using the stress-strain relations, the strains in composite material are written as

$$\begin{aligned}\varepsilon_{\theta\theta} &= \frac{b}{4G_{23}}\left[-\frac{2}{b}-(\eta-3)a_3\frac{b}{r^2}+3c_3\frac{b^3}{r^4}\right]\cos 2\theta\\ \varepsilon_{rr} &= \frac{b}{4G_{23}}\left[\frac{2}{b}-(\eta+1)a_3\frac{b}{r^2}-3c_3\frac{b^3}{r^4}\right]\cos 2\theta\\ \gamma_{r\theta} &= -\frac{b}{4G_{23}}\left[\frac{4}{b}+4a_3\frac{b}{r^2}+6c_3\frac{b^3}{r^4}\right]\sin 2\theta\end{aligned} \tag{7.257}$$

where, $\eta = 3-4v_{23}$. Now using the strain displacement relations as in Equation (7.172) the displacement components in composite material are obtained as

(7.258)

$$u^{(c)} = 0$$
$$v^{(c)} = \frac{b}{4G_{23}}\left[-\frac{2r}{b} - (\eta - 1)a_3\frac{b}{r} + c_3\frac{b^3}{r^3}\right]\sin 2\theta$$
$$w^{(c)} = \frac{b}{4G_{23}}\left[\frac{2r}{b} + (\eta + 1)a_3\frac{b}{r} + c_3\frac{b^3}{r^3}\right]\cos 2\theta$$

Here, the polar coordinates are used. Further, as $r \to \infty$ the above equation leads to the imposed state of simple shear deformation. It should be noted that the displacements in this equation satisfy the equilibrium equations.

Previous Next

In a similar way, we can derive the displacement field in matrix phase. It is given as

$$\begin{aligned} v^{(m)} &= \frac{b}{4G^{(m)}}\left[\left(\eta^{(m)}+3\right)a_2\frac{r^3}{b^3}-d_2\frac{r}{b}-\left(\eta^{(m)}-1\right)c_2\frac{b}{r}+b_2\frac{b^3}{r^3}\right]\sin 2\theta \\ w^{(m)} &= \frac{b}{4G^{(m)}}\left[\left(\eta^{(m)}-3\right)a_2\frac{r^3}{b^3}+d_2\frac{r}{b}+\left(\eta^{(m)}+1\right)c_2\frac{b}{r}+b_2\frac{b^3}{r^3}\right]\cos 2\theta \end{aligned} \qquad (7.259)$$

The displacement in fibre phase is given as

$$\begin{aligned} v^{(f)} &= \frac{b}{4G_{23}^{(f)}}\left[\left(\eta^{(f)}+3\right)a_1\frac{r^3}{b^3}-d_1\frac{r}{b}\right]\sin 2\theta \\ w^{(f)} &= \frac{b}{4G_{23}^{(f)}}\left[\left(\eta^{(f)}-3\right)a_1\frac{r^3}{b^3}+d_1\frac{r}{b}\right]\cos 2\theta \end{aligned} \qquad (7.260)$$

where,

$$\eta^{(m)} = 3-4v^{(m)} \text{ and } \eta^{(f)} = 3-4v_{23}^{(f)} \qquad (7.261)$$

There are eight unknowns to be determined. These are $a_1, a_2, a_3, b_2, c_2, c_3, d_1$ and d_2. Now, we need to develop eight equations in these unknowns to solve for these unknowns. These equations can be developed using the continuity of displacements and stress at the fibre and matrix interface and that between the matrix and the equivalent homogeneous material.

Previous Next

The displacement continuity at $r = a$ gives the relations

$$\begin{aligned}
&a_2\left[G_{23}^{(f)}\left(\eta^{(m)}-3\right)-G^{(m)}\left(\eta^{(f)}-3\right)\right]+c_2\left(\frac{b}{a}\right)^4\left[G_{23}^{(f)}\left(\eta^{(m)}+1\right)-G^{(m)}\left(\eta^{(f)}+1\right)\right]\\
&+d_2\left(\frac{b}{a}\right)^2\left(G_{23}^{(f)}-G^{(m)}\right)+b_2\left(\frac{b}{a}\right)^6\left(G_{23}^{(f)}+G^{(m)}\eta^{(f)}\right)=0\\
&a_2\left[G_{23}^{(f)}\left(\eta^{(m)}+3\right)-G^{(m)}\left(\eta^{(f)}+3\right)\right]-c_2\left(\frac{b}{a}\right)^4\left[G_{23}^{(f)}\left(\eta^{(m)}-1\right)-G^{(m)}\left(\eta^{(f)}-1\right)\right]\\
&-d_2\left(\frac{b}{a}\right)^2\left(G_{23}^{(f)}-G^{(m)}\right)+b_2\left(\frac{b}{a}\right)^6\left(G_{23}^{(f)}+G^{(m)}\eta^{(f)}\right)=0
\end{aligned} \tag{7.262}$$

The displacement continuity at $r = b$ gives

$$\begin{aligned}
G^{(m)}\left[2+a_3(\eta+1)+c_3\right]-G_{23}^{*}\left[a_2\left(\eta^{(m)}-3\right)+d_2+c_2\left(\eta^{(m)}+1\right)+b_2\right]=0\\
G^{(m)}\left[-2-a_3(\eta+1)+c_3\right]-G_{23}^{*}\left[a_2\left(\eta^{(m)}+3\right)-d_2-c_2\left(\eta^{(m)}-1\right)+b_2\right]=0
\end{aligned} \tag{7.263}$$

The stress continuity at $r = a$ gives the following relations

$$\begin{aligned}
&d_1-d_2+4c_2\left(\frac{b}{a}\right)^2+3b_2\left(\frac{b}{a}\right)^4=0\\
&a_1-a_2+c_2\left(\frac{b}{a}\right)^4+b_2\left(\frac{b}{a}\right)^6=0
\end{aligned} \tag{7.264}$$

And the stress continuity at $r = b$ leads to

$$\begin{aligned}
&2-4a_3-3c_3-d_2+4c_2+3b_2=0\\
&2+2a_3+3c_3+6a_2-d_2-2c_2-3b_2=0
\end{aligned} \tag{7.265}$$

Now the effective properties are determined from the equivalence of strain energy in heterogeneous media and in the equivalent homogeneous media. Thus, we write this condition as

$$U = U_{\text{Equivalent Homogeneous}} \tag{7.266}$$

The strain energies in the above equation can be given in terms of Eshelby formula.

$$U = U^{\circ} - \frac{1}{2}\int_S \left(T_i^{\circ}u_i - T_i u_i^{\circ}\right)ds \tag{7.267}$$

where U is the strain energy of a homogeneous medium containing an inclusion under the conditions of applied displacements. S is the surface of the inclusion and U° is the strain energy in the same medium when the medium does not contain any inclusion. Further, T_i° and u_i° are the tractions and displacements on the surface when the medium has no inclusion. and are the tractions and displacements on the surface when the medium has inclusion. It should be noted that the context of three phase cylinder model the inclusion here it means the concentric cylinders. Thus, U refers to the strain energy of the Figure 7.12(a). U° refers to the strain energy of the Figure 7.12(b) when the inclusion is replaced by the equivalent homogeneous material outside. However, this energy is $U_{\text{Equivalent Homogeneous}}$.

Thus, we can write

$$U^{\circ} = U_{\text{Equivalent Homogeneous}} \tag{7.268}$$

Thus, from Equation (7.266) and Equation (7.268) we can write that

$$\int_S \left(T_i^{\circ} u_i - T_i u_i^{\circ}\right) dS = 0 \tag{7.269}$$

Using the definitions of stresses and displacements this equation is written as

$$\int_0^{2\pi} \left[\sigma_{rr}^{\circ} w + \tau_{r\theta}^{\circ} v - \sigma_{rr} w^{\circ} - \tau_{r\theta} v^{\circ}\right]_{r=b} b d\theta = 0 \tag{7.270}$$

The various quantities in above equation are given as

$$\begin{aligned} v^{\circ} &= -\frac{r}{2G_{23}^*} \sin 2\theta \\ w^{\circ} &= \frac{r}{2G_{23}^*} \cos 2\theta \end{aligned} \tag{7.271}$$

This expression is derived from the condition that the displacements must be bounded at $r = 0$. Thus, from Equation (7.256), $a_3 = c_3 = 0$. Thus, we can write

$$\begin{aligned} v &= -\frac{r}{2G_{23}} \sin 2\theta \\ w &= \frac{r}{2G_{23}} \cos 2\theta \end{aligned} \tag{7.272}$$

Similarly, with $a_3 = c_3 = 0$ in Equation (7.256), the stresses become

$$\begin{aligned} \sigma_{rr}^{\circ} &= \cos 2\theta \\ \tau_{r\theta}^{\circ} &= -\sin 2\theta \end{aligned} \tag{7.273}$$

The stresses at $r = b$ can be given again using Equation (7.256) as

$$\begin{aligned} \sigma_{rr} &= \left(1 - 2a_3 - \frac{3}{2}c_3\right) \cos 2\theta \\ \tau_{r\theta} &= -\left(1 + a_3 + \frac{3}{2}c_3\right) \sin 2\theta \end{aligned} \tag{7.274}$$

Finally, v and w are evaluated at $r = b$ from Equation (7.258) and are given as

$$\begin{aligned} v &= \frac{b}{4G_{23}^*}\left[-2 - a_3(\eta - 1) + c_3\right] \sin 2\theta \\ w &= \frac{b}{4G_{23}^*}\left[2 + a_3(\eta + 1) + c_3\right] \cos 2\theta \end{aligned} \tag{7.275}$$

Previous Next

Now using all the required terms in Equation (7.270) and carrying out the integration, we get

$$a_3 = 0 \qquad (7.276)$$

The solution for obtained by solving eight simultaneous equations is thus set equal to zero. This leads to the following equation.

$$A\left(\frac{G_{23}^*}{G^{(m)}}\right)^2 + B\left(\frac{G_{23}^*}{G^{(m)}}\right) + D = 0 \qquad (7.277)$$

where

$$\begin{aligned} A &= 3V_f\left(1-V_f^2\right)\left(\frac{G_{23}^{(f)}}{G^{(m)}}-1\right)\left(\frac{G_{23}^{(f)}}{G^{(m)}}+\eta^{(f)}\right) \\ &+\left[\frac{G_{23}^{(f)}}{G^{(m)}}\eta^{(m)}+\eta^{(f)}\eta^{(m)}-V_f^3\left(\frac{G_{23}^{(f)}}{G^{(m)}}\eta^{(m)}-\eta^{(f)}\right)\right]\left[\eta^{(m)}V_f\left(\frac{G_{23}^{(f)}}{G^{(m)}}-1\right)-\left(\frac{G_{23}^{(f)}}{G^{(m)}}\eta^{(m)}+1\right)\right] \end{aligned} \qquad (7.278)$$

$$\begin{aligned} B &= -6V_f\left(1-V_f^2\right)\left(\frac{G_{23}^{(f)}}{G^{(m)}}-1\right)\left(\frac{G_{23}^{(f)}}{G^{(m)}}+\eta^{(f)}\right) \\ &+\left[\frac{G_{23}^{(f)}}{G^{(m)}}\eta^{(m)}+\left(\frac{G_{23}^{(f)}}{G^{(m)}}-1\right)V_f+1\right]\left[\left(\eta^{(m)}-1\right)\left(\frac{G_{23}^{(f)}}{G^{(m)}}+\eta^{(f)}\right)-2V_f^3\left(\frac{G_{23}^{(f)}}{G^{(m)}}\eta^{(m)}-\eta^{(f)}\right)\right] \\ &+V^{(f)}\left(\frac{G_{23}^{(f)}}{G^{(m)}}-1\right)\left(\eta^{(m)}+1\right)\left[\frac{G_{23}^{(f)}}{G^{(m)}}+\eta^{(f)}+V_f^3\left(\frac{G_{23}^{(f)}}{G^{(m)}}\eta^{(m)}-\eta^{(f)}\right)\right] \end{aligned} \qquad (7.279)$$

and

$$\begin{aligned} D &= 3V_f\left(1-V_f^2\right)\left(\frac{G_{23}^{(f)}}{G^{(m)}}-1\right)\left(\frac{G_{23}^{(f)}}{G^{(m)}}+\eta^{(f)}\right) \\ &+\left[\frac{G_{23}^{(f)}}{G^{(m)}}\eta^{(m)}+V_f\left(\frac{G_{23}^{(f)}}{G^{(m)}}-1\right)+1\right]\left[\frac{G_{23}^{(f)}}{G^{(m)}}+\eta^{(f)}+V_f^3\left(\frac{G_{23}^{(f)}}{G^{(m)}}\eta^{(m)}-\eta^{(f)}\right)\right] \end{aligned} \qquad (7.280)$$

Previous Next

Further, in the case of dilute suspension, that is, a single inclusion or fibre in an infinite equivalent homogeneous medium (that is, a condition leading to a low fibre volume fraction) the following relation is given

$$\frac{G_{23}^*}{G_m} = 1 + \frac{V_f}{\frac{G_m}{\left(G_{23}^{(f)} - G_m\right)} + \frac{\left(k_m + \frac{7}{3}G_m\right)}{\left(2k_m + \frac{8}{3}G_m\right)}} \tag{7.281}$$

where k_m is the bulk modulus of the matrix relating hydrostatic stress to the change in volume. The bulk modulus relates the hydrostatic stress and the change in volume as

$$\sigma_{kk} = 3k_m \varepsilon_{kk} \tag{7.282}$$

It should be noted from Equation (7.281) that there is no correlation with rule of mixture for transverse shear modulus.

Note: The remaining effective properties like $E_2^*, \upsilon_{23}^*, \upsilon_{12}^*$ and υ_{21}^* can be obtained from the relations given in Equation (7.167).

Previous Next

Module 7: Micromechanics

Lecture 33: Three Phase CCA Model: Effective Transverse Shear Modulus

Home Work:

1. What is a three-phase cylinder model? Why is it required?
2. Write a short note on determination of transverse shear modulus using three phase cylinder model.

References

- *Z Hashin, BW Rosen. The elastic moduli of fibre-reinforced materials. J. Appl. Mech. 1964, Vol. 31, pp. 223-232.*
- *Z Hashin. Analysis of properties of fiber composites with anisotropic constituents. J. Appl. Mech. 1979, Vol. 46, pp. 543-550.*
- *RM Christensen. Mechanics of Composite Materials. Krieger Publishing Company, Florida, 1991.*
- *Z. Hashin: On elastic behaviour of fibre reinforced materials of arbitrary transverse phase geometry. J. Mech. Phys. Solids, 1965, Vol. 13, pp. 119-134.*
- *Z. Hashin: Analysis of composite materials-a survey. J. Mech. Phys. Solids, 1983, Vol. 50, pp. 481-505.*
- *YC Fung. Foundations of Solid Mechanics. Prentice Hall International, Inc., 1965.*
- *PC Chou, NJ Pagano. Elasticity: Tensor, Dyadic and Engineering Approaches, General Publishing Company, Ltd. Canada, 1967.*
- *CT Herakovich. Mechanics of Fibrous Composites, John Wiley & Sons, Inc. New York, 1998.*
- *MW Hyer, AM Waas. Micromechanics of linear elastic continuous fibre composites, in Comprehensive Composite Materials. Vol. 1: Fiber Reinforcements and General Theory of Composites, Eds. A Kelly, C Zweben. Elsevier, 2000.*
- *RM Christensen, KH Lo. Solutions for effective shear properties in three phase sphere and cylinder models. J. Appl. Mech. Phys. Solids, 1979, Vol. 27(4), pp. 315-330.*
- *GN Savin. Stress Concentration Around Holes, Pergamon Press, New York, 1961.*
- *Z Hashin. Viscoelastic fiber reinforced materials. AIAA J, 1966, Vol. 4, pp. 1411-1417.*
- *Z Hashin, S Shtrikman. A variational approach to the theory of elastic behaviour of multiphase materials. J. Mech. Phys. Solids, 1963, Vol. 11, pp. 127-140.*

Previous Next