Module 5: Laminate Theory

Lecture 18: Laminate Engineering Constants

The Lecture Contains:

Introduction:

In the previous lecture, we have introduced the laminate constitutive equation and classification of laminates. Further, we have seen the partially and fully inverted form of the constitutive equations. In this lecture we are going to see the laminate engineering constants and some numerical examples based on this.

Laminate Engineering Constants:

In this section we will develop the laminate engineering constants, similar to lamina engineering constants as we have done earlier. Here, we will consider only symmetric laminates as the extension-bending coupling is absent. We have constitutive equation from Equation. (5.28) as

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \epsilon^{(0)} \\ \kappa \end{Bmatrix}$$

For the symmetric laminate with $[B] = 0$, we get

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \begin{Bmatrix} \epsilon^{(0)} \\ \kappa \end{Bmatrix}$$

From the first of the above equation, we can write

$$\{N\} = [A]\{\epsilon^{(0)}\} \tag{5.54}$$

Let us define the laminate average stress $\{\overline{\sigma}\}$ as

$$\{\overline{\sigma}\} = \frac{1}{2H}[N] \tag{5.55}$$

Using Equation (5.54), we get

$$\{\overline{\sigma}\} = \frac{1}{2H}[A]\{\epsilon^{(0)}\} \tag{5.56}$$

The strains can be given by inverting the above equation as

$$\{\epsilon^{(0)}\} = 2H[A]^{-1}\{\overline{\sigma}\} \tag{5.57}$$

Let us define, $[a^*] = 2H[A]^{-1}$ as laminate compliance. Thus,

$$\{\epsilon^{(0)}\} = [a^*]\{\overline{\sigma}\} \tag{5.58}$$

The expanded form of above equation is written as

$$\begin{Bmatrix} \epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = \begin{bmatrix} a_{11}^{*} & a_{12}^{*} & a_{16}^{*} \\ a_{12}^{*} & a_{22}^{*} & a_{26}^{*} \\ a_{16}^{*} & a_{26}^{*} & a_{66}^{*} \end{bmatrix} \begin{Bmatrix} \overline{\sigma}_{xx} \\ \overline{\sigma}_{yy} \\ \overline{\tau}_{xy} \end{Bmatrix} \tag{5.59}$$

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In-plane Laminate Engineering Constants:

Now we express the laminate engineering constants with the help of thought experiments as follows.

Effective in-plane longitudinal modulus E_x:

The state of stress applied is

$$\overline{\sigma}_{xx} \neq 0 \text{ and } \overline{\sigma}_{yy} = \overline{\tau}_{xy} = 0 \tag{5.60}$$

For this state of stress, we get the relation as

$$\epsilon_{xx}^{(0)} = a_{11}^{*}\overline{\sigma}_{xx} \tag{5.61}$$

Let us define effective in-plane longitudinal modulus as

$$E_x = \frac{\overline{\sigma}_{xx}}{\epsilon_{xx}^{(0)}} \tag{5.62}$$

Using Equation (5.61) we get

$$E_x = \frac{1}{a_{11}^{*}} \tag{5.63}$$

Effective in-plane Poisson's ratio ν_{xy}:

Let us define the effective in-plane Poisson's ratio as

$$\nu_{xy} = -\frac{\epsilon_{yy}^{(0)}}{\epsilon_{xx}^{(0)}} \tag{5.64}$$

For the state of stress in Equation (5.60) we have

$$\epsilon_{yy}^{(0)} = a_{12}^{*}\overline{\sigma}_{xx} \tag{5.65}$$

From Equation (5.61) and Equation (5.65), we can write for Equation (5.64) as

$$\nu_{xy} = -\frac{a_{12}^{*}}{a_{11}^{*}} \tag{5.66}$$

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Coefficient of mutual influence of first kind $\eta_{xy,x}$:

Let us define the coefficient of mutual influence for the loading in Equation (5.60) as

$$\eta_{xy,x} = \frac{\gamma_{xy}^{(0)}}{\epsilon_{xx}^{(0)}} \tag{5.67}$$

For the loading of Equation (5.60) we have

$$\gamma_{xy}^{(0)} = a_{16}^{*}\overline{\sigma}_{xx} \tag{5.68}$$

Thus

$$\eta_{xy,x} = \frac{a_{16}^{*}}{a_{11}^{*}} \tag{5.69}$$

Effective in-plane transverse modulus E_y:

The state of stress applied is

$$\overline{\sigma}_{xx} = \overline{\tau}_{xy} = 0 \text{ and } \overline{\sigma}_{yy} \neq 0 \tag{5.70}$$

For this state of stress, we get the relation as

$$\begin{aligned} \epsilon_{xx}^{(0)} &= a_{12}^{*}\overline{\sigma}_{yy} \\ \epsilon_{yy}^{(0)} &= a_{22}^{*}\overline{\sigma}_{yy} \\ \gamma_{xy}^{(0)} &= a_{26}^{*}\overline{\sigma}_{yy} \end{aligned} \tag{5.71}$$

Let us define the effective in-plane transverse modulus as

$$E_y = \frac{\overline{\sigma}_{yy}}{\epsilon_{yy}^{(0)}} \tag{5.72}$$

This can be written as using the second of Equation (5.71) as

$$E_y = \frac{1}{a_{22}^{*}} \tag{5.73}$$

Effective in-plane Poisson's ratio ν_{yx}:

Let us define this Poisson's ratio as follows

$$\nu_{yx} = -\frac{\epsilon_{xx}^{(0)}}{\epsilon_{yy}^{(0)}} \tag{5.74}$$

Using the first and second of Equation (5.71) we can write

$$\nu_{yx} = -\frac{a_{12}^*}{a_{22}^*} \tag{5.75}$$

Coefficient of mutual influence of first kind $\eta_{xy,y}$:

Let us define the coefficient of mutual influence for the loading in Equation (5.70) as

$$\eta_{xy,y} = \frac{\gamma_{xy}^{(0)}}{\epsilon_{yy}^{(0)}} \tag{5.76}$$

For the second and third of Equation (5.71) we have

$$\eta_{xy,y} = \frac{a_{26}^*}{a_{22}^*} \tag{5.77}$$

In-plane effective shear modulus G_{xy}:

Let the loading on laminate be

$$\overline{\sigma}_{xx} = \overline{\sigma}_{yy} = 0 \text{ and } \overline{\tau}_{xy} \neq 0 \tag{5.78}$$

Thus, for this loading we can write from Equation. (5.59) as

$$\begin{aligned} \epsilon_{xx}^{(0)} &= a_{16}^*\overline{\tau}_{xy} \\ \epsilon_{yy}^{(0)} &= a_{26}^*\overline{\tau}_{xy} \\ \gamma_{xy}^{(0)} &= a_{66}^*\overline{\tau}_{xy} \end{aligned} \tag{5.79}$$

Let us define the inplane effective shear modulus for laminate as

$$G_{xy} = \frac{\overline{\tau}_{xy}}{\gamma_{xy}^{(0)}} \tag{5.80}$$

With the use of Equation (5.79), we can write

$$G_{xy} = \frac{1}{a_{66}^{*}} \tag{5.81}$$

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Coefficient of mutual influence of second kind $\eta_{x,xy}$ ***and*** $\eta_{y,xy}$**:**

Let us define these coefficients as follows

$$\eta_{x,xy} = \frac{\epsilon_{xx}^{(0)}}{\gamma_{xy}^{(0)}} \qquad \eta_{y,xy} = \frac{\epsilon_{yy}^{(0)}}{\gamma_{xy}^{(0)}} \tag{5.82}$$

Using Equation (5.79), these coefficients are given as

$$\eta_{x,xy} = \frac{a_{16}^*}{a_{66}^*} \qquad \eta_{y,xy} = \frac{a_{26}^*}{a_{66}^*} \tag{5.83}$$

Reciprocal relationship for Poisson's ratios**:**

We have reciprocal relations for lamina. Similarly, we can have reciprocal relations for laminate as well.

From Equation (5.63) and Equation (5.66), we can write

$$\frac{\nu_{xy}}{E_x} = \left(-\frac{a_{12}^*}{a_{11}^*}\right) a_{11}^* = -a_{12}^* \tag{5.84}$$

Similarly, from Equation (5.73) and Equation (5.75), we can write

$$\frac{\nu_{yx}}{E_y} = \left(-\frac{a_{12}^*}{a_{22}^*}\right) a_{22}^* = -a_{12}^* \tag{5.85}$$

Thus, combining Equation (5.83) and Equation (5.84), we get the required reciprocal relation for laminate as

$$\frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y} \tag{5.86}$$

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Flexural Engineering Constants of a Laminate:

For a symmetric laminate, we have $[B]=0$. Hence, from the second of Eq. (5.28) we write

$$\{M\}=[D]\{\kappa\} \tag{5.87}$$

Let us define the laminate average moments as $\{\overline{M}\}=\frac{12}{(2H)^3}[M]$

$$\{\overline{M}\}=\frac{12}{(2H)^3}[D]\{\kappa\}$$

The inverse of this equation is written as

$$\{\kappa\}=\frac{(2H)^3}{12}[D]^{-1}\{\overline{M}\}=[d^*]\{\overline{M}\}$$

Writing this in expanded form

$$\begin{Bmatrix}\kappa_{xx}\\ \kappa_{yy}\\ \kappa_{xy}\end{Bmatrix}=\begin{bmatrix}d_{11}^* & d_{12}^* & d_{16}^*\\ d_{12}^* & d_{22}^* & d_{26}^*\\ d_{16}^* & d_{26}^* & d_{66}^*\end{bmatrix}\begin{Bmatrix}\overline{M}_{xx}\\ \overline{M}_{yy}\\ \overline{M}_{xy}\end{Bmatrix} \tag{5.88}$$

Now the applied resultant moments be

$$\overline{M}_{xx}\neq 0, \overline{M}_{yy}=\overline{M}_{xy}=0 \tag{5.89}$$

Effective Flexural Longitudinal Young's Modulus E_x^f:

Let us define the effective flexural Young's modulus as

$$E_x^f=\frac{\overline{M}_{xx}}{\kappa_{xx}} \tag{5.90}$$

Thus, using Equation (5.87) and Equation (5.88),

$$E_x^f=\frac{1}{d_{11}^*} \tag{5.91}$$

Flexural Poisson's Ratio ν_{xy}^f:

Let us define flexural Poisson's ratio as

$$\nu_{xy}^{f} = -\frac{\kappa_{yy}}{\kappa_{xx}} \tag{5.92}$$

Thus, from Equation (5. 87) and Equation (5.88),

$$\nu_{xy}^{f} = -\frac{d_{12}^{*}}{d_{11}^{*}} \tag{5.93}$$

In a similar way, one can show that the other flexural moduli can be given by following relations.

$$E_{y}^{f} = \frac{1}{d_{22}^{*}}$$

$$\nu_{yx}^{f} = -\frac{d_{12}^{*}}{d_{22}^{*}} \tag{5.94}$$

$$G_{xy}^{f} = \frac{1}{d_{66}^{*}}$$

Further, it can be shown that the reciprocal relation also holds true for flexural Poisson's ratios as

$$\frac{\nu_{xy}^{f}}{E_{x}^{f}} = \frac{\nu_{yx}^{f}}{E_{y}^{f}} \tag{5.95}$$

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Examples:

Note: In the following examples, each lamina has a thickness of 1 *mm* and material is AS4/3501-6 Epoxy from Soden et al [4].

Example 5.8: For the laminate in Example 5.2 (that is, $[0/90/0]$), find the effective in-plane and flexural laminate engineering constants.

Solution: This is a symmetric matrix. Hence, $[B] = 0$. *A* and *D* matrices are given as:

$$[A] = \begin{bmatrix} 264.43 & 9.30 & 0 \\ 9.30 & 148.82 & 0 \\ 0 & 0 & 19.80 \end{bmatrix} GPa - mm$$

$$[D] = \begin{bmatrix} 275.82 & 6.97 & 0 \\ 6.97 & 34.56 & 0 \\ 0 & 0 & 14.85 \end{bmatrix} GPa - mm^3$$

Now we have

$$[A^*] = [A]^{-1} = \begin{bmatrix} 0.00378 & -0.00023 & 0 \\ -0.00023 & 0.00672 & 0 \\ 0 & 0 & 0.05050 \end{bmatrix} \frac{1}{GPa - mm}$$

$$[a^*] = 2H[A^*]^{-1} = 3\begin{bmatrix} 0.00378 & -0.00023 & 0 \\ -0.00023 & 0.00672 & 0 \\ 0 & 0 & 0.05050 \end{bmatrix} \frac{1}{GPa}$$

Thus, we can find the in-plane laminate engineering constants as

$$E_x = \frac{1}{a_{11}^*} = \frac{1}{0.01134} = 88.18\ GPa$$

$$E_y = \frac{1}{a_{22}^*} = \frac{1}{0.02016} = 49.61\ GPa$$

$$G_{xy} = \frac{1}{a_{66}^*} = \frac{1}{0.1515} = 6.60\ GPa$$

$$\nu_{xy} = -\frac{a_{12}^*}{a_{11}^*} = -\frac{-0.00069}{0.01134} = 0.0608$$

$$\nu_{yx} = -\frac{a_{12}^*}{a_{22}^*} = -\frac{-0.00069}{0.02016} = 0.0342$$

$$\eta_{xy,x} = \frac{a_{16}^*}{a_{11}^*} = \frac{0}{0.01134} = 0$$

$$\eta_{xy,y} = \eta_{x,xy} = \eta_{y,xy} = 0$$

Now,

$$[d^*] = \frac{(3)^3}{12}[D]^{-1} = \begin{bmatrix} 0.01123 & -0.00226 & 0 \\ -0.00226 & 0.08964 & 0 \\ 0 & 0 & 0.20763 \end{bmatrix} \frac{1}{GPa}$$

Now, we calculate the flexural engineering constants as

$$E_x^f = \frac{1}{d_{11}^*} = \frac{1}{0.01123} = 89.04\ GPa$$

$$E_y^f = \frac{1}{d_{22}^*} = \frac{1}{0.08964} = 11.15\ GPa$$

$$G_{xy}^f = \frac{1}{d_{66}^*} = \frac{1}{0.20763} = 4.81\ GPa$$

$$\nu_{xy}^f = -\frac{d_{12}^*}{d_{11}^*} = -\frac{-0.00226}{0.01123} = 0.2012$$

$$\nu_{yx}^f = -\frac{d_{12}^*}{d_{22}^*} = -\frac{-0.00226}{0.08964} = 0.0252$$

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Homework:

1. Derive the effective in-plane engineering constants for a laminate.

2. Derive the effective flexural engineering constants for a laminate.

3. For the composite material T300/5208, calculate the effective in-plane and engineering constants for the ollowing laminates. The thickness of each lamina is 1 mm.

    a. $[0/(90)_2]_S$

    b. $[\pm 45]_S$

    c. $[\mp 45]_S$

    d. $[0/45/0]_S$

    e. $[0/45/90]_S$

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- *PD Soden, MJ Hinton, AS Kaddour. Lamina properties, lay-up configurations and loading conditions for a range of fibre-reinforced composite laminates. Composite Science and Technology, 1998, Vol. 58, pp. 1011-1022.*

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