Introduction

In this lecture we will introduce some more micromechanical methods to predict the effective properties of the composite. Here we will introduce expressions for the effective properties without the detailed derivations.

In the present lecture we will study the self consistent method, Mori-Tanaka method and some relations based on semi-empirical method introduced by Halpin-Tsai.

The Lecture Contains

- Self Consistent Method
- Mori-Tanaka Method
- Halpin-Tsai Semi-Empirical Relations
- Homework
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Self Consistent Method

The self consistent method is based on the solution to an auxiliary inclusion problem where a single ellipsoidal inclusion is embedded in an infinite medium. In this system it is assumed that the bond between inclusion and the infinite medium is perfect. Therefore, there is displacement and traction continuity across the interface of the two phases. One can determine the stresses and strains by applying uniform stresses or strains to the system at infinity. It was shown by Eshelby [1] that in these types of problems, the stress and strain fields in the inclusion are uniform. Further, the elastic properties can be determined by finding the relation between far-field stresses and strains in the homogeneous medium and stresses and strains in the inclusion, or the stress or strain concentration factors.

The problem of determining the effective properties of such a system was dealt in depth by Hill [2] and Budiansky [3]. In this approach the average stress and strain fields in the fiber are taken to be equal to those in the inclusion problem. Further, the infinite medium is taken to be homogeneous with the same properties of the composite.

For fibrous composites, which are transversely isotropic in nature with both fibre and matrix phases also transversely isotropic, the self consistent estimates of Hill [2] of the overall moduli give the following relations.

$$\begin{aligned}
&\frac{V_f k_f}{k_f + m^*} + \frac{V_m k_m}{k_m + m^*} = 2\left[\frac{V_f m_m}{m_m - m^*} + \frac{V_m m_f}{m_f - m^*}\right] \\
&\frac{1}{2P^*} = \frac{V_f}{P^* - P_m} + \frac{V_m}{P^* - P_f} \\
&\frac{1}{k^* + m^*} = \frac{V_f}{k_f + m^*} + \frac{V_m}{k_m + m^*}
\end{aligned} \qquad (7.283)$$

It was shown by Hill [4] that regardless of the method used to obtain the estimates, only three of the five overall moduli of such composites are actually independent. Then the moduli k^*, l^* and n^* are related through so called universal relations in terms of overall moduli and phase moduli and their volume fractions as,

$$\frac{k^* - k_f}{l^* - l_f} = \frac{k^* - k_m}{l^* - l_m} = \frac{l^* - V_f l_f - V_m l_m}{n^* - V_f n_f - V_m n_m} = \frac{k_f - k_m}{l_f - l_m} \qquad (7.284)$$

Therefore, only one of the three moduli is independent. This fact is clear from the relation in Equation (7.283). This equation gives a cubic equation in m^* and quadratic equations for k^* and p^*, again, in terms of m^*. Hence, if k^* is known, then l^* and n^* can easily be obtained from Eq. (7.284).

Note: When one or both phases are isotropic, then there are only two independent moduli in such phase. Further, one can write then k^*, l^*, m^*, n^* and p^* in terms of engineering constants.

Mori-Tanaka Method

The original method was proposed by Mori and Tanaka [5] in 1973. Further, Benveniste [6] proposed a simpler version of the same model. The key assumption in this model is that the average strain in the inclusion, that is fibre, is related to the average strain in the matrix by a fourth order tensor. This fourth order tensor gives the relation between the uniform strain in the inclusion embedded in an all matrix material. Further, this material is subjected to uniform strain at infinity.

The strain concentration factors in fibre are given as

$$A_{ijkl}^{(f)} = T_{ijrs}\left[V_f T_{rskl} + V_m I_{rskl}\right]^{-1} \quad (7.285)$$

where,

$$T_{ijkl} = \left[S_{ijmn}\left(C_{mnrs}^{(f)}\right)^{-1}\left(C_{rskl}^{(f)} - C_{rskl}^{(m)}\right) + I_{ijkl}\right]^{-1} \quad (7.286)$$

Here, T_{ijkl} is the fourth order tensor which relates the average strain in the inclusion to the average strain in the matrix. S_{ijkl} is Eshelby's tensor, $C_{ijkl}^{(f)}$ and $C_{ijkl}^{(m)}$ are the stiffness tensors of fibre and matrix materials, respectively. Dvorak et al [7-9] have given the explicit relations in terms of Hill's moduli as

$$\begin{aligned}
k &= \frac{V_f k_f (k_m + m_m) + V_m k_m (k_f + m_m)}{V_f (k_m + m_m) + V_m (k_f + m_m)} \\
l &= \frac{V_f l_f (k_m + m_m) + V_m l_m (k_f + m_m)}{V_f (k_m + m_m) + V_m (k_f + m_m)} \\
n &= V_m n_m + V_f n_f + \left(1 - V_f l_f - V_m l_m\right)\left(\frac{l_f - l_m}{k_f - m_m}\right) \\
m &= \frac{m_f m_m (k_m + 2m_m) + k_m m_m (V_f m_f + V_m m_m)}{k_m m_m + (k_m + 2m_m)(V_f m_m + V_m m_f)} \\
P &= \frac{2V_f p_m p_f + V_m\left(p_m p_f + p_m^2\right)}{2V_f p_m + V_m\left(p_f + p_m\right)}
\end{aligned} \quad (7.287)$$

Halpin-Tsai Semi-Empirical Relations

A set of semi-empirical relations have been developed by Halpin and Tsai for easy design procedure. These relations were developed by curve fitting to the results that are based on elasticity. These relations are called semi-empirical relations because the parameters involved in these relations have some physical significance. In the following we give these relations.

The longitudinal Young's modulus is same as given by rule of mixtures using strength of materials approach. Thus,

$$E_1^* = E_1^{(f)} V_f + E^{(m)} V_m \tag{7.288}$$

Further, the axial Poisson's ratio is the same as given by the rule of mixtures using strength of materials approach. Thus, the axial Poisson's ratio is

$$v_{12}^* = \nu_{12}^{(f)} V_f + \nu^{(m)} V_m \tag{7.289}$$

The Hill's Moduli are given as

$$\frac{M^*}{M^{(m)}} = \frac{1 + \zeta \eta V_f}{1 - \eta V_f} \tag{7.290}$$

where

$$\eta = \frac{\dfrac{M^{(f)}}{M^{(m)}} - 1}{\dfrac{M^{(f)}}{M^{(m)}} + \zeta} \tag{7.291}$$

Further, M^* stands for k^*, m^* or G_{12}^* and $M^{(f)}$ and $M^{(m)}$ stands for corresponding values for fibre and matrix, respectively. Here, ζ is a measure of reinforcement geometry which depends upon loading conditions and geometries of the inclusion, that is, fibre.

The parameters used in the Equation (7.290) have physical significance. The limiting values give following significant information:

1. $\eta = 1$ corresponds to a situation that the inclusions are rigid.
2. $\eta = 0$ corresponds to a homogeneous material
3. And for voids , $\frac{M^{(f)}}{M^{(m)}} = 0$, then . $\eta = -\frac{1}{\zeta}$

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The limiting values of ζ are given as follows:

- $\zeta = 0$. For this value, Equation (7.290) becomes

$$\frac{M^*}{M^{(m)}} = \frac{1}{1-\eta V_f} = \frac{1}{M^{(m)}\left(\frac{V_f}{M^{(f)}} + \frac{V_m}{M^{(m)}}\right)} \tag{7.292}$$

This gives,

$$\frac{1}{M^*} = \frac{V_f}{M^{(f)}} + \frac{V_m}{M^{(m)}} \tag{7.293}$$

It is easy to see that this is a series connected model which gives the lower bound of a composite modulus.

$\zeta = \infty, \eta = 0$. For this case, we get

$$\zeta\eta = \frac{M^{(f)}}{M^{(m)}} . 1 \tag{7.294}$$

Thus, Equation (7.290) becomes

$$\frac{M^*}{M^{(m)}} = V_f M^{(f)} + V_m M^{(m)} \tag{7.295}$$

This is the parallel connected model. This gives the upper bound of a composite modulus. Thus, ζ is regarded as a reinforcement measure. This factor covers all possible range of the composite moduli as it varies from zero to infinity. Once this factor is known, the composite moduli are determined from the generalized formula.

For example, in case of $M^* = E_2^*$ for a circular fibres in a square array, $\zeta = 2$ and for rectangular fibres cross section of length a and width b in a hexagonal array, $\zeta = 2\frac{a}{b}$ where b is in the direction of loading. Similarly, for $M^* = G_{12}^*$ for circular fibres in a square array $\zeta = 1$, and for rectangular cross-section with length a and width b in a hexagonal array, $\zeta = \sqrt{3}\log_e\left(\frac{a}{b}\right)$, where a is in the loading direction.

Note: In case of transversely isotropic material in 23 plane, the constitutive relations are given as

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_L} & \frac{-\nu_L}{E_L} & \frac{-\nu_L}{E_L} & 0 & 0 & 0 \\ & \frac{1}{E_T} & \frac{-\nu_T}{E_T} & 0 & 0 & 0 \\ & & \frac{1}{E_T} & 0 & 0 & 0 \\ & & & \frac{1}{G_T} & 0 & 0 \\ & & & & \frac{1}{G_L} & 0 \\ Sym & & & & & \frac{1}{G_L} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} \qquad (7.296)$$

and

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} n & l & l & 0 & 0 & 0 \\ & (k+m) & (k-m) & 0 & 0 & 0 \\ & & (k+m) & 0 & 0 & 0 \\ & & & m & 0 & 0 \\ & & & & p & 0 \\ Sym & & & & & p \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} \qquad (7.297)$$

Here, the moduli $E_L(=E_1)$ and $G_L(=G_{12})$ refer to the values in the longitudinal or axial direction of straining and $E_T(=E_2)$ and $G_T(=G_{23})$ refer to the values in transverse plane. Further, the Poisson's ratios are defined as $\nu_L = \nu_{12} = \frac{-\varepsilon_{22}}{\varepsilon_{11}}$ and $\nu_T = \nu_{23} = \frac{-\varepsilon_{33}}{\varepsilon_{22}}$ under the uniaxial tensions in 1 and 2 directions, respectively.

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The relationship between the Hill's moduli, $k, l, n, m,$ and p and the engineering moduli are given as

$$\begin{aligned} k &= \left[\frac{1}{G_T} - \frac{4}{E_T} + \frac{4\nu_L^2}{E_L}\right]^{-1} \\ l &= 2k\nu_L \\ n &= E_L + 4k\nu_L^2 \\ m &= G_T \\ p &= G_L \end{aligned} \tag{7.298}$$

Some additional useful relations are

$$\begin{aligned} E_T &= 2(1+\nu_T)G_T = \frac{4km}{(k+qm)} \\ \nu_T &= \frac{(k-qm)}{(k+qm)} \\ q &= 1 + \frac{4k\nu_L^2}{E_L} \end{aligned} \tag{7.299}$$

Further, if the phase is isotropic then with bulk modulus k and shear modulus G, we have

$$\begin{aligned} k &= \frac{G}{1-2\nu} \quad l = k - \frac{2G}{3} \\ n &= k + \frac{4G}{3} \quad m = p = G \end{aligned} \tag{7.300}$$

Module 7: Micromechanics

Lecture 34: Self Consistent, Mori -Tanaka and Halpin -Tsai Models

Home Work:

1. What is meant by self consistent method?
2. Write a short note on Mori-Tanaka method.
3. Write a short note on Halpin-Tsai semi-empirical models.

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