The Lecture Contains:

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Introduction:

In the previous lecture we have introduced the laminate theory. In this lecture we are going to develop laminate constitute equations for classical laminate theory. In the previous lecture we have introduced the in-plane stress resultants and resultant moments. These resultant quantities will be related to mid-plane strains and curvatures. Further, we will introduce classification of laminates.

Laminate Constitutive Relations:

Using Equations (5.18) and (5.24) we can write a combined equation as

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \epsilon^{(0)} \\ \kappa \end{Bmatrix} \tag{5.28}$$

This equation is the fundamental equation in classical laminate theory and is known as constitutive equation. This equation can be written in expanded form as

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} \tag{5.29}$$

It should be noted that the matrices *A*, *B* and *D* are symmetric. Hence, the matrix in above equation is also a symmetric matrix. The inverse constitutive relations of Equation (5.28) can be written as

$$\begin{Bmatrix} \epsilon^{(0)} \\ \kappa \end{Bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{Bmatrix} N \\ M \end{Bmatrix} \tag{5.30}$$

The matrix $\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}$ is obtained by using individual relations for $\{N\}$ and $\{M\}$ as follows. We write for $\{N\}$ and $\{M\}$ as

$$\begin{aligned} \{N\} &= [A]\{\epsilon^{(0)}\} + [B]\{\kappa\} \\ \{M\} &= [B]\{\epsilon^{(0)}\} + [D]\{\kappa\} \end{aligned} \tag{5.31}$$

From the first of the above equation we can write

$$\{\epsilon^{(0)}\} = [A]^{-1}\{N\} - [A]^{-1}[B]\{\kappa\} \tag{5.32}$$

Putting this in second of Equation (5.31) we get

$$\{M\} = [B][A]^{-1}\{N\} + \left([D] - [B][A]^{-1}[B]\right)\{\kappa\} \tag{5.33}$$

We define

$$\begin{aligned} [A^*] &= [A]^{-1} & [B^*] &= -[A]^{-1}[B] \\ [C^*] &= [B][A]^{-1} & [D^*] &= [D] - [B][A]^{-1}[B] \end{aligned} \quad (5.34)$$

We make a note that $[C^*] = -[B^*]^T$ and we can write

$$\begin{aligned} [D^*] &= [D] + [B^*]^T[B] \\ &= [D] + [B][B^*] \end{aligned}$$

Using the above definitions, Equation (5.32) and Equation (5.33) can be written together as

$$\begin{Bmatrix} \epsilon^{(0)} \\ M \end{Bmatrix} = \begin{bmatrix} A^* & B^* \\ C^* & D^* \end{bmatrix} \begin{Bmatrix} N \\ \kappa \end{Bmatrix} \quad (5.35)$$

The above equation is called as partially inverted constitutive equation for laminate. From the second of the above equation we write

$$\{\kappa\} = -[D^*]^{-1}[C^*]\{N\} + [D^*]^{-1}\{M\} \quad (5.36)$$

Putting this in Equation (5.35) we can get for $\{\epsilon^{(0)}\}$ as

$$\{\epsilon^{(0)}\} = ([A^*] - [B^*][D^*]^{-1}[C^*])\,\{N\} + [B^*][D^*]^{-1}\{M\} \quad (5.37)$$

Let us define

$$\begin{aligned} [A'] &= [A^*] - [B^*][D^*]^{-1}[C^*], & [B'] &= [B^*][D^*]^{-1} \\ [C'] &= -[D^*]^{-1}[C^*], & [D'] &= [D^*]^{-1} \end{aligned} \quad (5.38)$$

Combining Equations (5.37) and (5.36) and using the definitions in Equation (5.38), we can write

$$\begin{Bmatrix} \epsilon^{(0)} \\ \kappa \end{Bmatrix} = \begin{bmatrix} A' & B' \\ B'^T & D' \end{bmatrix} \begin{Bmatrix} N \\ M \end{Bmatrix} \quad (5.39)$$

This equation is the fully inverted form of laminate constitutive equation. Using Equation (5.34) in Equation (5.38) we can write the above equation in terms of *A*, *B* and *D* matrices as

$$\begin{aligned} [A'] &= [A]^{-1} + [A]^{-1}[B][D^*]^{-1}[B][A]^{-1} \\ [B'] &= -[A]^{-1}[B][D^*]^{-1} \\ [C'] &= -[D^*]^{-1}[B][A]^{-1} \\ [D'] &= ([D] - [B][A]^{-1}[B])^{-1} \end{aligned} \quad (5.40)$$

From this equation it is easy to deduce that

$$[C'] = [B']^T \qquad (5.41)$$

The full matrix $\begin{bmatrix} A' & B' \\ B'^T & D' \end{bmatrix}$ is symmetric. This also follows from the fact that this is an inverse of a symmetric matrix, that is $\begin{bmatrix} A & B \\ B & D \end{bmatrix}$, and the inverse of a symmetric matrix is also a symmetric matrix.

Equation (5.28) and Equation (5.39) are very important equations in laminate analysis. These equations relate the mid-plane strains and curvatures with resultant in-plane forces and moments and vice versa.

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Classification of Laminates:

In this section we are going to classify the laminates depending upon the stacking sequence nature. This classification is very helpful in the laminate analysis as some of the coupling terms become zero under specific laminate sequence and their arrangement with respect to the midplane.

Symmetric Laminates:

A laminate is called symmetric when the material, angle and thickness of the layers are the same above and below the mid-plane. **For example** laminate $[30/45/0]_S$ is shown in Figure 5.6(a).

For symmetric laminates the matrix *B* is zero. This can be proved as follows:

Consider two layers *r* and *s* which have the same material, angle and thickness and are located symmetrically with respect to the mid-plane as shown in Figure 5.7. For these layers we can write the relation about the reduced stiffness matrix entries as

$$\bar{Q}_{ij}(r) = \bar{Q}_{ij}(s) \qquad (5.28)$$

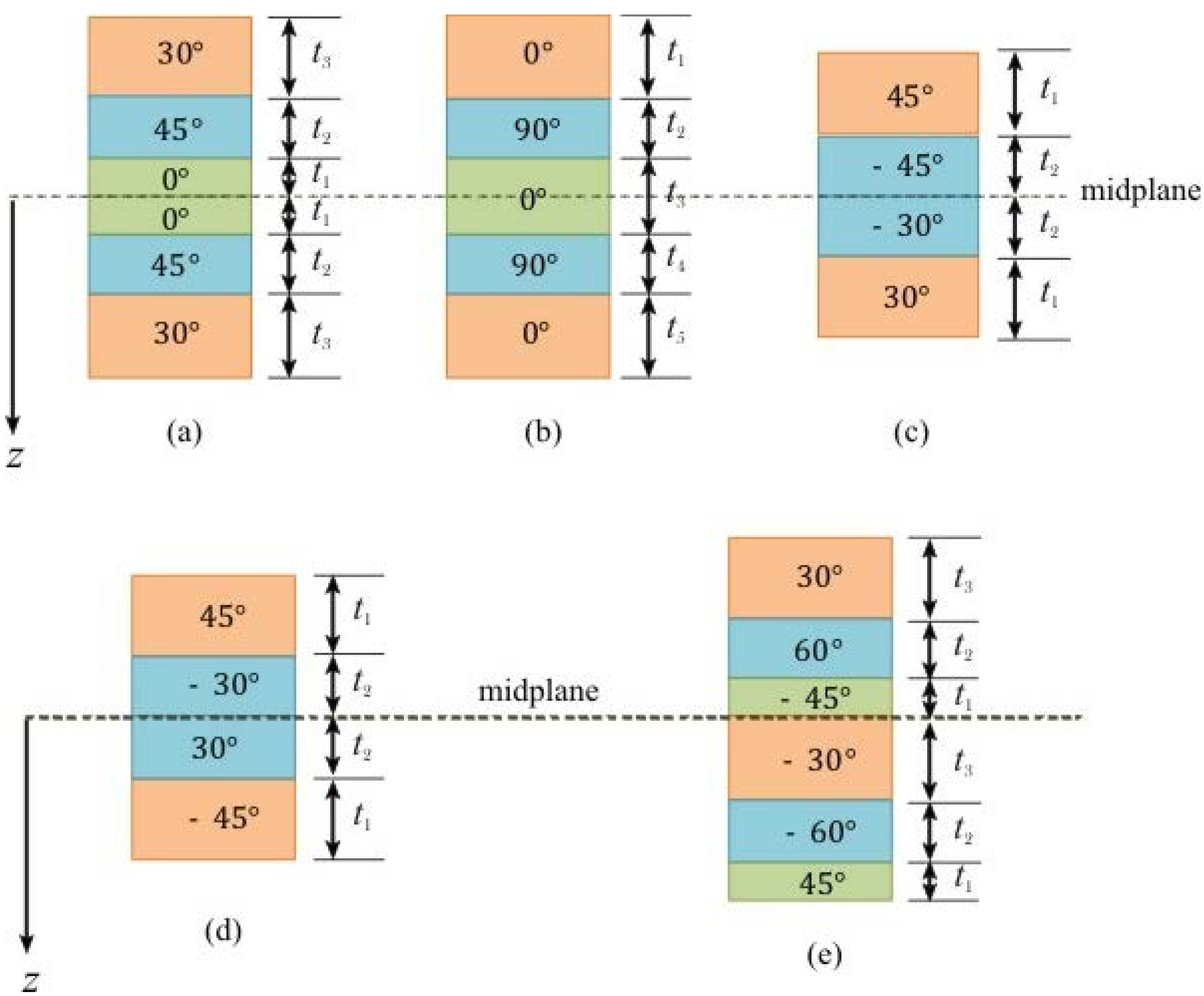

Figure 5.6: Classification of laminates examples (a) Symmetric laminate (b) Cross-ply laminate (c) Angle-ply laminate (d) Anti-symmetric laminate and (e)

Balanced laminate

The symmetry of location of these layers results in the following relation

$$z_r = -z_{s-1}$$
$$z_{r-1} = -z_s \qquad (5.43)$$

For these two layers, the contribution of to B matrix of the laminate is

$$B_{ij}(r+s) = \frac{1}{2}\overline{Q}_{ij}(r)(z_r^2 - z_{r-1}^2) + \frac{1}{2}\overline{Q}_{ij}(s)(z_s^2 - z_{s-1}^2) \qquad (5.44)$$

which upon substituting Equations (5.42) and (5.43) becomes

$$B_{ij}(r+s) = \frac{1}{2}\overline{Q}_{ij}(r)(z_r^2 - z_{r-1}^2 + z_s^2 - z_{s-1}^2) = 0 \qquad (5.45)$$

From this derivation it is very clear that the contribution of any pair of symmetric layers to B matrix is always zero. Thus, the B matrix is zero for symmetric laminates. However, one can show that the matrices A and D are not zero for symmetric laminates.

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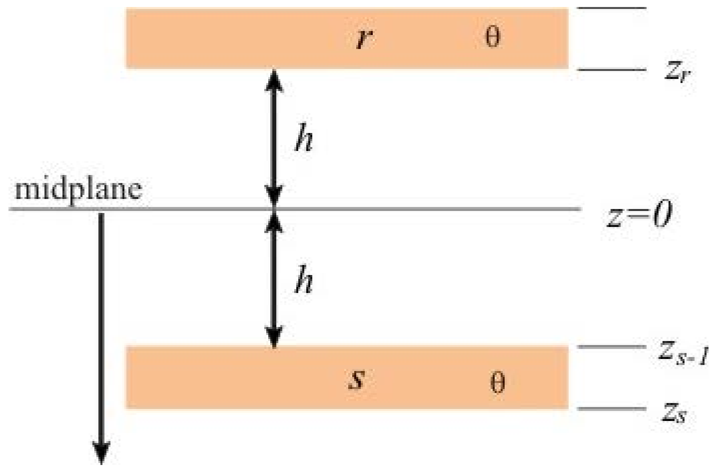

Figure 5.7: Coordinates for a pair of symmetric layers

The constitutive equation for symmetric laminates (with $[B]=0$) becomes

$$\begin{aligned}\{N\} &= [A]\{\epsilon^{(0)}\} \\ \{M\} &= [D]\{\kappa\}\end{aligned} \tag{5.46}$$

The inverse constitutive relations can be given as

$$\begin{aligned}\{\epsilon^{(0)}\} &= [A]^{-1}\{N\} \\ \{\kappa\} &= [D]^{-1}\{M\}\end{aligned} \tag{5.47}$$

This equation is consistent with Equation (5.39) through Equation (5.40). Setting $[B]=0$ in Equation (5.40), we get

$$\begin{aligned}[A'] &= [A]^{-1} \\ [B'] &= [0] \\ [C'] &= [0] \\ [D'] &= [D]^{-1}\end{aligned} \tag{5.48}$$

Thus, we can write Equation (5.39) or Equation (5.47) as

$$\begin{Bmatrix}\epsilon^{(0)} \\ \kappa\end{Bmatrix} = \begin{bmatrix}A^{-1} & 0 \\ 0 & D^{-1}\end{bmatrix}\begin{Bmatrix}N \\ M\end{Bmatrix} \tag{5.49}$$

***Note*:** For symmetric laminates *B* matrix is zero. It means that there is no coupling between extension and bending action. Thus, the applied stresses will produce only in-plane and shear strains and it will not produce any curvatures. Thus, it is easy to understand that the mid-plane strains will be the strains in each ply.

Note: For symmetric laminates, the *A* and *D* matrices can be given as

$$
\begin{aligned}
A_{ij}(r+s) &= \overline{Q}_{ij}(r)(z_r - z_{r-1} + z_s - z_{s-1}) = 2\overline{Q}_{ij}(r)(z_r - z_{r-1}) \neq 0 \\
D_{ij}(r+s) &= \tfrac{1}{3}\overline{Q}_{ij}(r)(z_r^3 - z_{r-1}^3 + z_s^3 - z_{s-1}^3) = \tfrac{2}{3}\overline{Q}_{ij}(r)(z_r^3 - z_{r-1}^3) \neq 0
\end{aligned}
\tag{5.50}
$$

For symmetric laminates, the uncoupling between extension and bending makes the analysis of laminates simpler. This is very useful because during thermal cooling down in the processing of such laminates there will not be any twisting due thermal loads.

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***Cross-Ply Laminates*:**

A laminate is called cross-ply laminate if all the plies used to fabricate the laminate are only 0° and 90°

For example $[0/90/0/90/0]$ is shown in Figure 5.6 (b).

For a cross ply laminate the terms $A_{16} = A_{26} = B_{16} = B_{26} = D_{16} = D_{26} = 0$. This is because these terms involve the terms $\overline{Q}_{16}$ and $\overline{Q}_{26}$ which have the products of *mn* terms. This product is zero for any cross-ply. Thus, the terms $\overline{Q}_{16}$ and $\overline{Q}_{26}$ are identically zero for each ply.

***Note*:** For a cross-ply following relations hold true. The readers should verify these relations from earlier lectures on planar constitutive relations.

$$\begin{aligned} &\overline{Q}_{11}(0) = \overline{Q}_{22}(90), \quad \overline{Q}_{22}(0) = \overline{Q}_{11}(90) \\ &\overline{Q}_{12}(0) = \overline{Q}_{12}(90), \quad \overline{Q}_{66}(0) = \overline{Q}_{66}(90) \end{aligned} \tag{5.51}$$

***Angle-Ply Laminates*:**

A laminate is called angle-ply laminate if it has plies of the same thickness and material and are oriented at $+\theta$ and – θ. **For example** $[45/-45/-30/30]$ is shown in Figure 5.6(c).

For angle-ply laminates the terms $A_{16} = A_{26}$ are zero. This can be justified by that fact that $\overline{Q}_{16}$ and $\overline{Q}_{26}$ have the term *mn*. Due to this term $\overline{Q}_{16}$ and $\overline{Q}_{26}$ have opposite signs for layers with $+\theta$ and – θfibre orientation. Since the thicknesses and materials of these layers are same, by the definition the terms $A_{16} = A_{26}$ are zero for the laminate.

***Note*:** For angle-ply laminates the following relations are very useful in computing $[A], [B]$ and $[D]$.

$$\begin{aligned} &\overline{Q}_{11}(+\theta) = \overline{Q}_{11}(-\theta), \quad \overline{Q}_{22}(+\theta) = \overline{Q}_{22}(-\theta) \\ &\overline{Q}_{12}(+\theta) = \overline{Q}_{12}(-\theta), \quad \overline{Q}_{66}(+\theta) = \overline{Q}_{66}(-\theta) \\ &\overline{Q}_{16}(+\theta) = -\overline{Q}_{16}(-\theta), \quad \overline{Q}_{26}(+\theta) = -\overline{Q}_{26}(-\theta) \end{aligned} \tag{5.52}$$

***Anti-symmetric Laminates*:**

A laminate is called anti-symmetric when the material and thickness of the plies are same above and below the mid-plane but the orientation of the plies at same distance above and below the mid-plane have opposite signs. **For example**, $[45/-30/30/-45]$ is shown in Figure 5.6(d).

For anti-symmetric laminates the terms $A_{16} = A_{26} = D_{16} = D_{26} = 0$. The proof is left to the readers as an exercise.

***Balanced Laminates*:**

A laminate is called balanced laminate when it has pairs of plies with same thickness and material and the angles of plies are and . However, the balanced laminate can also have layers

$+\theta$ $-\theta$

oriented at 0° and 90°. For this laminate also $A_{16} = A_{26}$ are zero. It should be noted that angle-ply laminates are balanced laminates. For example, $[30/60/-45/-30/-60/45]$ is shown in Figure 5.6(e).

***Specially Orthotropic Laminates*:**

The laminates for which the terms $A_{16} = A_{26}$ are zero are called specially orthotropic laminates. It is clear that such laminates do not show coupling between in-plane extensional and shear responses.

Note that the cross-ply, angle-ply and anti-symmetric laminates are specially orthotropic laminates. These laminates by their design have $A_{16} = A_{26} = 0$. For cross-ply laminates, the terms $\overline{Q}_{16}$ and $\overline{Q}_{26}$ are identically zero. Hence, there is no restriction on the lamina thickness for cross-ply laminate to be a specially orthotropic laminate. However, for an angle ply and anti-symmetric laminates the thicknesses of a pair of $+\theta$ and – θ laminate should be same.

Other specially orthotropic laminates includes the combination of cross-ply and angle ply laminates .

For example,

$$[0/\pm\theta/90]_s \quad [90/\pm\theta/0]_s$$
$$[\pm\theta/90]_s \quad [\pm\theta/0/90]_s$$

***Quasi-Isotropic Laminates*:**

A laminate is called quasi-isotropic when its extensional stiffness matrix behaves like an isotropic material. This requires that $A_{11} = A_{22}$, $A_{16} = A_{26} = 0$ and $A_{66} = (A_{11} - A_{12})/2$. Further, this extensional stiffness matrix is independent of orientation of layers in laminate. This requires a laminate with $N \geq 3$ equal thickness layers and *N* equal angles between adjacent fibre orientations. The *N* equal angles, $\Delta\theta$ between the fibre orientations in this case can be given as

$$\Delta\theta = \frac{\pi}{N} \qquad (5.53)$$

The quasi-isotropic laminate with this construction for *N*=3, 4 and 6 will have fibre orientations as shown in Figure 5.8.

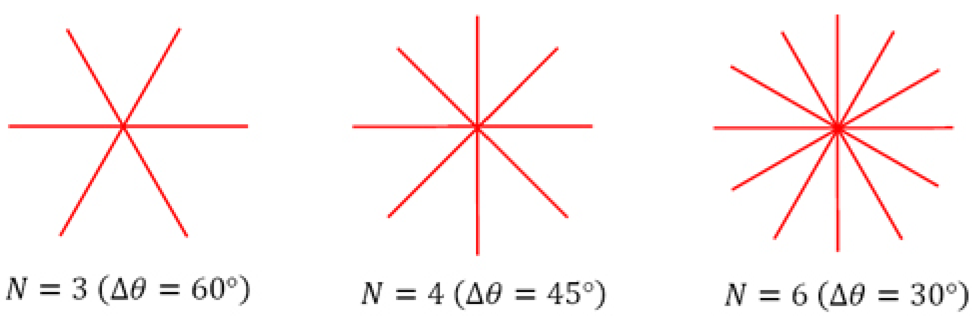

Figure 5.8: Fibre orientations in a typical quasi-isotropic laminates

It should be noted that the isotropy in these laminates is in-plane only. The matrices *B* and *D* may not behave like an isotropic material. Hence, such laminates are quasi-isotropic in nature.

Some examples of quasi-isotropic laminate are: $[0/\pm 60]_S$, $[0/\pm 45/90]_S$.

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Examples:

Calculate $[A]$, $[B]$ and $[D]$ for following laminate sequences.

***Note*:** In all following examples, each lamina has a thickness of 1 *mm* and material is AS4/3501-6 Epoxy from Soden et al [4].

Example 5.1: *Cross-ply laminate with two layers* $[0/90]$

Solution: The Q matrix for this material is calculated as

$$[Q]=\begin{bmatrix}126.68 & 3.10 & 0\\ 3.10 & 11.07 & 0\\ 0 & 0 & 6.60\end{bmatrix}GPa$$

Now, $\overline{Q}$ matrix for $0°$ and $90°$ is calculated as

$$\left[\overline{Q}(0)\right]=\begin{bmatrix}126.68 & 3.10 & 0\\ 3.10 & 11.07 & 0\\ 0 & 0 & 6.60\end{bmatrix}GPa,\quad \left[\overline{Q}(90)\right]=\begin{bmatrix}11.07 & 3.10 & 0\\ 3.10 & 126.68 & 0\\ 0 & 0 & 6.60\end{bmatrix}GPa$$

Equation (5.51) can be used for $\left[\overline{Q}(90)\right]$ calculation.

For this laminate, $z_0=-1, z_1=0 \text{ and } z_2=1$, as shown in Figure 5.9(a). The entries of $[A]$, $[B]$ and $[D]$ are calculated as:

$$A_{ij}=\sum_{k=1}^{2}\left[\overline{Q}\right]^k(t_k-t_{k-1})=\left[\overline{Q}(0)\right]\left(0-(-1)\right)+\left[\overline{Q}(90)\right](1-0)$$

that is,

$$A_{ij}=1\left[\overline{Q}(0)\right]+\ 1\left[\overline{Q}(90)\right]=\begin{bmatrix}137.75 & 6.20 & 0\\ 6.20 & 137.75 & 0\\ 0 & 0 & 13.2\end{bmatrix}GPa-mm$$

$$B_{ij}=\frac{1}{2}\sum_{k=1}^{2}\left[\overline{Q}\right]^k(t_k^2-t_{k-1}^2)=\frac{1}{2}\left\{\left[\overline{Q}(0)\right]\left(0-(1)\right)+\left[\overline{Q}(90)\right](1-0)\right\}$$

which gives

$$B_{ij}=-\frac{1}{2}\left[\overline{Q}(0)\right]+\frac{1}{2}\left[\overline{Q}(90)\right]=\begin{bmatrix}-57.805 & 0 & 0\\ 0 & 57.805 & 0\\ 0 & 0 & 0\end{bmatrix}GPa-mm^2$$

and

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{2}\left[\overline{Q}\right]^{k}(t_k^3 - t_{k-1}^3) = \frac{1}{3}\{[\overline{Q}(0)]\big(0-(-1)\big) + [\overline{Q}(90)](1-0)\}$$

which gives,

$$D_{ij} = \frac{1}{3}\left\{[\overline{Q}(0)] + [\overline{Q}(90)]\right\} = \begin{bmatrix} 45.916 & 2.066 & 0 \\ 2.066 & 45.916 & 0 \\ 0 & 0 & 4.4 \end{bmatrix} GPa - mm^3$$

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***Example 5.2*: *Cross-ply laminate with three layers* $[0/90/0]$**

Solution: $[Q]$, $\left[\bar{Q}(0)\right]$ and $\left[\bar{Q}(90)\right]$ will be the same as in Example 5.1. For this laminate $z_0 = -1.5, z_1 = -0.5, z_2 = 0.5$ and $z_3 = 1.5$ as shown in Figure 5.9(b). The entries of $[A]$, $[B]$ and $[D]$ are calculated as:

$$A_{ij} = \sum_{k=1}^{3} \left[\bar{Q}\right]^k (t_k - t_{k-1})$$
$$= \left[\bar{Q}(0)\right]\left(-0.5-(-1.5)\right) + \left[\bar{Q}(90)\right]\left(0.5-(-0.5)\right) + \left[\bar{Q}(0)\right](1.5-0.5)$$

that is,

$$A_{ij} = 2\left[\bar{Q}(0)\right] + 1\left[\bar{Q}(90)\right] = \begin{bmatrix} 264.43 & 9.30 & 0 \\ 9.30 & 264.43 & 0 \\ 0 & 0 & 19.80 \end{bmatrix} GPa-mm$$

Now we calculate

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{3} \left[\bar{Q}\right]^k (t_k^2 - t_{k-1}^2)$$
$$= \frac{1}{2}\left\{\left[\bar{Q}(0)\right]\left(0.25-(2.25)\right) + \left[\bar{Q}(90)\right]\left(0.25-(0.25)\right) + \left[\bar{Q}(0)\right](2.25-0.25)\right\}$$

That results in $[B] = [0]\ GPa-mm^2$.

Note: $[0/90/0]$ is a symmetric laminate. Hence, $[B] = [0]$ can be directly written without any calculations.

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{3} \left[\bar{Q}\right]^k (t_k^3 - t_{k-1}^3)$$
$$= \frac{1}{3}\left\{\left[\bar{Q}(0)\right]\left(-0.125-(-3.375)\right) + \left[\bar{Q}(90)\right]\left(0.125-(-0.125)\right) + \left[\bar{Q}(0)\right](3.375-0.125)\right\}$$

Thus,

$$D_{ij} = \frac{2}{3}\ 3.25\left[\bar{Q}(0)\right] + \frac{1}{3}0.25\left[\bar{Q}(90)\right] = \begin{bmatrix} 275.395 & 6.975 & 0 \\ 6.975 & 34.542 & 0 \\ 0 & 0 & 14.85 \end{bmatrix} GPa-mm^3$$

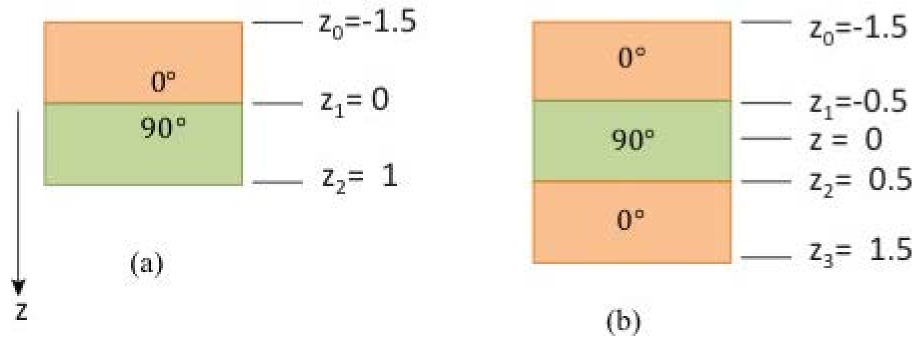

Figure 5.9: Example problems (a) Example 5.1 $[0/90]$ laminate (b) Example 5.2 $[0/90/0]$ laminate

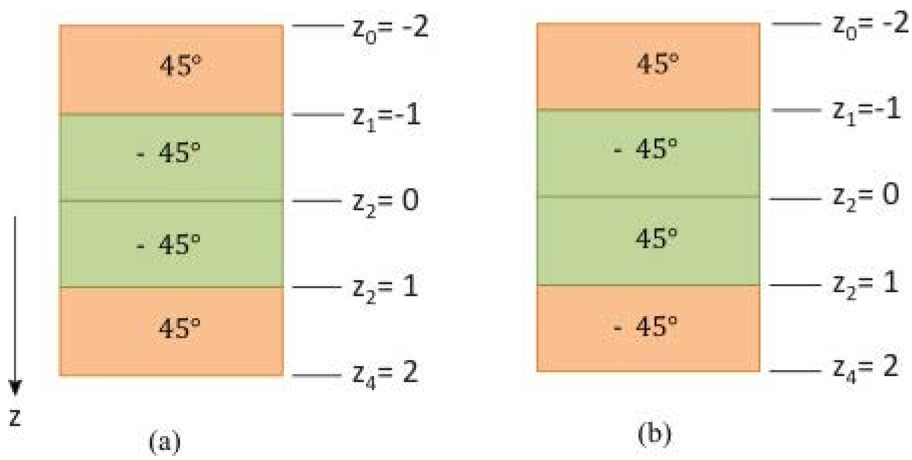

Figure 5.10: Example problems (a) Example 5.3 $[\pm 45]_S$ laminate (b) Example 5.4 $[45/-45/45/-45]$ laminate

Example 5.3: Angle-ply symmetric laminate $[\pm 45]_S$.

Solution: The coordinates for this laminate are shown in Figure 5.10 (a). For this laminate $z_0 = -2, z_1 = -1, z_2 = 0, z_3 = 1, z_2 = 2$. $[Q]$ is the same as in Example 5.1. Now, $[\overline{Q}]$ for $+45°$ and $-45°$ is calculated below.

$$[\overline{Q}(+45)] = \begin{bmatrix} 42.64 & 29.44 & 28.95 \\ 29.44 & 42.64 & 28.95 \\ 28.95 & 28.95 & 32.94 \end{bmatrix} GPa, \quad [\overline{Q}(-45)] = \begin{bmatrix} 42.64 & 29.44 & -28.95 \\ 29.44 & 42.64 & -28.95 \\ -28.95 & -28.95 & 32.94 \end{bmatrix} GPa$$

$$\begin{aligned} A_{ij} &= 1\left(\left[\overline{Q}_{ij}(+45)\right] + \left[\overline{Q}_{ij}(-45)\right] + \left[\overline{Q}_{ij}(-45)\right] + \left[\overline{Q}_{ij}(+45)\right]\right) \\ &= 2\left(\left[\overline{Q}_{ij}(+45)\right] + \left[\overline{Q}_{ij}(-45)\right]\right) \end{aligned}$$

Thus,

$$[A] = \begin{bmatrix} 170.56 & 117.76 & 0 \\ 117.76 & 170.56 & 0 \\ 0 & 0 & 131.76 \end{bmatrix} GPa - mm$$

Since the laminate is symmetric, $[B] = [0]\ GPa - mm^2$.

$$\begin{aligned} D_{ij} &= \frac{1}{3}\Big(\left[\overline{Q}_{ij}(+45)\right]((-1)^3 - (-2)^3) + \left[\overline{Q}_{ij}(-45)\right]((0)^3 - (-1)^3) + \\ &\qquad \left[\overline{Q}_{ij}(-45)\right]((1)^3 - (0)^3) + \left[\overline{Q}_{ij}(+45)\right]((2)^3 - (1)^3)\Big) \\ &= \frac{14}{3}\left[\overline{Q}_{ij}(+45)\right] + \frac{2}{3}\left[\overline{Q}_{ij}(-45)\right] \end{aligned}$$

Putting the values of $[\overline{Q}(+45)]$ and $[\overline{Q}(-45)]$, we get

$$[D] = \begin{bmatrix} 227.413 & 157.013 & 115.80 \\ 157.013 & 227.413 & 115.80 \\ 115.800 & 115.800 & 175.68 \end{bmatrix} GPa - mm^3$$

Note: In this example, the middle two $-45°$ layers of 1 *mm* thickness can be treated as one layer of $-45°$layer with 2 *mm* thickness. The *A*, *B* and *D* matrices should be the same. The readers are suggested to check this.

Example 5.4: ***Angle-ply anti-symmetric laminate*** $[45/-45/45/-45]$

Solution: The coordinates for this laminate is shown in Figure 5.10(b). Here, $z_0 = -2, z_1 = -1, z_2 = 0, z_3 = 1, z_2 = 2$. $[Q]$ is the same as in Example 5.1. $[\overline{Q}]$ for $+45°$ and $-45°$ are the same as in Example 5.3.

$[A]$ is the same as in Example 5.3.

$$B_{ij}=\frac{1}{2}\sum_{k=1}^{4}\left[\overline{Q}\right]^{k}(t_k^2-t_{k-1}^2)$$
$$=\frac{1}{2}\Big(\left[\overline{Q}_{ij}(+45)\right]((-1)^2-(-2)^2)+\left[\overline{Q}_{ij}(-45)\right]((0)^2-(-1)^2)+$$
$$\left[\overline{Q}_{ij}(45)\right]((1)^2-(0)^2)+\left[\overline{Q}_{ij}(-45)\right]((2)^2-(1)^2)\Big)$$

This gives,

$$[B]=\begin{bmatrix}0 & 0 & -57.90\\ 0 & 0 & -57.90\\ -57.90 & -57.90 & 0\end{bmatrix} GPa-mm^2$$

Now we calculate matrix *D* as

$$D_{ij}=\frac{1}{3}\sum_{k=1}^{4}\left[\overline{Q}\right]^{k}(t_k^3-t_{k-1}^2)$$
$$=\frac{1}{3}\Big(\left[\overline{Q}_{ij}(+45)\right]((-1)^3-(-2)^3)+\left[\overline{Q}_{ij}(-45)\right]((0)^3-(-1)^3)+$$
$$\left[\overline{Q}_{ij}(45)\right]((1)^3-(0)^3)+\left[\overline{Q}_{ij}(-45)\right]((2)^3-(1)^3)\Big)$$

Putting $\left[\overline{Q}_{ij}(+45)\right]$ and $\left[\overline{Q}_{ij}(-45)\right]$ gives

$$[D]=\begin{bmatrix}227.413 & 157.013 & 0\\ 157.013 & 227.413 & 0\\ 0 & 0 & 175.68\end{bmatrix} GPa-mm^3$$

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Example 5.5: $[(0)_2/(45)_3]$ ***laminate***

Solution: This is a general laminate and does not follow in any category of classification. Hence, we can call it as an unsymmteric laminate. Further, in this laminate there are two layers of $0°$ and 3 layers of $45°$ fibre orientation. Since, thickness of each layer is 1 *mm*, we can consider this laminate as one layer of fibre orientation with thickness of 2 *mm* and one layer of $45°$ fibre orientation with thickness of 3 *mm*. This is shown in Figure 5.11.

Here, $z_0 = -2.5, z_1 = -0.5$ and $z_2 = 2.5$. $\left[\overline{Q}_{ij}(0)\right]$ and $\left[\overline{Q}_{ij}(+45)\right]$ are the same as in earlier examples.

A matrix is calculated as

$$
\begin{aligned}
A_{ij} &= \sum_{k=1}^{2} \left[\overline{Q}\right]^k (t_k - t_{k-1}) \\
&= \left[\overline{Q}_{ij}(0)\right]\left(-0.5-(-2.5)\right) + \left[\overline{Q}_{ij}(+45)\right]\left(2.5-(-0.5)\right) \\
&= 2\left[\overline{Q}_{ij}(0)\right] + 3\left[\overline{Q}_{ij}(+45)\right]
\end{aligned}
$$

This gives us

$$
[A] = \begin{bmatrix} 381.28 & 94.52 & 86.85 \\ 94.52 & 150.06 & 86.85 \\ 86.85 & 86.85 & 112.02 \end{bmatrix} GPa - mm
$$

Matrix *B* is calculated as

$$
\begin{aligned}
B_{ij} &= \frac{1}{2}\sum_{k=1}^{2} \left[\overline{Q}_{ij}\right]^k (t_k^2 - t_{k-1}^2) \\
&= \frac{1}{2}\left\{\left[\overline{Q}_{ij}(0)\right]\left((-0.5)^2-(-2.5)^2\right) + \left[\overline{Q}_{ij}(+45)\right]\left((2.5)^2-(-0.5)^2\right)\right\} \\
&= \frac{6}{2}\left\{-\left[\overline{Q}_{ij}(0)\right] + \left[\overline{Q}_{ij}(+45)\right]\right\}
\end{aligned}
$$

Thus,

$$
[B] = \begin{bmatrix} -252.12 & 79.02 & 86.85 \\ 79.02 & 94.71 & 86.85 \\ 86.85 & 86.85 & 79.02 \end{bmatrix} GPa - mm^2
$$

Now matrix *D* is calculated as

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{2}\left[\overline{Q}_{ij}\right]^{k}(t_k^3 - t_{k-1}^3)$$
$$= \frac{1}{3}\left\{\left[\overline{Q}_{ij}(0)\right]((-0.5)^3 - (-2.5)^3) + \left[\overline{Q}_{ij}(+45)\right]((2.5)^3 - (-0.5)^3)\right\}$$
$$= \frac{15.5}{3}\left\{\left[\overline{Q}_{ij}(0)\right] + \left[\overline{Q}_{ij}(+45)\right]\right\}$$

which gives,

$$[D] = \begin{bmatrix} 874.82 & 168.12 & 149.57 \\ 168.12 & 94.68 & 149.57 \\ 149.57 & 149.57 & 204.29 \end{bmatrix} GPa - mm^3$$

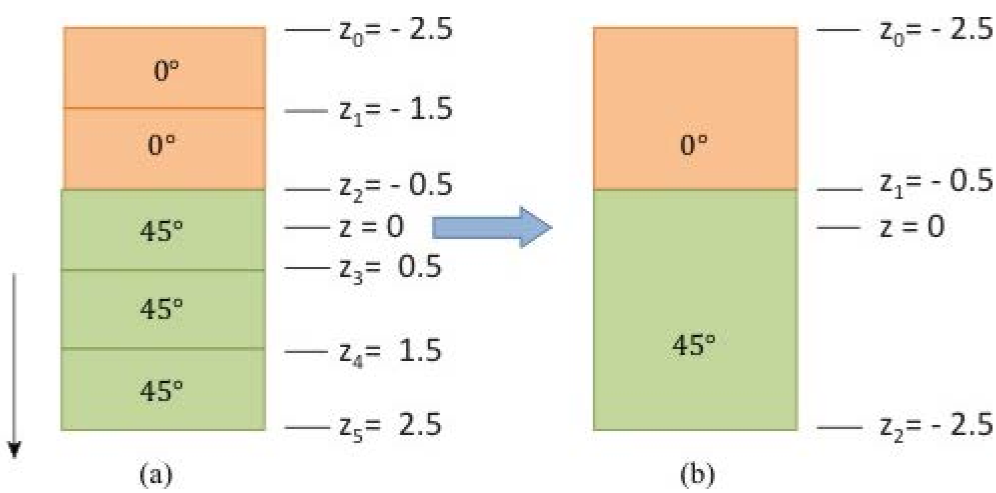

Figure 5.11: Example problem 5.5, (a) Actual laminate (b) Equivalent laminate

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Example 5.6: Find the partially and fully inverted form of laminate constitutive equation for laminate in Example 5.5.

Solution:
First we find $[A]^{-1}$ as

$$[A^*] = [A^{-1}] = 10^{-3}\begin{bmatrix} 3.27 & -1.07 & -1.70 \\ -1.07 & 12.44 & -8.81 \\ -1.70 & -8.81 & 17.08 \end{bmatrix}$$

$[B^*]$ is calculated as

$$[B^*] = -[A^{-1}][B] = -10^{-3}\begin{bmatrix} 3.27 & -1.07 & -1.70 \\ -1.07 & 12.44 & -8.81 \\ -1.70 & -8.81 & 17.08 \end{bmatrix}\begin{bmatrix} -252.12 & 79.02 & 86.85 \\ 79.02 & 94.71 & 86.85 \\ 86.85 & 86.85 & 79.02 \end{bmatrix}$$

$$[B^*] = \begin{bmatrix} 1.061 & -0.009 & -0.056 \\ -0.489 & -0.327 & -0.291 \\ -1.218 & -0.514 & -0.436 \end{bmatrix}$$

$[C^*]$ is given as $[C^*] = [B][A^{-1}] = -[B^*]^T$

$$[C^*] = \begin{bmatrix} -1.061 & 0.489 & 1.218 \\ 0.009 & 0.327 & 0.514 \\ 0.056 & 0.291 & 0.436 \end{bmatrix}$$

$[D^*]$ is calculated as $[D^*] = [D] - [B][A^{-1}][B] = [D] + [B][B^*]$

Thus,

$$[D^*] = \begin{bmatrix} 874.82 & 168.12 & 149.57 \\ 168.12 & 94.68 & 149.57 \\ 149.57 & 149.57 & 204.29 \end{bmatrix}$$

$$+\begin{bmatrix} -252.12 & 79.02 & 86.85 \\ 79.02 & 94.71 & 86.85 \\ 86.85 & 86.85 & 79.02 \end{bmatrix}\begin{bmatrix} 1.061 & -0.009 & -0.056 \\ -0.489 & -0.327 & -0.291 \\ -1.218 & -0.514 & -0.436 \end{bmatrix}$$

$$= \begin{bmatrix} 466.88 & 102.22 & 105.33 \\ 102.22 & 204.66 & 82.11 \\ 105.33 & 82.11 & 142.40 \end{bmatrix}$$

Now, the partially inverted constitutive equation for laminate can be written as

$$\begin{Bmatrix} \epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} 0.00327 & -0.00107 & -0.00170 & 1.061 & -0.009 & -0.056 \\ -0.00107 & 0.01244 & -0.00881 & -0.489 & -0.327 & -0.291 \\ -0.00170 & -0.00881 & 0.01708 & -1.218 & -0.514 & -0.436 \\ -1.061 & 0.489 & 1.218 & 466.88 & 102.22 & 105.33 \\ 0.009 & 0.327 & 0.514 & 102.22 & 204.66 & 82.11 \\ 0.056 & 0.291 & 0.436 & 105.33 & 82.11 & 142.40 \end{bmatrix} \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix}$$

Now we will obtain the fully inverted laminate constitutive equation.

$$[D'] = [D^{*-1}] = \begin{bmatrix} 0.00264 & -0.00069 & -0.00155 \\ -0.00069 & 0.00654 & -0.00325 \\ -0.00155 & -0.00325 & 0.01005 \end{bmatrix}$$

$$[B'] = [B^*][D^{*-1}]$$

$$= \begin{bmatrix} 1.061 & -0.009 & -0.056 \\ -0.489 & -0.327 & -0.291 \\ -1.218 & -0.514 & -0.436 \end{bmatrix} \begin{bmatrix} 0.00264 & -0.00069 & -0.00155 \\ -0.00069 & 0.00654 & -0.00325 \\ -0.00155 & -0.00325 & 0.01005 \end{bmatrix}$$

$$= \begin{bmatrix} 0.00289 & -0.00061 & -0.00218 \\ -0.00061 & -0.00085 & -0.00109 \\ -0.00218 & -0.00109 & -0.00081 \end{bmatrix}$$

$$[C'] = [B']^T = \begin{bmatrix} 0.00289 & -0.00061 & -0.00218 \\ -0.00061 & -0.00085 & -0.00109 \\ -0.00218 & -0.00109 & -0.00081 \end{bmatrix}$$

$$[A'] = [A^*] - [B^*][D^*]^{-1}[C^*] = [A^*] - [B'][C^*]$$

$$= \begin{bmatrix} 0.00327 & -0.00107 & -0.00170 \\ -0.00107 & 0.01244 & -0.00881 \\ -0.00170 & -0.00881 & 0.01708 \end{bmatrix}$$

$$- \begin{bmatrix} 0.00289 & -0.00061 & -0.00218 \\ -0.00061 & -0.00085 & -0.00109 \\ -0.00218 & -0.00109 & -0.00081 \end{bmatrix} \begin{bmatrix} -1.061 & 0.489 & 1.218 \\ 0.009 & 0.327 & 0.514 \\ 0.056 & 0.291 & 0.436 \end{bmatrix}$$

$$= \begin{bmatrix} 0.00647 & -0.00165 & -0.00396 \\ -0.00165 & 0.01334 & -0.00714 \\ -0.00396 & -0.00714 & 0.02066 \end{bmatrix}$$

Thus, the fully inverted laminate constitutive equation can be written as

$$\begin{Bmatrix} \epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \\ \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \end{Bmatrix} = \begin{bmatrix} 0.00647 & -0.00165 & -0.00396 & 0.00289 & -0.00061 & -0.00218 \\ -0.00165 & 0.01334 & -0.00714 & -0.00061 & -0.00085 & -0.00109 \\ -0.00396 & -0.00714 & 0.02066 & -0.00218 & -0.00109 & -0.00081 \\ 0.00289 & -0.00061 & -0.00218 & 0.00264 & -0.00069 & -0.00155 \\ -0.00061 & -0.00085 & -0.00109 & -0.00069 & 0.00654 & -0.00325 \\ -0.00218 & -0.00109 & -0.00081 & -0.00155 & -0.00325 & 0.01005 \end{bmatrix} \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}$$

Example 5.7: Consider Example 5.3. Let this laminate be subjected to the forces $N_{xx} = 1000\ N/mm$, $N_{yy} = 500\ N/mm$ and $N_{xy} = 100\ N/mm$. Calculate global strains and stresses in each ply.

Solution: The laminate in this example is a symmetric laminate. Hence, *B* matrix is zero. It means that there is no coupling between extension and bending actions. Thus, the applied stresses will produce only in-plane and shear strains and it will not produce any curvatures. Thus, it is easy to understand that the mid-plane strains will be the strains in each ply.

We can find the mid-plane strains as follows:

$$\begin{aligned}\{N\} &= [A]\{\epsilon^{(0)}\} + [B]\{\kappa\} \\ &= [A]\{\epsilon^{(0)}\}\end{aligned}$$

This gives

$$\{\epsilon^{(0)}\} = [A]^{-1}\{N\}$$

Thus,

$$\begin{Bmatrix}\epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)}\end{Bmatrix} = 10^{-3}\begin{bmatrix}0.01120 & -0.00773 & 0 \\ -0.00773 & 0.01120 & 0 \\ 0 & 0 & 0.00759\end{bmatrix}\begin{Bmatrix}1000 \\ 500 \\ 100\end{Bmatrix} = 10^{-3}\begin{Bmatrix}7.335 \\ -2.130 \\ 0.759\end{Bmatrix}$$

The strains are same in all layers. However, the stresses in each layer will be different as their stiffnesses are different.

Stresses in $+45°$ layer are

$$\begin{Bmatrix}\sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy}\end{Bmatrix}_{(+45)} = 10^{-3}\begin{bmatrix}42.63 & 29.43 & 28.94 \\ 29.43 & 42.63 & 28.94 \\ 28.94 & 28.94 & 32.93\end{bmatrix}\begin{Bmatrix}7.335 \\ -2.130 \\ 0.759\end{Bmatrix} = \begin{Bmatrix}0.2719 \\ 0.1471 \\ 0.1756\end{Bmatrix} GPa$$

And stresses in $-45°$ layer are

$$\begin{Bmatrix}\sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy}\end{Bmatrix}_{(-45)} = 10^{-3}\begin{bmatrix}42.63 & 29.43 & -28.94 \\ 29.43 & 42.63 & -28.94 \\ -28.94 & -28.94 & 32.93\end{bmatrix}\begin{Bmatrix}7.335 \\ -2.130 \\ 0.759\end{Bmatrix} = \begin{Bmatrix}0.2281 \\ 0.1031 \\ -0.1256\end{Bmatrix} GPa$$

Now, let us find the strains and stresses in principal material directions as well for these laminae.

Let us transform the strains in $+45°$ layer as

$$\{\epsilon\}_{12} = [T_2(+45)]\{\epsilon\}_{xy}$$

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ -1.0 & 1.0 & 0.0 \end{bmatrix} \begin{Bmatrix} \epsilon_{xx}^{(0)} \\ \epsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix}$$

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{Bmatrix} 0.00298 \\ 0.00222 \\ -0.00946 \end{Bmatrix}$$

Similarly, the strains in $-45°$ layer in principal directions are

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{Bmatrix} 0.00222 \\ 0.00298 \\ 0.00946 \end{Bmatrix}$$

Now, stresses in principal directions in $+45°$ layer are

$$\{\sigma\}_{12} = [T_1(+45)]\{\sigma\}_{xy}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} = \begin{Bmatrix} 0.29728 \\ 0.12166 \\ -0.12500 \end{Bmatrix} GPa$$

And stresses in principal material directions for $-45°$ layer are

$$\{\sigma\}_{12} = [T_1(-45)]\{\sigma\}_{xy}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} = \begin{Bmatrix} 0.29114 \\ 0.03990 \\ 0.06250 \end{Bmatrix} GPa$$

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Homework:

1. Write laminate constitutive equation and obtain its partially and fully inverted form.

2. What are the types of laminate?

3. Differentiate between symmetric and unsymmteric laminates.

4. What is an unsymmteric and antisymmetric laminate? Are they the same?

5. For antisymmetric laminates show that the terms $A_{16}, A_{26}, D_{16}, D_{26}$ are zero.

6. Show that for a symmetric laminate there is no coupling between extension and bending responses.

7. Classify the following laminates
    a. $[-60/30/-30/60]$
    b. $[45/0/45]$
    c. $[\pm 22.5/\mp 22.5]$
    d. $[90/0]$
    e. $[0/90/0/90]$
    f. $[0/90/0/90/0]$

8. Write an example for following laminates:
    a. Antisymmetric laminate
    b. Cross-ply
    c. Cross-ply symmetric
    d. Angle ply symmetric
    e. Balanced angle ply
    f. Quasi-isotropic

g. Specially orthotropic

9. For the composite material T300/5208, calculate the $[A], [B]$ and $[D]$ for the following laminates. The thickness of each lamina is 0.1 mm.

   a. $\left[(0/90)_2\right]$

   b. $\left[(0/90)\right]_s$

   c. $[\pm 45]$

   d. $[\pm 45]_s$

   e. $[0/+30/-30/30/0]$

10. For the above laminate sequences calculate the compliance relation (for midplane strains and curvatures). Develop a computer code for this.

11. Using the code developed in exercise (10), verify the solutions given for Example 5.6 and Example 5.7.

12. Show that the T300/5208 $[0/\pm 60]$ laminate is a quasi-isotropic laminate. Is it an isotropic laminate?

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References:

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