Module 4: Plane Stress Constitutive Equations

Lecture 14: 2-Dimensional Lamina Analysis

**The Lecture Contains**:

## Introduction

In the previous chapter, we have developed 3D constitutive equations. While analyzing composites, most of the times a planar state of stress actually exists. It is noted that a typical unidirectional lamina has very small thickness compared to its planar (*xy*) dimensions. Thus, it is appropriate to assume a planar state of stress in a lamina. In this chapter, we are going to derive a constitutive equation for plane stress problem in unidirectional laminar composite.

## Plane Stress for Monoclinic (or Rotated Orthotropic) Material

3*D* constitutive equation for a single layer of a unidirectional composite with a fiber orientation $\theta$ relative to the global coordinate is

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & 0 & 0 & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{23} & 0 & 0 & \bar{S}_{26} \\ \bar{S}_{13} & \bar{S}_{23} & \bar{S}_{33} & 0 & 0 & \bar{S}_{36} \\ 0 & 0 & 0 & \bar{S}_{44} & \bar{S}_{45} & 0 \\ 0 & 0 & 0 & \bar{S}_{45} & \bar{S}_{55} & 0 \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{36} & 0 & 0 & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} \quad (4.1)$$

For a state of plane stress, we have

$$\sigma_{zz} = \tau_{yz} = \tau_{xz} = 0 \quad (4.2)$$

Thus, it is easy to see that the two out of plane shear strains $\gamma_{yz} = \gamma_{xz} = 0$ are zero. We can write these strains using Equation (4.1) as

$$\begin{aligned} \gamma_{yz} &= \bar{S}_{44}\tau_{yz} + \bar{S}_{45}\tau_{xz} = 0 \\ \gamma_{xz} &= \bar{S}_{45}\tau_{yz} + \bar{S}_{55}\tau_{xz} = 0 \end{aligned} \quad (4.3)$$

The out of plane normal strain $\varepsilon_{zz}$ is expressed using Equation (4.1) and Equation (4.2) as

$$\varepsilon_{zz} = \bar{S}_{13}\,\sigma_{xx} + \bar{S}_{23}\,\sigma_{yy} + \bar{S}_{36}\,\tau_{xy} \quad (4.4)$$

**Note that this strain component is not zero.**

In plane components of strain for a plane stress state can be written using Equation (4.1) as

(4.5)

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}$$

From 3*D* constitutive equation (Equation (3.72)) for the transformed stiffness, we can write $\sigma_{zz}$ as

$$\sigma_{zz} = \bar{C}_{13}\,\varepsilon_{xx} + \bar{C}_{23}\,\varepsilon_{yy} + \bar{C}_{33}\,\varepsilon_{zz} + \bar{C}_{36}\,\gamma_{xy} = 0 \tag{4.6}$$

From this equation, we can get the out of plane transverse normal strain as

$$\varepsilon_{zz} = -\frac{\bar{C}_{13}\,\varepsilon_{xx} + \bar{C}_{23}\,\varepsilon_{yy} + \bar{C}_{36}\,\gamma_{xy}}{\bar{C}_{33}} \tag{4.7}$$

Thus, the out of plane normal strain is expressed in terms of in-plane strain components and known stiffness coefficients.

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## Reduced Transformed Stiffness Matrix

The equation for in-plane components of stress in terms of the transformed stiffness coefficients is

$$\begin{aligned}\sigma_{xx} &= \bar{C}_{11}\,\varepsilon_{xx} + \bar{C}_{12}\,\varepsilon_{yy} + \bar{C}_{13}\,\varepsilon_{zz} + \bar{C}_{16}\,\gamma_{xy} \\ \sigma_{yy} &= \bar{C}_{12}\,\varepsilon_{xx} + \bar{C}_{22}\,\varepsilon_{yy} + \bar{C}_{23}\,\varepsilon_{zz} + \bar{C}_{26}\,\gamma_{xy} \\ \tau_{xy} &= \bar{C}_{16}\,\varepsilon_{xx} + \bar{C}_{26}\,\varepsilon_{yy} + \bar{C}_{36}\,\varepsilon_{zz} + \bar{C}_{66}\,\gamma_{xy}\end{aligned} \tag{4.8}$$

Substituting $\varepsilon_{zz}$ from Equation (4.7) into Equation (4.8) and upon simplification, we get

$$\begin{aligned}\sigma_{xx} &= \left(\bar{C}_{11} - \frac{\bar{C}_{13}\bar{C}_{13}}{\bar{C}_{33}}\right)\varepsilon_{xx} + \left(\bar{C}_{12} - \frac{\bar{C}_{13}\bar{C}_{23}}{\bar{C}_{33}}\right)\varepsilon_{yy} + \left(\bar{C}_{16} - \frac{\bar{C}_{13}\bar{C}_{36}}{\bar{C}_{33}}\right)\gamma_{xy} \\ \sigma_{yy} &= \left(\bar{C}_{12} - \frac{\bar{C}_{13}\bar{C}_{23}}{\bar{C}_{33}}\right)\varepsilon_{xx} + \left(\bar{C}_{22} - \frac{\bar{C}_{23}\bar{C}_{23}}{\bar{C}_{33}}\right)\varepsilon_{yy} + \left(\bar{C}_{26} - \frac{\bar{C}_{23}\bar{C}_{36}}{\bar{C}_{33}}\right)\gamma_{xy} \\ \tau_{xy} &= \left(\bar{C}_{16} - \frac{\bar{C}_{13}\bar{C}_{36}}{\bar{C}_{33}}\right)\varepsilon_{xx} + \left(\bar{C}_{26} - \frac{\bar{C}_{23}\bar{C}_{36}}{\bar{C}_{33}}\right)\varepsilon_{yy} + \left(\bar{C}_{66} - \frac{\bar{C}_{36}\bar{C}_{36}}{\bar{C}_{33}}\right)\gamma_{xy}\end{aligned} \tag{4.9}$$

The above equation is written in matrix form as

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{Bmatrix} = \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26}\\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix} \tag{4.10}$$

where the transformed reduced stiffness coefficients $\bar{Q}_{ij}$ are defined as

$$\bar{Q}_{ij} = \bar{C}_{ij} - \frac{\bar{C}_{i3}\bar{C}_{3j}}{\bar{C}_{33}} \qquad i, j = 1, 2, 6 \tag{4.11}$$

***Note***: Transformed reduced stiffness matrix is symmetric.

***Note***: It is very important to note that the transformed reduced stiffness terms for plane stress, $\bar{Q}_{ij}$, are not simply the corresponding terms, $\bar{C}_{ij}$ taken from the 3*D* stiffness matrix. This should be clear from the fact that the inverse of a $3\times 3$ matrix is different from that of a $6\times 6$. This can easily be seen from Equation (4.11). The readers should easily understand that when $\bar{Q}_{ij}$ terms are used to define a constitutive equation, then it is a reduced transformed constitutive equation.

## Plane Stress for Orthotropic Material

Let us recall the constitutive equation for orthotropic material in principal directions.We can write the constitutive equation using compliance matrix as (Equation (3.45))

$$\overline{Q}_{ij} = \overline{C}_{ij} - \frac{\overline{C}_{i3}\,\overline{C}_{3j}}{\overline{C}_{33}} \qquad i, j = 1, 2, 6$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ & & & S_{44} & 0 & 0 \\ & symmetric & & & S_{55} & 0 \\ & & & & & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} \quad (4.12)$$

We have planar state of stress for orthotropic lamina. Then we have out of plane transverse stress components zero, that is,

$$\sigma_{33} = \tau_{23} = \tau_{13} = 0 \qquad (4.13)$$

Let us write the out of plane transverse shear strains using this information and Equation (4.12) as

$$\begin{aligned} \gamma_{23} &= S_{44}\,\tau_{23} = 0 \\ \gamma_{13} &= S_{55}\,\tau_{13} = 0 \end{aligned} \qquad (4.14)$$

Thus, the out of plane transverse shear strains are zero. Now, let us write the out of plane transverse normal strain using Equation (4.12) as

$$\varepsilon_{33} = S_{13}\,\sigma_{11} + S_{23}\,\sigma_{22} + S_{33}\,\sigma_{33} \tag{4.15}$$

Using and Equation (4.13) in the above equation, we get

$$\varepsilon_{33} = S_{13}\,\sigma_{11} + S_{23}\,\sigma_{22} \tag{4.16}$$

and the inplane strain components are given as

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} \tag{4.17}$$

This equation is called reduced constitutive equation using compliance matrix.

We have the 3*D* constitutive equation using stiffness matrix in principal material directions as (Equation . (3.26))

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & symmetric & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} \quad (4.18$$

We have from the condition of plane stress problem that $\sigma_{33} = 0$. Thus, using Equation. (4.18), we can write

$$\sigma_{33} = C_{13}\,\varepsilon_{11} + C_{23}\,\varepsilon_{22} + C_{33}\,\varepsilon_{33} = 0 \tag{4.19}$$

This leads to non-zero transverse normal strain $\varepsilon_{33}$ as

$$\varepsilon_{33} = -\frac{C_{13}\,\varepsilon_{11} + C_{23}\,\varepsilon_{22}}{C_{33}} \tag{4.20}$$

Using Equation. (4.18), we can write the inplane stress components as

$$\begin{aligned}\sigma_{11} &= C_{11}\,\varepsilon_{11} + C_{12}\,\varepsilon_{22} + C_{13}\,\varepsilon_{33} \\ \sigma_{22} &= C_{12}\,\varepsilon_{11} + C_{22}\,\varepsilon_{22} + C_{23}\,\varepsilon_{33} \\ \tau_{12} &= C_{66}\,\gamma_{12}\end{aligned} \tag{4.21}$$

Putting the expression for $\varepsilon_{33}$ from Equation. (4.20) in above equation, we get

$$\begin{aligned}\sigma_{11} &= \left(C_{11} - \frac{C_{13}\,C_{13}}{C_{33}}\right)\varepsilon_{11} + \left(C_{12} - \frac{C_{13}\,C_{23}}{C_{33}}\right)\varepsilon_{22} \\ \sigma_{22} &= \left(C_{12} - \frac{C_{13}\,C_{23}}{C_{33}}\right)\varepsilon_{11} + \left(C_{22} - \frac{C_{23}\,C_{23}}{C_{33}}\right)\varepsilon_{22} \\ \tau_{12} &= C_{66}\,\gamma_{12}\end{aligned} \tag{4.22}$$

This equation is written in matrix form as

$$\begin{Bmatrix}\sigma_{11} \\ \sigma_{22} \\ \tau_{12}\end{Bmatrix} = \begin{bmatrix}Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12}\end{Bmatrix} \tag{4.23}$$

where, the $Q_{ij}$ terms can be written using index notations as follows

$$Q_{ij} = C_{ij} - \frac{C_{i3}\,C_{3j}}{C_{33}} \qquad (i, j = 1, 2, 6) \tag{4.24}$$

***Note***: The reduced stiffness matrix is symmetric.

***Note***: The readers should again understand the difference between $Q_{ij}$ and $C_{ij}$ terms. They are not the same.

The inversion of Equation (4.23) should give us Equation (4.17), that is,

$$\begin{Bmatrix}\varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12}\end{Bmatrix} = \begin{bmatrix}S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66}\end{bmatrix}\begin{Bmatrix}\sigma_{11} \\ \sigma_{22} \\ \tau_{12}\end{Bmatrix}$$

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Let us compare this equation with corresponding 3*D* equation (Eq. (4.12). It can easily be seen that the compliance terms of the constitutive equation are identical for 3*D* and plane stress problems. Thus, we can write for the plane stress problems as

$$[Q]=[S]^{-1} \tag{4.25}$$

It is easy to invert a $3\times3$ matrix. In fact, you need to invert a $2\times2$ matrix. Thus, we can write the individual reduced stiffness entries in terms of compliance entries as

$$\begin{aligned} Q_{11}&=\frac{S_{22}}{S_{11}S_{22}-S_{12}^2}, \qquad Q_{22}=\frac{S_{11}}{S_{11}S_{22}-S_{12}^2} \\ Q_{12}&=\frac{-S_{12}}{S_{11}S_{22}-S_{12}^2}, \qquad Q_{66}=\frac{1}{S_{66}} \end{aligned} \tag{4.26}$$

## Compliance and Stiffness Coefficients Using Engineering Constants

Let us write the compliance and stiffness matrices using engineering constants. It is easy to see that the individual entries of the compliance matrix in plane stress problem are same as the 3*D* compliance. Thus, we write for the plane stress problem the compliance entries as

$$\begin{aligned} S_{11}&=\frac{1}{E_1}, \qquad S_{12}=\frac{-\nu_{21}}{E_2}=S_{21}=\frac{-\nu_{12}}{E_1} \\ S_{22}&=\frac{1}{E_2}, \qquad S_{66}=\frac{1}{G_{12}} \end{aligned} \tag{4.27}$$

Here, we have used the property that compliance matrix is symmetric, that is, $S_{ij}=S_{ji}$. Using $S_{12}=S_{21}$, we can develop the reciprocal relationship for 2*D* case as

$$\frac{\nu_{21}}{E_2}=\frac{\nu_{12}}{E_1} \tag{4.28}$$

***Note***: It is easy to see for a plane stress problem of an orthotropic material that only four of the five material constants are independent.

We can write the individual terms of reduced stiffness matrix in principal material directions by using Equation (4.26) and Equation (4.27) as

(4.29)

$$Q_{11} = \frac{E_1}{1-\nu_{12}\,\nu_{21}}, \qquad Q_{12} = \frac{\nu_{12}\,E_2}{1-\nu_{12}\,\nu_{21}} = \frac{\nu_{21}\,E_1}{1-\nu_{12}\,\nu_{21}}$$

$$Q_{22} = \frac{E_2}{1-\nu_{12}\,\nu_{21}}, \qquad Q_{66} = G_{12}$$

***Note***: For a transversely isotropic material there is no reduction of the number of independent constants for plane stress problem.

One can write the constitutive equation in material coordinates, using Equation (3.42), Equation (3.43) and introducing the corresponding reduction in out of plane direction as

$$\begin{aligned} \varepsilon_{11} &= \frac{\sigma_{11}}{E_1} - \frac{\nu_{21}}{E_2}\sigma_{22} \\ \varepsilon_{22} &= -\frac{\nu_{12}}{E_1}\sigma_{11} + \frac{\sigma_{22}}{E_2} \\ \gamma_{12} &= \frac{\tau_{12}}{G_{12}} \end{aligned} \tag{4.30}$$

It is easy to write the compliance coefficients in Equation (4.27) from these relations. Further, we can write the above relations in inverted form as

$$\begin{aligned} \sigma_{11} &= \frac{E_1}{1-\nu_{12}\nu_{21}}\varepsilon_{11} + \frac{\nu_{12}\,E_2}{1-\nu_{12}\nu_{21}}\varepsilon_{22} \\ \sigma_{22} &= \frac{\nu_{12}\,E_2}{1-\nu_{12}\nu_{21}}\varepsilon_{11} + \frac{E_2}{1-\nu_{12}\nu_{21}}\varepsilon_{22} \\ \tau_{12} &= G_{12}\,\gamma_{12} \end{aligned} \tag{4.31}$$

These relations lead to individual reduced stiffness coefficients given in Equation (4.29).

### 2*D* Transformations about an Axis:

In planar stress condition we need to transform the stresses in plane. Let us write, similar to Equation (3.63), the transformation equation for stresses as

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} = [T_1] \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} \tag{4.32}$$

where $[T_1]$ is the transformation matrix for stress tensor. For the above equation, using Equation (3.64), this matrix can be written as

$$[T_1] = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{bmatrix} \tag{4.33}$$

Similarly, we can write the strain transformation equation in the following form.

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} = [T_2] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \tag{4.34}$$

where $[T_2]$ is the transformation matrix for strain tensor. We can find this matrix using Equation (3.69) and the above relations as

$$[T_2] = \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2-n^2 \end{bmatrix} \tag{4.35}$$

***Note***: The transformation matrices $[T_1]$ and $[T_2]$ are not symmetric. There is a difference of factor 2 in two entries of these matrices.

***Note***: The transformation matrices $[T_1]$ and $[T_2]$ can be inverted using following relation

$$[T_i(\theta)]^{-1} = [T_i(-\theta)] \qquad i = 1, 2 \tag{4.36}$$

The readers should verify this result.

***Note***: We have used the same matrix notation for stress and strain transformation matrices ( $\left[T_1\right]$ and $\left[T_2\right]$) in 3*D* case and plane stress case. However, the readers should note the corresponding differences.

## Lamina Constitutive Relations in Global Coordinates:

The plane stress constitutive equation in principal material coordinates is

$$\{\sigma\}_{123} = [Q]\{\varepsilon\}_{123} \qquad (4.37)$$

Let us write the stresses and strains in terms of components in global directions using Equation (4.32) and Equation (4.34). The above equation can be re-written to give stresses in global coordinates as

$$\{\sigma\}_{xyz} = [T_1]^{-1}[Q][T_2]\{\varepsilon\}_{xyz} \qquad (4.38)$$

We define the plane stress transformed reduced stiffness matrix $\left[\overline{Q}\right]$ as

$$\left[\overline{Q}\right] = [T_1]^{-1}[Q][T_2] \qquad (4.39)$$

Introducing this definition in Equation (4.38), we get

$$\{\sigma\}_{xyz} = \left[\overline{Q}\right]\{\varepsilon\}_{xyz} \qquad (4.40)$$

The above equation is written in expanded form as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \qquad (4.41)$$

***Note***: $\left[\overline{Q}\right]$ is a symmetric matrix. Further, it is a fully populated matrix with non zero $\overline{Q}_{16}$, $\overline{Q}_{26}$ coefficients.

Thus, using Equation (4.33), Equation (4.35 in Equation (4.39), we can write the individual terms in expanded form as

$$\{\sigma\}_{xyz} = \left[\overline{Q}\right]\{\varepsilon\}_{xyz}$$

(4.42)

$$\bar{Q}_{11} = Q_{11}m^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}n^4$$
$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})m^2n^2 + Q_{12}(n^4 + m^4)$$
$$\bar{Q}_{22} = Q_{11}n^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}m^4$$
$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})m^3n + (Q_{12} - Q_{22} + 2Q_{66})n^3m$$
$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})mn^3 + (Q_{12} - Q_{22} + 2Q_{66})nm^3$$
$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})m^2n^2 + Q_{66}(n^4 + m^4)$$

***Note***: Reduced stiffness coefficients are fourth order in the sine and cosine functions.

***Note***: $\bar{Q}_{16}, \bar{Q}_{26}$ are very important. They define the coupling between in-plane normal and shear responses. Figure 4.1 shows response of an isotropic and orthotropic material under traction. The behaviour of an orthotropic lamina loaded along fibre direction and perpendicular to fibre direction is essentially similar to an isotropic material. However, for an off axis lamina, the behaviour clearly shows the coupling between normal and shear terms.

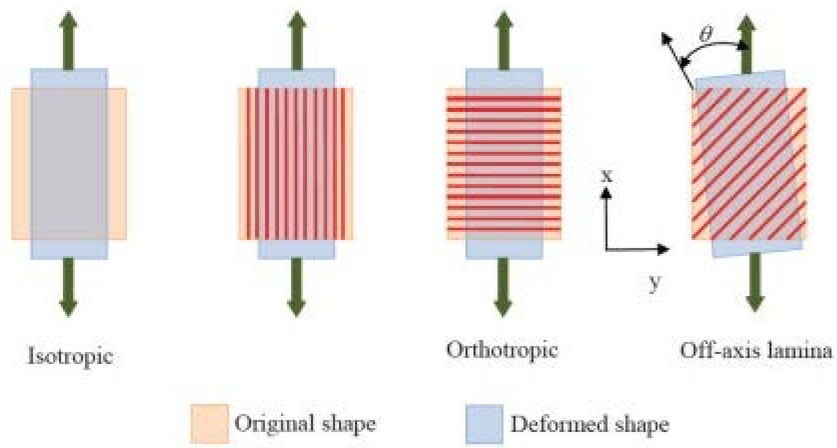


**Figure 4.1: Normal shear coupling in orthotropic lamina**

The transformed plane stress constitutive equation can also be given in inverted form of Equation (4.40) as

$$\{\varepsilon\}_{xyz} = [\bar{S}]\{\sigma\}_{xyz} \qquad (4.43)$$

where

$$\begin{aligned}[\bar{S}] &= [\bar{Q}]^{-1} \\ &= [T_2]^{-1}[Q]^{-1}[T_1]\end{aligned} \qquad (4.44)$$

Using $[T_1]$ from Equation (4.33) and $[T_2]$ from Equation (4.35) in the above equation, we get the individual coefficients of transformed reduced compliance matrix as

$$\begin{aligned}
\bar{S}_{11} &= m^4 S_{11} + m^2 n^2 (2S_{12} + S_{66}) + n^4 S_{22} \\
\bar{S}_{12} &= (S_{11} + S_{22} - S_{66}) n^2 m^2 + S_{12} (n^4 + m^4) \\
\bar{S}_{16} &= nm \left[ (2S_{11} - 2S_{12} - S_{66}) m^2 + (2S_{12} - 2S_{22} + S_{66}) n^2 \right] \\
\bar{S}_{22} &= n^4 S_{11} + (2S_{12} + S_{66}) m^2 n^2 + m^4 S_{22} \\
\bar{S}_{26} &= nm \left[ (2S_{11} - 2S_{12} - S_{66}) n^2 + (2S_{12} - 2S_{22} + S_{66}) m^2 \right] \\
\bar{S}_{66} &= 4n^2 m^2 (S_{11} - 2S_{12} + S_{22}) + S_{66} (n^2 + m^2)^2
\end{aligned} \tag{4.45}$$

***Note***: The same notation has been used for compliance matrices in principal directions $[S]$ and transformed directions $[\bar{S}]$ in 2*D* and 3*D*. This is because the corresponding terms are identical. However, for the stiffness coefficients these are different in 2*D* and 3*D*.

***Note***: One can see the difference between the stiffness values by algebra involved. The inverse of the $3 \times 3$ compliance matrix for plane stress case is different from the inverse of the $6 \times 6$ matrix for 3*D* case.

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## Thermal Effects:

Thermal strains in principal material coordinates are proportional to the temperature change $\Delta T$. These are given using coefficient of thermal expansion as

$$\left\{\varepsilon^{(T)}\right\}_{123} = \{\alpha\}_{123}\Delta T \tag{4.46}$$

where $\{\alpha\}_{123} = \{\alpha_1 \quad \alpha_2 \quad 0\}^T$.

Transformation of the thermal strains $\left\{\varepsilon^{(T)}\right\}_{123}$ to the strains $\left\{\varepsilon^{(T)}\right\}_{xyz}$ in global coordinates gives

$$\begin{Bmatrix} \varepsilon_{xx}^{(T)} \\ \varepsilon_{yy}^{(T)} \\ \gamma_{xy}^{(T)} \end{Bmatrix} = [T_2]^{-1} \begin{Bmatrix} \varepsilon_{11}^{(T)} \\ \varepsilon_{22}^{(T)} \\ \gamma_{12}^{(T)} \end{Bmatrix} = [T_2]^{-1} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{Bmatrix} \Delta T \tag{4.47}$$

Let $\{\alpha\}_{xyz} = [T_2]^{-1}\{\alpha\}_{123}$. Thus, Equation (4.47) becomes

$$\begin{Bmatrix} \varepsilon_{xx}^{(T)} \\ \varepsilon_{yy}^{(T)} \\ \gamma_{xy}^{(T)} \end{Bmatrix} = \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ \alpha_{xy} \end{Bmatrix} \Delta T = \begin{Bmatrix} m^2\alpha_1 + n^2\alpha_2 \\ n^2\alpha_1 + m^2\alpha_2 \\ 2mn(\alpha_1 - \alpha_2) \end{Bmatrix} \Delta T \tag{4.48}$$

or

$$\left\{\varepsilon^{(T)}\right\}_{xyz} = \{\alpha\}_{xyz}\Delta T \tag{4.49}$$

## Thermo-Elastic Constitutive Equation:

The total strain due to mechanical and thermal loading in principal material directions is given as

$$\{\varepsilon\}_{123} = \left\{\varepsilon^{(\sigma)}\right\}_{123} + \left\{\varepsilon^{(T)}\right\}_{123} \tag{4.50}$$

We can write for the mechanical strains as

$$\left\{\varepsilon^{(\sigma)}\right\}_{123} = [S]\{\sigma\}_{123} \tag{4.51}$$

Thus, Equation (4.50) becomes

$$\{\varepsilon\}_{123} = [S]\{\sigma\}_{123} + \left\{\varepsilon^{(T)}\right\}_{123} \tag{4.52}$$

Re-arranging the above equation, we can write for the stresses as

$$\{\sigma\}_{123} = [Q]\left(\{\varepsilon\}_{123} - \left\{\varepsilon^{(T)}\right\}_{123}\right) \tag{4.53}$$

The above equation can be written in global coordinate system as

$$\begin{aligned}\{\sigma\}_{xyz} &= \left[\overline{Q}\right]\left(\{\varepsilon\}_{xyz} - \left\{\varepsilon^{(T)}\right\}_{xyz}\right)\\ &= \left[\overline{Q}\right]\left(\{\varepsilon\}_{xyz} - \{\alpha\}_{xyz}\Delta T\right)\end{aligned}$$

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## Moisture Effect:

The hygroscopic expansion in principal material direction is proportional to the amount of percentage weight of moisture absorbed. Further, the hygroscopic expansion will be in principal normal directions only. This expansion will not lead to any shear. Thus, we write the hygral strains in principal directions for planar problem as

$$\left\{\varepsilon^{(H)}\right\}_{123} = \{\beta\}_{123}\Delta\mathrm{M} = \begin{Bmatrix} \beta_1 \\ \beta_2 \\ 0 \end{Bmatrix}\Delta\mathrm{M} \tag{4.54}$$

Here, $\{\beta\}_{123}$ denotes the coefficient of hygroscopic expansion in principal material directions for planar problem and $\Delta\mathrm{M}$ denotes the amount by percentage weight of moisture absorbed.
Now let us transform the hygroscopic strains in global coordinate system as

$$\left\{\varepsilon^{(H)}\right\}_{xyz} = [T_2]^{-1}\left\{\varepsilon^{(H)}\right\}_{123} \tag{4.55}$$

Using Equation (4.54), we can write

$$\left\{\varepsilon^{(H)}\right\}_{xyz} = [T_2]^{-1}\{\beta\}_{123}\Delta\mathrm{M} = \{\beta\}_{xyz}\Delta\mathrm{M} \tag{4.56}$$

where

$$\{\beta\}_{xyz} = \begin{Bmatrix} \beta_{xx} \\ \beta_{yy} \\ \beta_{xy} \end{Bmatrix} = [T_2]^{-1}\{\beta\}_{123} = \begin{Bmatrix} m^2\beta_1 + n^2\beta_2 \\ n^2\beta_1 + m^2\beta_2 \\ 2mn(\beta_1 - \beta_2) \end{Bmatrix} \tag{4.57}$$

It is clearly seen from Equation (4.48) and Equation (4.57) that $\{\alpha\}_{xyz}$ and $\{\beta\}_{xyz}$ behave in a similar way.

## Hygro-Thermo-Elastic Constitutive Equations:

When hygral and thermal effects are present along with mechanical strains, then the total strain in principal material direction is given as

$$\{\varepsilon\}_{123} = \left\{\varepsilon^{(\sigma)}\right\}_{123} + \left\{\varepsilon^{(T)}\right\}_{123} + \left\{\varepsilon^{(H)}\right\}_{123} \tag{4.58}$$

Using Hooke's low for mechanical strain and solving for stress the hygro-thermal constitutive equation, we get

$$\begin{aligned}\{\sigma\}_{123} &= [Q]\left(\{\varepsilon\}_{123} - \left\{\varepsilon^{(T)}\right\}_{123} - \left\{\varepsilon^{(H)}\right\}_{123}\right) \\ &= [Q]\left(\{\varepsilon\}_{123} - \{\alpha\}_{123}\Delta T - \{\beta\}_{123}\Delta \mathrm{M}\right)\end{aligned} \tag{4.59}$$

Equation (4.59) can be written to give stresses in global directions as

$$\begin{aligned}\{\sigma\}_{xyz} &= [\overline{Q}]\left(\{\varepsilon\}_{xyz} - \left\{\varepsilon^{(T)}\right\}_{xyz} - \left\{\varepsilon^{(H)}\right\}_{xyz}\right) \\ &= [\overline{Q}]\left(\{\varepsilon\}_{xyz} - \{\alpha\}_{xyz}\Delta T - \{\beta\}_{xyz}\Delta \mathrm{M}\right)\end{aligned} \tag{4.60}$$

where $\{\alpha\}_{xyz}$ and $\{\beta\}_{xyz}$ are as given in Equation (4.48) and Equation (4.57), respectively.

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## Examples:

***Example 4.1***: Calculate $\overline{Q}_{ij}$ for AS4-3501-6 Epoxy material for fibre orientation of $60°$.

***Solution***:

Calculate the Poisson's ratio $\nu_{21}$ as

$$\nu_{21} = \frac{E_2}{E_1}\nu_{12} = \frac{11\times10^3}{126\times10^3}0.28 = 0.02444$$

Calculate the reduced stiffness matrix entries in principal material directions as

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}} = 126868.3433\,MPa, \qquad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} = 3101.1697\,MPa$$

$$Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}} = 11075.8077\,MPa, \qquad Q_{66} = G_{12} = 6600\,MPa$$

Now, calculate $[T_1]^{-1}$ and $[T_2]$ as from Equation (4.33), Equation (4.35) and Equation (4.36) with $m = \cos 60° = 0.5$ and $n = \sin 60° = 0.866$.

$$[T_1]^{-1} = \begin{bmatrix} 0.25 & 0.75 & -0.866 \\ 0.75 & 0.25 & 0.866 \\ 0.433 & -0.433 & -0.5 \end{bmatrix} \text{ and}$$

$$[T_2] = \begin{bmatrix} 0.25 & 0.75 & 0.433 \\ 0.75 & 0.25 & -0.433 \\ -0.866 & 0.866 & -0.5 \end{bmatrix}$$

Now, $\left[\overline{Q}\right] = [T_1]^{-1}[Q][T_2]$. This gives us

$$\left[\overline{Q}\right] = \begin{bmatrix} 20272.37 & 22852.79 & 13666.24 \\ 22852.79 & 78168.64 & 36473.39 \\ 13666.24 & 36473.39 & 26351.56 \end{bmatrix} MPa$$

***Example 4.2***: In the above example, if the state of strain at a point in principal material directions is $\{\varepsilon\}_{123} = \{2 \quad 1 \quad 1\}^T \times 10^{-4}$, then find the corresponding state of stress in global directions for fibre orientation of $60°$.

***Solution***:

We find stresses in global directions as

$$\{\sigma\}_{xyz} = \left[\overline{Q}\right]\{\varepsilon\}_{xyz}$$

We calculate the strains in global directions as

$$\{\varepsilon\}_{xyz} = [T_2]^{-1}\{\varepsilon\}_{123}$$

And

$$[T_2]^{-1} = \begin{bmatrix} 0.25 & 0.75 & -0.433 \\ 0.75 & 0.25 & 0.433 \\ 0.866 & -0.866 & -0.5 \end{bmatrix}$$

Thus,

$$\{\varepsilon\}_{xyz} = \{0.817 \quad 2.183 \quad 0.366\}^T \times 10^{-4}$$

And

$$\{\varepsilon\}_{xyz} = \{7.145 \quad 20.266 \quad 10.043\}^T \; MPa$$

***Example 4.3***: In Example 1, the coefficients of thermal expansion in principal material directions are $\{\alpha\}_{123} = \{-1 \quad 26 \quad 0\} \times 10^{-6} / {}^\circ C$. Calculate the stresses developed due to temperature rise of $30°C$ in principal material directions as well as in global material directions.

***Solution*:**

Stresses in principal material directions due to thermal strains alone are given as

$$\{\sigma\}_{123} = [Q]\left\{\varepsilon^{(T)}\right\}_{123} = [Q]\{\alpha\}_{123}\Delta T$$

Here, $\Delta T = 30°C$. Thus,

$$\{\sigma\}_{123} = \{-1.387 \quad 8.546 \quad 0\}^T \; MPa$$

And stresses in global directions are

$$\{\sigma\}_{xyz} = [T_1]^{-1}\{\sigma\}_{123} = \{6.063 \quad 1.096 \quad -4.301\}^T \; MPa$$

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## Homework:

1. Verify the result given in Equation (4.36).
2. Using the invariance property of strain energy density function, show that:

$$[T_1]^{-1} = [T_2]^T \text{ and } [T_2]^{-1} = [T_1]^T$$

3. Using the relation between $Q_{ij}$ and $C_{ij}$ as given in Equation (4.24) and $C_{ij}$ in terms of engineering constants, show that $Q_{ij}$ are as given in Equation (4.29).
4. Write the compliance coefficients in Equation (4.45) in terms of engineering constants.
5. Using Equation (4.24) in Equation (4.42) obtain the individual terms of $\overline{Q}_{ij}$ in terms of $C_{ij}$.
6. For fibre orientation $\theta = 30°$ and $\theta = 45°$ obtain $\left[\overline{Q}\right]$ matrix for materials given in Table 3.1.
7. The $\left[\overline{Q}\right]$ matrix for a composite with fibre orientation of $\theta = 60°$ is given as

$$\left[\overline{Q}\right] = \begin{bmatrix} 18409 & 10436 & 3193 \\ 10436 & 33524 & 9896 \\ 3193 & 9896 & 11635 \end{bmatrix} MPa$$

   Find all engineering constants for this material.
8. Write a computer code to calculate reduced transformed stiffness and compliance matrix for any angle of fibre orientation with respect to global coordinate system.
9. Extend the code written for the above problem to plot the variation of $\overline{Q}_{ij}$ terms for orientation of fibres between $-90° < \theta < 90°$. Plot the variation for materials given in Table 3.1.
10. Write a computer code to plot the variation of thermal and hygroscopic expansion coefficients with fibre orientation between $-90° < \theta < 90°$ for T300/5208 composite.

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