Module 2: Concepts of Solid Mechanics

Lecture 9: Basic Concepts

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In this lecture, we are going to introduce some concepts from solid mechanics which will be useful for better understanding of this course. It is presumed that the readers have some basic knowledge of linear algebra and solid mechanics.
In solid mechanics, each phase of a material is considered to be continuum, that is, there is no discontinuity in the material. Thus, in this course individual fibres and the matrix of a lamina/composite are considered to be continuum. Further, this results in saying that heterogeneous composite is also a continuum.

In this lecture, we will introduce some of the notations that will be followed for the rest of the course. Hence, the readers are advised to understand them clearly before they proceed to further lectures.

Concept of Tensors

Tensors are physical entities whose components are the coefficients of a linear relationship between vectors.
The list of some of the tensors used in this course is given in Table 2.1.

Table 2.1 List of some of the tensor quantities

	Quantity	Live subscripts
α	Scalar (zeroth order tensor)	0
v_i	Vector (first order tensor)	1
$\sigma_{ij}, \varepsilon_{ij}$	Second order tensor	2
C_{ijkl}	Fourth order tensor	4

It is often needed to transform a tensorial quantity from one coordinate system to another coordinate system. This transformation of a tensor is done using direction cosines of the angle measured from initial coordinate system to final coordinate system. Let us use axes x_i as the initial coordinate axes and x_i' as the final coordinate axes (denoted here by symbol prime – $'$). Now, we need to find the direction cosines (denoted here by a_{ij}) for this transformation relation. Let us use the convention for direction cosines that the first subscript (that is, i) of a_{ij} corresponds to the initial axes and the second subscript (that is, j) corresponds to final axes. The direction cosine correspondence with this convention in *3D* Cartesian coordinate system is given in Table 2.2. The corresponding Cartesian coordinate systems are shown in Figure 2.1.

Table 2.2 Direction cosines for 3D Cartesian coordinate system

From/To	x_1'	x_2'	x_3'
x_1	a_{11}	a_{12}	a_{13}
x_2	a_{21}	a_{22}	a_{23}

x_3	a_{31}	a_{32}	a_{33}

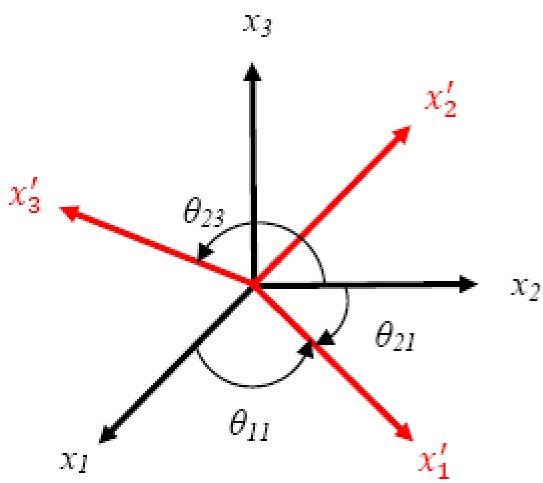

Figure 2.1 Rectangular or Cartesian coordinate systems

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Let us derive the direction cosines for a transformation in a plane. Let the coordinate axes x_1-x_2 (that is, plane 1-2) are rotated about the third axis x_3 by an angle θ as shown in Figure 2.2. Thus, from the figure it is easy to see that $\theta_{11} = \theta_{22} = \theta$. A careful observation of the figure shows that the angle between x_1 and x_2' is not the same as the angle between x_2 and x_1'. It means that the direction cosines $a_{12} \neq a_{21}$.

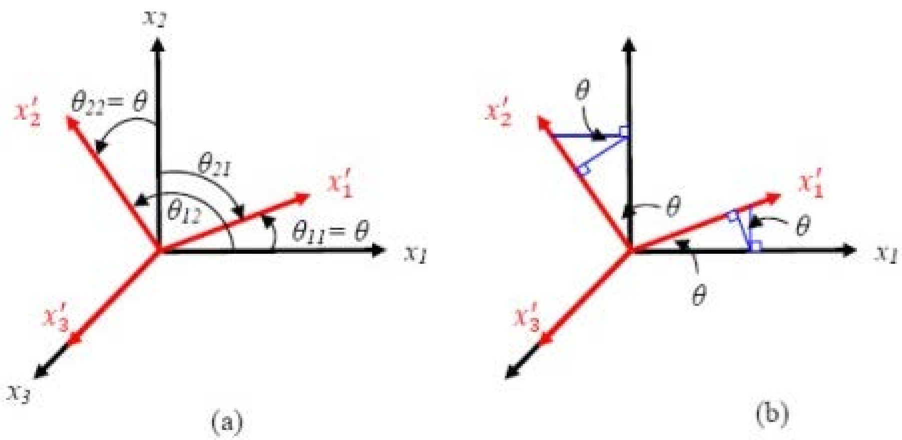

Figure 2.2 Transformation about x_3 axis

Now, we will find all the direction cosines. The list is given below.

$$\begin{aligned}
&a_{11} = \cos\theta_{11} = \cos\theta, && a_{21} = \cos\theta_{21} = \sin\theta, && a_{31} = \cos 90^\circ = 0\\
&a_{12} = \cos\theta_{12} = \cos(90^\circ + \theta) = -\sin\theta, && a_{22} = \cos\theta_{22} = \cos\theta, && a_{32} = \cos 90^\circ = 0\\
&a_{13} = \cos 90^\circ = 0, && a_{23} = \cos 90^\circ = 0, && a_{33} = \cos 0^\circ = 1
\end{aligned}$$

The above can be written in a matrix form as

$$a_{ij} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (2.1)$$

The matrix of direction cosines given above in Eq. (2.1) is also written using short forms for $\cos\theta = m$ and $\sin\theta = n$. Then Equation (2.1) becomes

$$a_{ij} = \begin{bmatrix} m & -n & 0 \\ n & m & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (2.2)$$

***Note*:** Some of the books and research articles also use $\cos\theta = c$ and $\sin\theta = s$.

***Note*:** This matrix is also called **Rotation Matrix**.

***Note*:** The above direction cosine matrix can be obtained from the relation between unrotated and rotated coordinates. For the transformation shown in Figure 2.2 (a) one can write this relation using the geometrical relations shown in Figure 2.2 (b) as

$$x_1' = x_1 \cos\theta_{11} + x_2 \sin\theta_{22}$$
$$x_2' = -x_1 \sin\theta_{11} + x_2 \cos\theta_{22}$$
$$x_3' = x_3$$

Now the direction cosines are given by the following relation:

$$a_{ji} = \frac{\partial x_i'}{\partial x_j}$$

Now we will use the direction cosines to transform a vector, a second order tensor and a fourth order tensor from initial coordinate (unprimed) system to a vector, a second order tensor and a fourth order tensor in final coordinate (primed) system.

First, let us do it for a vector. Let P_i and P_i' denote the components of a vector ***P*** in unprimed and primed coordinate axes. Then the components of this vector in rotated coordinate system are given in terms of components in unrotated coordinate system and corresponding direction cosines as

$$P_i' = a_{ji} P_j \qquad (2.3)$$

Now, putting the direction cosines in terms of angles and summing over the repeated index *j* (=1, 2, 3) in Equation (2.3) we get

$$P_i' = a_{1i} P_1 + a_{2i} P_2 + a_{3i} P_3$$
$$P_i' = \cos\theta_{1i}\, P_1 + \cos\theta_{2i}\, P_2 + \cos\theta_{3i}\, P_3 \qquad (2.4)$$

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Let us assume that, the unprimed and primed coordinate systems are as shown in Figure 2.2. The transformation matrix for this rotation is given in Equation (2.1). Then, the components P_i' can be given as

$$\begin{aligned} P_1' &= \cos\theta\, P_1 + \sin\theta\, P_2 \\ P_2' &= -\sin\theta\, P_1 + \cos\theta\, P_2 \\ P_2' &= P_3 \end{aligned}$$

Note: In two dimensional case, the above transformation is written as

$$P_i' = a_{ji} P_j = a_{1i} P_1 + a_{2i} P_2 = \cos\theta_{1i}\, P_1 + \cos\theta_{2i}\, P_2$$

Equation (2.3) can also be written in an inverted form to give the components P_i in unrotated axes in terms of components P_i' in rotated axes system as

$$P_i = a_{ji}^{-1} P_j' \qquad (2.5)$$

The rotation matrix a_{ij} in Equation (2.2) has a property that

$$a_{ji}^{-1} = a_{ij} = \left(a_{ji}\right)^T \qquad (2.6)$$

Now, we will extend the concept to transform a second order tensor. Let us transform the stress tensor σ_{ij} as follows

$$\begin{aligned} \sigma_{ij}' &= a_{ki}\, a_{lj}\, \sigma_{kl} \\ \sigma_{ij}' &= a_{1i}\, a_{1j}\, \sigma_{11} + a_{1i}\, a_{2j}\, \sigma_{12} + a_{1i}\, a_{3j}\, \sigma_{13} \\ &\quad + a_{2i}\, a_{1j}\, \sigma_{21} + a_{2i}\, a_{2j}\, \sigma_{22} + a_{2i}\, a_{3j}\, \sigma_{23} \\ &\quad + a_{3i}\, a_{1j}\, \sigma_{31} + a_{3i}\, a_{2j}\, \sigma_{32} + a_{3i}\, a_{3j}\, \sigma_{33} \\ \sigma_{ij}' &= \cos\theta_{1i}\, \cos\theta_{1j}\, \sigma_{11} + \cos\theta_{1i}\, \cos\theta_{2j}\, \sigma_{12} + \cos\theta_{1i}\, \cos\theta_{3j}\, \sigma_{13} \\ &\quad + \cos\theta_{2i}\, \cos\theta_{1j}\, \sigma_{21} + \cos\theta_{2i}\, \cos\theta_{2j}\, \sigma_{22} + \cos\theta_{2i}\, \cos\theta_{3j}\, \sigma_{23} \\ &\quad + \cos\theta_{3i}\, \cos\theta_{1j}\, \sigma_{31} + \cos\theta_{3i}\, \cos\theta_{2j}\, \sigma_{32} + \cos\theta_{3i}\, \cos\theta_{3j}\, \sigma_{33} \end{aligned} \qquad (2.7)$$

The transformation of a fourth order tensor C_{ijkl} is given as

$$C_{ijkl}' = a_{pi}\, a_{qj}\, a_{rk}\, a_{sl}\, C_{pqrs} \qquad (2.8)$$

The readers are suggested to write the final form of Equation (2.8) using similar procedure used to get the last of Equation (2.7).

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Deformation of a Body

When a deformable body is subjected to external forces, a body may translate, rotate and deform as well. Thus, after deformation the body occupies a new region. The initial region occupied by the body is called **Reference Configuration** and the new region occupied by the body after translation, rotation and deformation is called **Deformed Configuration**. Let us consider a point *P* in reference configuration. Its position with respect to origin of a reference axes system $(\boldsymbol{r})$ is shown in Figure 2.3. The point *P* occupies a new position P' and its position vector $\boldsymbol{r}'$ is also given.

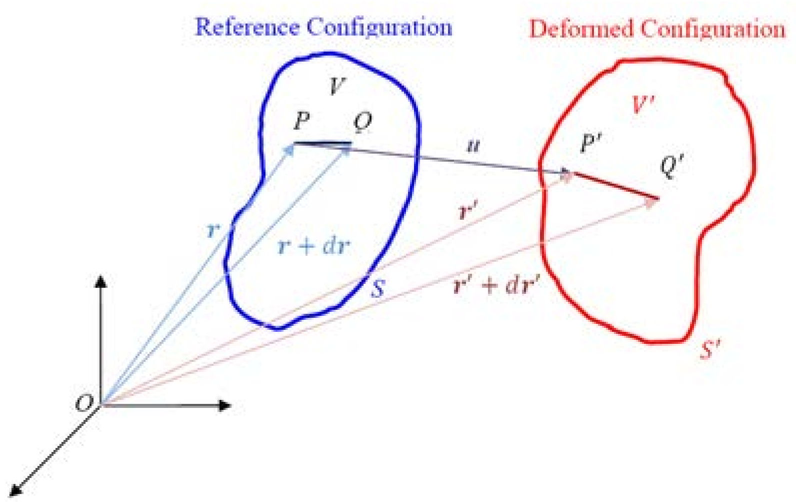

Figure 2.3 Reference and deformed configurations

The deformation map is defined as

$$\boldsymbol{r}' = \boldsymbol{r}'(\boldsymbol{r}) \qquad (2.9)$$

Thus, deformation map is a vector valued function. Similarly, for deformation of a point *Q* to Q', we can write

$$\boldsymbol{r}' + d\boldsymbol{r}' = \boldsymbol{r}'(\boldsymbol{r} + d\boldsymbol{r}) \qquad (2.10)$$

We can find the deformation $d\boldsymbol{r}'$ as

$$d\boldsymbol{r}' = \boldsymbol{r}'(\boldsymbol{r} + d\boldsymbol{r}) - \boldsymbol{r}'(\boldsymbol{r}) \approx (\nabla \boldsymbol{r}')d\boldsymbol{r} \qquad (2.11)$$

where $(\nabla \boldsymbol{r}') = \mathbf{F} = \frac{\partial \mathbf{r}'}{\partial \mathbf{r}}$ is called ***Deformation Gradient***. In component form, one can write

$$\mathrm{d}r_i' = F_{ij}\, dr_j \qquad (2.12)$$

Now, let us give the deformation map for the displacement of a point. Let us consider the point *P* in reference configuration again. It undergoes a deformation $\boldsymbol{u}(\boldsymbol{r})$ and occupies a new position P'. Thus, we can write this deformation as follows

$$\boldsymbol{u}(\boldsymbol{r}) = \boldsymbol{r}' - \boldsymbol{r} \quad \text{or} \quad \boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{u}(\boldsymbol{r}) \qquad (2.13)$$

This gives us the deformation gradient as

$$\boldsymbol{F} = \frac{\partial(\boldsymbol{r}')}{\partial(\boldsymbol{r})} = \boldsymbol{I} + \nabla\boldsymbol{u} \qquad (2.14)$$

or in component form

$$F_{ij} = \delta_{ij} + u_{i,j} \qquad (2.15)$$

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Now, we will define strain tensor. We are going to find $|d\boldsymbol{r}'|^2 - |d\boldsymbol{r}|^2$. We know that $|d\boldsymbol{r}'|^2 = d\boldsymbol{r}' \cdot d\boldsymbol{r}' = d\boldsymbol{r} \cdot \boldsymbol{F}^T \boldsymbol{F}\, d\boldsymbol{r}$. Thus,

$$\begin{aligned} |d\boldsymbol{r}'|^2 - |d\boldsymbol{r}|^2 &= d\boldsymbol{r} \cdot \boldsymbol{F}^T \boldsymbol{F}\, d\boldsymbol{r} - d\boldsymbol{r} \cdot d\boldsymbol{r} \\ &= d\boldsymbol{r} \cdot \boldsymbol{F}^T \boldsymbol{F}\, d\boldsymbol{r} - d\boldsymbol{r} \cdot \boldsymbol{I}\, d\boldsymbol{r} \\ &= d\boldsymbol{r} \cdot (\boldsymbol{F}^T \boldsymbol{F} - \boldsymbol{I})\, d\boldsymbol{r} \\ &= d\boldsymbol{r} \cdot 2\, \boldsymbol{E}\, d\boldsymbol{r} \end{aligned} \quad (2.16)$$

where ***E*** is **Lagrangian Strain Tensor**. Now using the last two of Equation (2.16) for $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T \boldsymbol{F} - \boldsymbol{I})$ we get,

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^T \boldsymbol{F} - \boldsymbol{I}) = \frac{1}{2}[(\boldsymbol{I} + (\nabla \boldsymbol{u})^{\mathrm{T}})(\boldsymbol{I} + \nabla \boldsymbol{u}) - \boldsymbol{I}] = \frac{1}{2}[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}} + (\nabla \boldsymbol{u})^{\mathrm{T}} \nabla \boldsymbol{u}] \quad (2.17)$$

This equation can be written in index form as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}\, u_{k,j}) \quad (2.18)$$

where $u_{i,j}$ is given as $u_{i,j} = \frac{\partial u_i}{\partial x_j}$. Thus, the strain components are nonlinear in u_i. Here, $u_i = (u_1, u_2, u_3) = (u, v, w)$ are the displacement components in three directions. For example, let us write the expanded form of strain components ε_{11} and ε_{23}.

$$\begin{aligned} \varepsilon_{11} &= \frac{1}{2}\left[\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1} + \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2\right] \\ &= \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2\right] \end{aligned} \quad (2.19)$$

Similarly,

$$\varepsilon_{23} = \frac{1}{2}\left[\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} + \frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3}\right] \quad (2.20)$$

The readers should observe that from the definition of strain tensor in Equation (2.18), the strain tensor is symmetric (that is, $\varepsilon_{ij} = \varepsilon_{ji}$). If the gradients of the displacements are very small the product terms in Equation (2.18) can be neglected. Then, the resulting strain tensor (called **Infinitesimal Strain Tensor**) is given as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (2.21)$$

The individual strain components are given as

(2.22)

$$\begin{aligned}
\varepsilon_{11} &= \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \\
\varepsilon_{22} &= \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{13} = \varepsilon_{31} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right), \\
\varepsilon_{33} &= \frac{\partial u_3}{\partial x_3}, \quad \varepsilon_{23} = \varepsilon_{32} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)
\end{aligned}$$

The readers are very well versed with these definitions. This strain tensor can be written in matrix form as

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \qquad (2.23)$$

Note: The shear strain components mentioned above are tensorial components. In actual practice, ***engineering shear strains*** (which are measured from laboratory tests) are used. These are denoted by γ_{ij}. The relation between tensorial and engineering shear strain components is

$$\gamma_{ij} = 2\varepsilon_{ij} \qquad (2.24)$$

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The engineering shear strain components are given as follows:

$$\begin{aligned}\gamma_{12} &= \gamma_{21} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\\ \gamma_{13} &= \gamma_{31} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\\ \gamma_{23} &= \gamma_{32} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\end{aligned} \qquad (2.25)$$

Using the engineering shear strain components, the strain tensor can be written in matrix form as

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{11} & \frac{\gamma_{12}}{2} & \frac{\gamma_{13}}{2} \\ \frac{\gamma_{21}}{2} & \varepsilon_{22} & \frac{\gamma_{23}}{2} \\ \frac{\gamma_{31}}{2} & \frac{\gamma_{32}}{2} & \varepsilon_{33} \end{bmatrix} \qquad (2.26)$$

Stress

Now, we will introduce the concept of stress. The components of stress at a point (also called **State of Stress**) are (in the limit) the forces per unit area which are acting on three mutually perpendicular planes passing through this point. This is represented in Figure 2.4. Stress tensor is a second order tensor and denoted as σ_{ij}. In this notation, the first subscript corresponds to the direction of the normal to the plane and the second subscript corresponds to the direction of the stress. For example, σ_{23} denotes the stress component acting on a plane which is perpendicular to direction 2 and stress is acting in direction 3. The tensile normal stress components $(\sigma_{11}, \sigma_{22}, \sigma_{33})$ are positive. The shear stress components $(i \neq j)$ are defined to be positive when the normal to the plane and the direction of the stress component are either both positive or both negative.
The readers should note that the state of stress shown in Figure 2.4 represents all stress components in positive sense. In this figure, the stress components are shown on positive faces only.

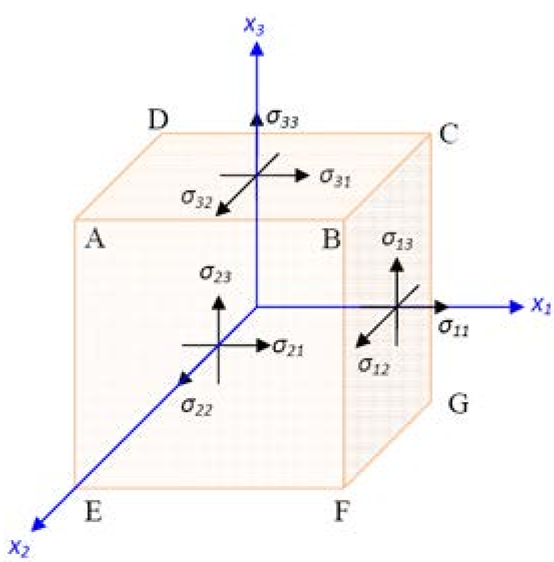

Figure 2.4 State of stress at a point

The stress tensor can be written in matrix form as follows:

$$[\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{2.27}$$

In general, instead of using global 1-2-3 coordinate system, *x*-*y*-*z* global coordinate system is used. Further, the shear stress components are shown using notation τ_{ij}. Thus, the stress tensor in this case can be written as

$$[\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \tag{2.28}$$

Note: The stress tensor will be symmetric, that is $\sigma_{ij} = \sigma_{ji}$ only when there are no distributed moments in the body. The readers are suggested to read more on this from any standard solid mechanics book. In this entire course, we will deal with symmetric stress-tensor.

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Equilibrium Equations

The equilibrium equations for a body to be in static equilibrium at a point are given in index notations as

$$\sigma_{ij,j} + b_i = 0 \qquad (2.29)$$

where, b_i are the body forces per unit volume. If the body forces are absent, then the equilibrium equation becomes

$$\sigma_{ij,j} = 0 \qquad (2.30)$$

The equilibrium equations, without body forces are written using *xyz* coordinates as follows:

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} &= 0 \\
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} &= 0 \\
\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= 0
\end{aligned} \qquad (2.31)$$

Boundary Conditions

The boundary conditions are very essential to solve any problem in solid mechanics. The boundary conditions are specified on the surface of the body in terms of components of displacement or traction. However, the combination of displacement and traction components is also specified. Figure 2.5 shows a body, where the displacement as well as traction components are used to specify the boundary conditions.

We define traction vector T_i for any arbitrary point (for example, point *P* in Figure 2.5) on surface as a vector consisting of three stress components acting on the surface at same point. Here, the three stress components are normal stress σ_{nn} and shear stress σ_{nt} and σ_{ns}. The traction vector at this point is written as

$$T_i = \sigma_{ji}\, n_j \qquad (2.32)$$

where n_i is the ith component of the unit normal to the surface at point *P*. For example, if this surface is perpendicular to axis 2, then $n_i = (0,1,0)$ and the components of traction acting at a point on this surface are given as follows

$$\begin{aligned}
T_1 &= \sigma_{21} \\
T_2 &= \sigma_{22} \\
T_3 &= \sigma_{23}
\end{aligned} \qquad (2.33)$$

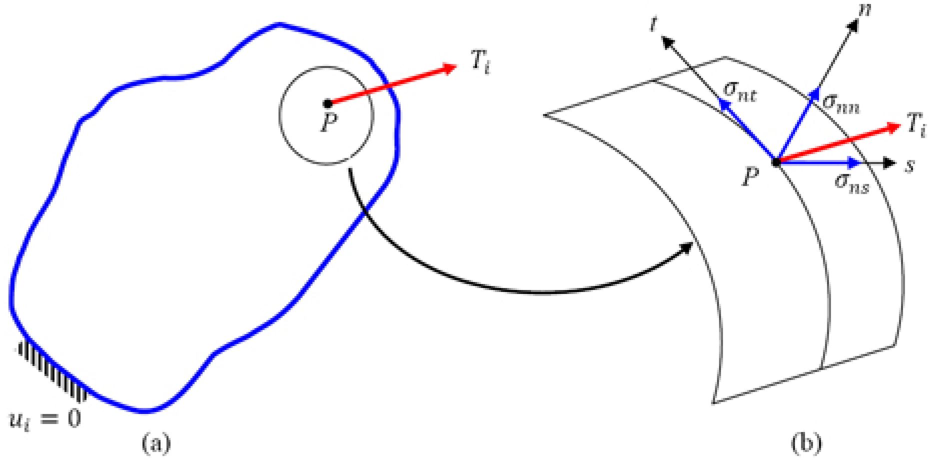

Figure 2.5: (a) A body showing displacement and traction boundary conditions, (b) Traction vector at any arbitrary point *P* on the surface of a body

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Constitutive Equations

The relationship between stress and strain is known as constitutive equation. The general form of this equation is

$$\sigma_{ij} = C_{ijkl}\,\epsilon_{kl} \qquad (2.34)$$

Here, C_{ijkl} are called **elastic constants**. This is also referred to as **elastic moduli or elastic stiffnesses**. This form of constitutive equation is known as generalized **Hooke's law**. Very soon, we will see this equation in detail for various material types.
The inverse of this equation can be written as

$$\epsilon_{ij} = C^{-1}_{ijkl}\,\sigma_{kl} = S_{ijkl}\,\sigma_{kl} \qquad (2.35)$$

where S_{ijkl} is known as **compliance**.

Plane Stress Problem

Plane stress problem corresponds to a situation where out of plane stress components are negligibly small. Thus, we can say that the state of stress is planar. The planar state of stress in *x*-*y* plane is shown in Figure 2.6. For the case shown in this figure, the normal and shear stress components in *z* directions, that is σ_{zz}, σ_{xz} and σ_{yz} are zero. Please note that the state of stress shown in this figure assumes the stress symmetry.

Note: A careful observation for strain components in *z* direction ($\epsilon_{zz}, \epsilon_{xz}$ and ϵ_{yz}) reveals that these need not be zero. This is a common mistake made by many readers. The magnitude of these strain components can be found with the help of constitutive equation given in Equation (2.34).

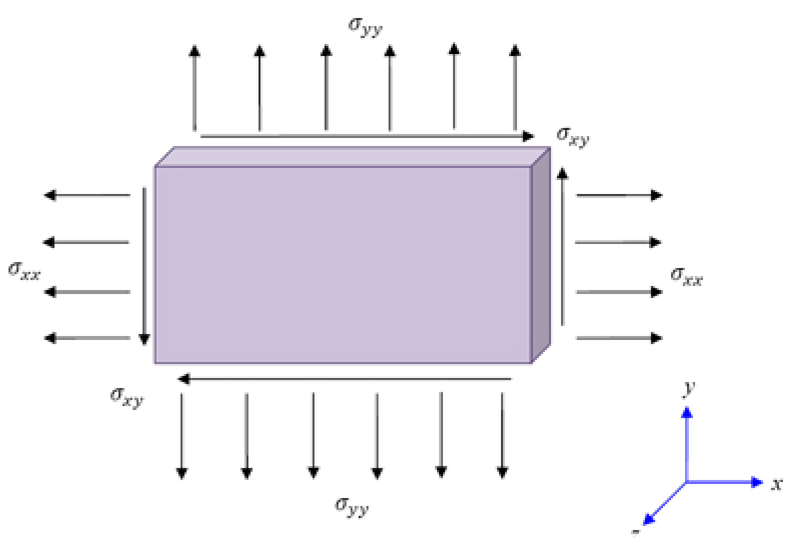

Figure 2.6: Plane stress problem

For plane stress problem the equilibrium equations take the following form

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$
$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \qquad (2.36)$$

Plane Strain Problem

Plane strain problem corresponds to a condition where all the out of plane strain components are negligibly small. Here, we denote $\epsilon_{xz}, \epsilon_{yz}$ and ϵ_{zz} as out of plane strain components. The readers are again cautioned to note that the out of plane stress components need not be zero. These depend upon the constitutive equation. Further, the equilibrium equation is same as Equation (2.36) and $\sigma_{zz} = f(x, y)$.

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Principles from Work and Energy

Strain Energy Density

The strain energy stored in a body per unit volume is called as **strain energy density**. In the absence of internal energy, the strain energy density for a linearly elastic body is given as

$$W = \frac{1}{2}\sigma_{ij}\, \epsilon_{ij} \qquad (2.37)$$

The expanded form of the above equation using symmetry of stress and strain components is

$$W = \frac{1}{2}(\sigma_{11}\, \epsilon_{11} + \sigma_{22}\, \epsilon_{22} + \sigma_{33}\, \epsilon_{33} + 2\sigma_{12}\, \epsilon_{12} + 2\sigma_{13}\, \epsilon_{13} + 2\sigma_{23}\, \epsilon_{23}) \qquad (2.38)$$

The readers should note that **strain energy density is a scalar quantity**. **Further, it is a positive definite quantity.**

Principle of Minimum of Total Potential Energy

The principle of minimum of total potential energy states that of all possible kinematically admissible displacement fields, the actual solution to the problem is one which minimizes the total potential energy (Π).
The total potential energy (for linearly elastic material) is defined as

$$\Pi = \frac{1}{2}\int_V C_{ijkl}\, \epsilon_{ij}\, \epsilon_{kl}\, dV - \int_S T_i\, u_i\, dS \qquad (2.39)$$

***Note*:** The kinematically admissible displacement field is a single valued and continuous displacement field that satisfies the displacement boundary condition.

Principle of Minimum of Total Complementary Potential Energy

The principle of minimum of total complementary potential energy states that of all possible statically admissible stress fields, the actual solution to the problem is one which minimizes the total complementary potential energy (Π^*).
The total complementary potential energy (for linearly elastic material) is defined as

$$\Pi^* = \frac{1}{2}\int_V S_{ijkl}\, \sigma_{ij}\, \sigma_{kl}\, dV - \int_S T_i^{\circ} u_i\, dS \qquad (2.40)$$

Note: The statically admissible stress field is one that satisfies both equilibrium equations and traction boundary condition.

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Homework

1. Verify the property given in Equation (2.6) for rotation matrix.

2. Using Equation (2.6), show that

$$a_{ik}a_{jk} = \delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$$

   where the term δ_{ij}, called **Kronecker delta,** has the value 1 on the diagonal and 0 on the off diagonal, that is, it represents an identity matrix when represented in matrix form.

3. Using relation for strain components (given in Equation (2.21)) write the expanded form of all strain components and understand the physical significance of all strain components. (The normal strain components denote the stretching of a line element, etc.)

4. Derive the principles of minimum of total potential and total complementary potential energy.

5. Derive the principle of virtual work.

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