Module 3: 3D Constitutive Equations

Lecture 11: Constitutive Relations: Transverse Isotropy and Isotropy

The Lecture Contains:

Transverse Isotropy:

Introduction:

In this lecture, we are going to see some more simplifications of constitutive equation and develop the relation for isotropic materials.

First we will see the development of transverse isotropy and then we will reduce from it to isotropy.

First Approach: Invariance Approach

This is obtained from an orthotropic material. Here, we develop the constitutive relation for a material with transverse isotropy in x_2-x_3 plane (this is used in lamina/laminae/laminate modeling). This is obtained with the following form of the change of axes.

$$\begin{aligned} x_1^{'} &= x_1 \\ x_2^{'} &= x_2 \cos\alpha + x_3 \sin\alpha \\ x_3^{'} &= -x_2 \sin\alpha + x_3 \cos\alpha \end{aligned} \quad (3.30)$$

Now, we have

$$\frac{\partial x_1^{'}}{\partial x_1} = 1, \quad \frac{\partial x_2^{'}}{\partial x_2} = \frac{\partial x_3^{'}}{\partial x_3} = \cos\alpha, \quad \frac{\partial x_2^{'}}{\partial x_3} = -\frac{\partial x_3^{'}}{\partial x_2} = \sin\alpha,$$
$$\frac{\partial x_1^{'}}{\partial x_2} = \frac{\partial x_1^{'}}{\partial x_3} = \frac{\partial x_2^{'}}{\partial x_1} = \frac{\partial x_3^{'}}{\partial x_1} = 0$$

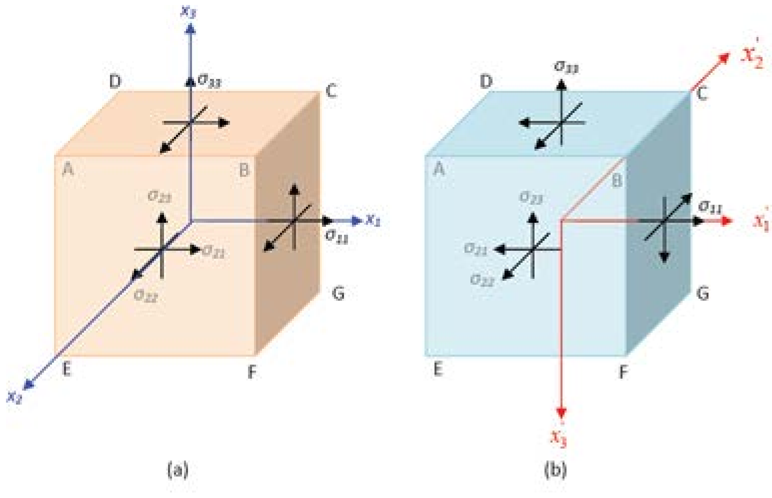

Figure 3.6: State of stress (a) in x_1, x_2, x_3 system (b) with x_1-x_2 and x_1-x_3 planes of symmetry

From this, the strains in transformed coordinate system are given as:

$$\begin{aligned}
\varepsilon_{11}^{'} &= \varepsilon_{11} \\
\varepsilon_{22}^{'} &= \varepsilon_{22}\cos^2\alpha + 2\varepsilon_{23}\cos\alpha\sin\alpha + \varepsilon_{33}\sin^2\alpha \\
\varepsilon_{33}^{'} &= \varepsilon_{22}\sin^2\alpha - 2\varepsilon_{23}\cos\alpha\sin\alpha + \varepsilon_{33}\cos^2\alpha \\
\varepsilon_{23}^{'} &= \left(\varepsilon_{33} - \varepsilon_{22}\right)\cos\alpha\sin\alpha + \varepsilon_{23}\left(\cos^2\alpha - \sin^2\alpha\right) \\
\varepsilon_{13}^{'} &= -\varepsilon_{12}\sin\alpha + \varepsilon_{13}\cos\alpha \\
\varepsilon_{12}^{'} &= \varepsilon_{12}\cos\alpha + \varepsilon_{13}\sin\alpha
\end{aligned} \tag{3.31}$$

Here, it is to be noted that the shear strains are the tensorial shear strain terms.

For any angle α,

$$\begin{aligned}
&\varepsilon_{22} + \varepsilon_{33} = \varepsilon_{22}^{'} + \varepsilon_{33}^{'}, \\
&\varepsilon_{22}\varepsilon_{33} - \left(\varepsilon_{23}\right)^2 = \varepsilon_{22}^{'}\varepsilon_{33}^{'} - \left(\varepsilon_{23}^{'}\right)^2, \\
&\left(\varepsilon_{12}\right)^2 + \left(\varepsilon_{13}\right)^2 = \left(\varepsilon_{12}^{'}\right)^2 + \left(\varepsilon_{13}^{'}\right)^2, \\
&\left|\varepsilon_{ij}\right| = \left|\varepsilon_{ij}^{'}\right|
\end{aligned} \tag{3.32}$$

and therefore, W must reduce to the form

$$W = W\left(\varepsilon_{22} + \varepsilon_{33}, \varepsilon_{22}\varepsilon_{33} - \varepsilon_{23}^2, \varepsilon_{33}, \varepsilon_{12}^2 + \varepsilon_{13}^2, \left|\varepsilon_{ij}\right|\right) \tag{3.33}$$

Then, for *W* to be invariant we must have

$$\begin{aligned}
&W\left(\varepsilon_{11}, \varepsilon_{22} + \varepsilon_{33}, \varepsilon_{22}\varepsilon_{33} - \varepsilon_{23}^2, \varepsilon_{12}^2 + \varepsilon_{13}^2, \left|\varepsilon_{ij}\right|\right) \\
&= W\left(\varepsilon_{11}^{'}, \varepsilon_{22}^{'} + \varepsilon_{33}^{'}, \varepsilon_{22}^{'}\varepsilon_{33}^{'} - \left(\varepsilon_{23}^{'}\right)^2, \left(\varepsilon_{12}^{'}\right)^2 + \left(\varepsilon_{13}^{'}\right)^2, \left|\varepsilon_{ij}^{'}\right|\right)
\end{aligned}$$

Now, let us write the left hand side of the above equation using the C_{ij} matrix as given in Equation (3.26) and engineering shear strains. In the following we do some rearrangement as

$$W = \begin{matrix} \left[C_{11}\varepsilon_{11}^2\right] + \left[2\varepsilon_{11}\left(C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33}\right)\right] + \left[4C_{55}\varepsilon_{13}^2 + 4C_{66}\varepsilon_{12}^2\right] + \\ \left[C_{22}\varepsilon_{22}^2 + C_{33}\varepsilon_{33}^2 + 2C_{23}\varepsilon_{22}\varepsilon_{33} + 4C_{44}\varepsilon_{23}^2\right] \end{matrix}$$

Similarly, we can write the right hand side of previous equation using rotated strain components. Now, for *W* to be invariant it must be of the form as in Equation (3.33).

1. If we observe the terms containing $\left(\varepsilon_{11}\right)^2$ and $\left(\varepsilon_{11}^{'}\right)^2$ in the first bracket, then we conclude that C_{11} is unchanged.
2. Now compare the terms in the second bracket. If we have $C_{12} = C_{13}$ then the first of Equation (3.32) is satisfied.

3. Now compare the third bracket. If we have $C_{55} = C_{66}$, then the third of Equation (3.32) is satisfied.
4. Now for the fourth bracket we do the following manipulations. Let us assume that $C_{22} = C_{33}$ and C_{23} is unchanged. Then we write the terms in fourth bracket as

$$\begin{aligned} &C_{22}\left(\varepsilon_{22}+\varepsilon_{33}\right)^2 - 2C_{22}\varepsilon_{22}\varepsilon_{33} + 2C_{23}\varepsilon_{22}\varepsilon_{33} + 4C_{44}\varepsilon_{23}^2 \\ &= C_{22}\left(\varepsilon_{22}+\varepsilon_{33}\right)^2 - 2\left[\left(C_{22}-C_{23}\right)\varepsilon_{22}\varepsilon_{33} + 2C_{44}\varepsilon_{23}^2\right] \end{aligned}$$

To have *W* to be invariant we need to have $C_{44} = \dfrac{C_{22}-C_{23}}{2}$ so that the third of Equation (3.32) is satisfied.

Thus, for transversely isotropic material (in plane x_2-x_3) the stiffness matrix becomes

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{22} & 0 & 0 & 0 \\ & & & \frac{1}{2}(C_{22}-C_{23}) & 0 & 0 \\ & \text{Symmetric} & & & C_{66} & 0 \\ & & & & & C_{66} \end{bmatrix} \qquad (3.34)$$

Thus, there are only 5 independent elastic constants for a transversely isotropic material.

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Second Approach: Comparison of Constants

This can also be verified from the elastic constants expressed in terms of engineering constants like E, ν and G. Recall the constitutive equation for orthotropic material expressed in terms of engineering constants. For the transversely isotropic materials the following relations hold.

$$E_2 = E_3, \qquad \nu_{12} = \nu_{13}$$

$$G_{12} = G_{13}, \quad G_{23} = \frac{E_2}{2(1+\nu_{23})}$$

When these relations are used in the constitutive equation for orthotropic material expressed in terms of engineering constants, the stiffness matrix relations in Equation (3.34) are verified.

Isotropic Bodies

If the function W remains unaltered in form under all possible changes to other rectangular Cartesian systems of axes, the body is said to be Isotropic. In this case, W is a function of the strain invariants. Alternatively, from the previous section, W must be unaltered in form under the transformations

$$\begin{aligned} x_1' &= x_1 \cos\alpha + x_2 \sin\alpha \\ x_2' &= -x_1 \sin\alpha + x_2 \cos\alpha \\ x_3' &= x_3 \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} x_3' &= x_3 \cos\alpha + x_1 \sin\alpha \\ x_1' &= -x_3 \sin\alpha + x_1 \cos\alpha \\ x_2' &= x_2 \end{aligned} \tag{3.36}$$

In other words, W when expressed in terms of ε_{ij}' must be obtained from Equation (3.33) simply by replacing ε_{ij} by ε_{ij}'. By analogy with the previous section it is seen that for this to be true under the transformation Equation (3.35). We can write

$$\frac{\partial x_1'}{\partial x_1} = \frac{\partial x_2'}{\partial x_2} = \cos\alpha, \quad \frac{\partial x_1'}{\partial x_2} = -\frac{\partial x_2'}{\partial x_1} = \sin\alpha,$$
$$\frac{\partial x_1'}{\partial x_3} = \frac{\partial x_2'}{\partial x_3} = \frac{\partial x_3'}{\partial x_1} = \frac{\partial x_3'}{\partial x_2} = 0, \quad \frac{\partial x_3'}{\partial x_3} = 1$$

And the transformed strains are given as

$$\begin{aligned}
\varepsilon_{11}' &= \varepsilon_{11}\cos^2\alpha + 2\varepsilon_{12}\sin\alpha\cos\alpha + \varepsilon_{22}\sin^2 \\
\varepsilon_{22}' &= \varepsilon_{11}\sin^2\alpha - 2\varepsilon_{12}\sin\alpha\cos\alpha + \varepsilon_{22}\cos^2\alpha \\
\varepsilon_{33}' &= \varepsilon_{33} \\
\varepsilon_{23}' &= -\varepsilon_{13}\sin\alpha + \varepsilon_{23}\cos\alpha \\
\varepsilon_{13}' &= \varepsilon_{13}\cos\alpha + \varepsilon_{23}\sin\alpha \\
\varepsilon_{12}' &= \left(\varepsilon_{22} - \varepsilon_{11}\right)\sin\alpha\cos\alpha + \varepsilon_{12}\left(\cos^2\alpha - \sin^2\alpha\right)
\end{aligned} \qquad (3.37)$$

Thus, for any angle α,

$$\begin{aligned}
&\varepsilon_{11} + \varepsilon_{22} = \varepsilon_{11}' + \varepsilon_{22}', \\
&\varepsilon_{11}\varepsilon_{22} - \left(\varepsilon_{12}\right)^2 = \varepsilon_{11}'\varepsilon_{22}' - \left(\varepsilon_{12}'\right)^2, \\
&\left(\varepsilon_{13}\right)^2 + \left(\varepsilon_{23}\right)^2 = \left(\varepsilon_{13}'\right)^2 + \left(\varepsilon_{23}'\right)^2, \\
&\left|\varepsilon_{ij}\right| = \left|\varepsilon_{ij}'\right|
\end{aligned} \qquad (3.38)$$

and therefore, *W* must reduce to the form

$$W = W\left(\varepsilon_{11} + \varepsilon_{22}, \varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}^2, \varepsilon_{33}, \varepsilon_{13}^2 + \varepsilon_{23}^2, \left|\varepsilon_{ij}\right|\right) \qquad (3.39)$$

Then, for *W* to be invariant we must have

$$\begin{aligned}
&W\left(\varepsilon_{11} + \varepsilon_{22}, \varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}^2, \varepsilon_{33}, \varepsilon_{13}^2 + \varepsilon_{23}^2, \left|\varepsilon_{ij}\right|\right) \\
&= W\left(\varepsilon_{11}' + \varepsilon_{22}', \varepsilon_{11}'\varepsilon_{22}' - \left(\varepsilon_{12}'\right)^2, \varepsilon_{33}', \left(\varepsilon_{13}'\right)^2 + \left(\varepsilon_{23}'\right)^2, \left|\varepsilon_{ij}'\right|\right)
\end{aligned}$$

Now, let us write the left hand side of above equation using the C_{ij} matrix as given in Equation (3.34) and engineering shear strains. In the following we do some rearrangement as

$$W = \begin{matrix}\left[C_{22}\varepsilon_{33}^2\right] + \left[2\varepsilon_{33}\left(C_{12}\varepsilon_{11} + C_{23}\varepsilon_{22}\right)\right] + \left[\dfrac{4}{2}\left(C_{22} - C_{23}\right)\varepsilon_{23}^2 + 4C_{66}\varepsilon_{13}^2\right] + \\ \left[C_{11}\varepsilon_{11}^2 + C_{22}\varepsilon_{22}^2 + 2C_{12}\varepsilon_{11}\varepsilon_{22} + 4C_{66}\varepsilon_{12}^2\right]\end{matrix} \qquad (3.40)$$

Similarly, we can write the right hand side of the previous equation using rotated strain components. Now, for *W* to be invariant it must be of the form as in Equation (3.39)

1. From the second bracket, if we propose $C_{23} = C_{12}$, then we can satisfy the first of Equation (3.38).
2. From the third bracket, third of Equation (3.38) holds true when $C_{66} = \dfrac{C_{22} - C_{23}}{2} = \dfrac{C_{22} - C_{12}}{2} = C_{44}$.
3. The fourth bracket is manipulated as follows:

$$
\begin{aligned}
& C_{11}\varepsilon_{11}^2 + C_{22}\varepsilon_{22}^2 + 2C_{12}\varepsilon_{11}\varepsilon_{22} + 4C_{66}\varepsilon_{12}^2 \\
&= C_{11}\left(\varepsilon_{11}+\varepsilon_{22}\right)^2 - 2C_{11}\varepsilon_{11}\varepsilon_{22} + 2C_{12}\varepsilon_{11}\varepsilon_{22} + 4C_{66}\varepsilon_{12}^2 \\
&= C_{11}\left(\varepsilon_{11}+\varepsilon_{22}\right)^2 - 2\left[\left(C_{11}-C_{12}\right)\varepsilon_{11}\varepsilon_{22} - 2C_{66}\varepsilon_{12}^2\right]
\end{aligned}
$$

Thus, to satisfy the second of Equation (3.38) we must have $C_{66} = \dfrac{C_{11}-C_{12}}{2}$. Further, we should have $C_{22} = C_{11}$. From our observation in 2, we can write $C_{44} = C_{55} = C_{66} = \dfrac{C_{11}-C_{12}}{2}$.

It follows automatically that W is unaltered in form under the transformation in Equation (3.36).

Thus, the stiffness matrix for isotropic material becomes as

$$
C_{ij} = \begin{bmatrix}
C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
 & C_{11} & C_{12} & 0 & 0 & 0 \\
 & & C_{11} & 0 & 0 & 0 \\
 & & & \frac{1}{2}(C_{11}-C_{12}) & 0 & 0 \\
 & \text{Symmetric} & & & \frac{1}{2}(C_{11}-C_{12}) & 0 \\
 & & & & & \frac{1}{2}(C_{11}-C_{12})
\end{bmatrix} \qquad (3.41)
$$

Thus, for an isotropic material there are only two independent elastic constants. It can be verified that W is unaltered in form under all possible changes to other rectangular coordinate systems, that is, it is the same function of $\varepsilon_{ij}^{'}$ as it is of ε_{ij} when x_i is changed to $x_i^{'}$.

Homework:

1. Starting with the stiffness matrix for transverse isotropic material, take the transformations about x_1 and x_2 and show that you get the stiffness matrix as given in Equation (3.41).

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References:

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