Module 3: 3D Constitutive Equations
Lecture 12: Constitutive Relations for Orthotropic Materials and Stress-Strain Transformations

The Lecture Contains:

- Engineering Constants
- Constitutive Equation for an Orthotropic Material
- Constraints on Engineering Constants in Orthotropic Materials
- Stress and Strain Transformation about an Axis
- Strain Transformation
- Homework
- References

In the previous lecture we have seen the constitutive equations for various types of (that is, nature of) materials. There are 81 independent elastic constants for generally anisotropic material and two for an isotropic material. Let us summarize the reduction of elastic constants from generally anisotropic to isotropic material.

1. For a generally anisotropic material there are 81 independent elastic constants.

2. With additional stress symmetry the number of independent elastic constants reduces to 54.

3. Further, with strain symmetry this number reduces to 36.

4. A hyperelastic material with stress and strain symmetry has 21 independent elastic constants. The material with 21 independent elastic constants is also called as anisotropic or aelotropic material.

5. Further reduction with one plane of material symmetry gives 13 independent elastic constants. These materials are known as monoclinic materials.

6. Additional orthogonal plane of symmetry reduces the number of independent elastic constants to 9. These materials are known as orthotropic materials. Further, if a material has two orthogonal planes of symmetry then it is also symmetric about third mutually perpendicular plane. A unidirectional lamina is orthotropic in nature.

7. For a transversely isotropic material there are 5 independent elastic constants. Plane 2-3 is transversely isotropic for the lamina shown in Figure 3.7.

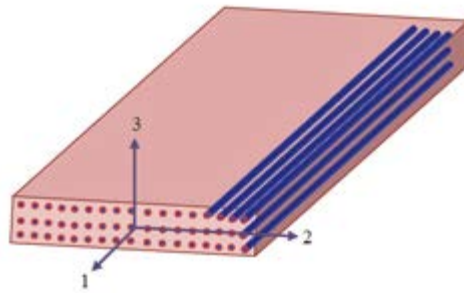8. For an isotropic material there are only 2 independent elastic constants.

Principal Material Directions:

The interest of this course is unidirectional lamina or laminae and laminate made from stacking of these unidirectional laminae. Hence, we will introduce the principal material directions for a unidirectional fibrous lamina. These are denoted by 1-2-3 directions. The direction 1 is along the fibre. The directions 2 and 3 are perpendicular to the direction 1 and mutually perpendicular to each other. The direction 3 is along the thickness of lamina. The principal directions for a unidirectional lamina are shown in Figure 3.7.

Engineering Constants:

The elastic constants which form the stiffness matrix are not directly measured from laboratory tests on a material. One can measure engineering constants like Young's modulus, shear modulus and Poisson's ratio from laboratory tests. The relationship between engineering constants and elastic constants of stiffness matrix is also not straight forward. This relationship can be developed with the help of relationship between engineering constants and compliance matrix coefficients.

In order to establish the relationship between engineering constants and the compliance coefficients, we consider an orthotropic material in the principal material directions. If this orthotropic material is subjected to a *3D* state of stress, the resulting strains can be expressed in terms of these stress components and engineering constants as follows:

Figure 3.7: Unidirectional lamina with principal material directions

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$$\varepsilon_{11} = \frac{\sigma_{11}}{E_1} - \frac{\nu_{21}}{E_2}\sigma_{22} - \frac{\nu_{31}}{E_3}\sigma_{33}$$

$$\varepsilon_{22} = -\frac{\nu_{12}}{E_1}\sigma_{11} + \frac{\sigma_{22}}{E_2} - \frac{\nu_{32}}{E_3}\sigma_{33} \qquad (3.42)$$

$$\varepsilon_{33} = -\frac{\nu_{13}}{E_1}\sigma_{11} - \frac{\nu_{23}}{E_2}\sigma_{22} + \frac{\sigma_{33}}{E_3}$$

whereas the engineering shear strain components are given as

$$\gamma_{23} = \frac{\tau_{23}}{G_{23}}$$

$$\gamma_{13} = \frac{\tau_{13}}{G_{13}} \qquad (3.43)$$

$$\gamma_{12} = \frac{\tau_{12}}{G_{12}}$$

Here, E_1, E_2 and E_3 are the Young's moduli in 1, 2 and 3 directions, respectively. Thus, E_1 represents the axial modulus and E_2 and E_3 represent in-plane transverse and out-of-plane transverse moduli, respectively. Note that axial direction is along the fibre direction.

G_{ij} represents the shear moduli. G_{12}, G_{13} are the axial shear moduli in two orthogonal planes that contain the fibers. G_{23} represents out-of-plane transverse shear modulus. Further, it should be noted that $G_{ij} = G_{ji}$.

The term ν_{ij} represents the Poisson's ratio. It is defined as follows

$$\nu_{ij} = \frac{-\varepsilon_{jj}}{\varepsilon_{ii}} \qquad i \neq j \text{ and no sum over } i \text{ and } j \qquad (3.44)$$

where ε_{ii} represents the strain in the direction of applied stress and ε_{jj} represents the strain the associated lateral direction. It should be noted that, in general $\nu_{ij} \neq \nu_{ji}$.

We will mimic some (thought) experiments that we actually do in laboratory to extract these engineering constants. For example, we find engineering constants of a transversely isotropic lamina.

Experiment 1: The lamina is loaded in traction along the axial direction as shown in Figure 3.8 (a) and the strains in along three principal directions are recorded as the load is varied. The slope of the axial stress versus axial strain curve yields the axial Young's modulus E_1. The ratios

$\frac{-\varepsilon_{22}}{\varepsilon_{11}}$ and $\frac{-\varepsilon_{33}}{\varepsilon_{11}}$ give the Poisson's ratios ν_{12} and ν_{13} respectively.

Experiment 2: The lamina is loaded in traction along direction 2. The two views of this loading

case are shown in Figure 3.8 (b). The slope of stress-strain curve in direction 2 gives the in-plane transverse Young's modulus E_2. Since, the material is isotropic in 2-3 plane, E_2 is also equal E_3. The strains in all three directions are measured. The ratios $\dfrac{-\varepsilon_{33}}{\varepsilon_{22}}$ and $\dfrac{-\varepsilon_{11}}{\varepsilon_{22}}$ give the Poisson's ratios ν_{23} and ν_{21}, respectively.

Experiment 3: The lamina is loaded in shear in plane *1-2* as shown in Figure 3.8 (c). The slope of the in-plane shear stress and engineering shear strain curve gives the shear modulus G_{12}. Please note that if we load the lamina in *1-3* plane by shear then also we will get this modulus because the behaviour of material in shear in these two planes is identical. Thus, by shear loading in plane *1-2* gives $G_{12} = G_{13}$.
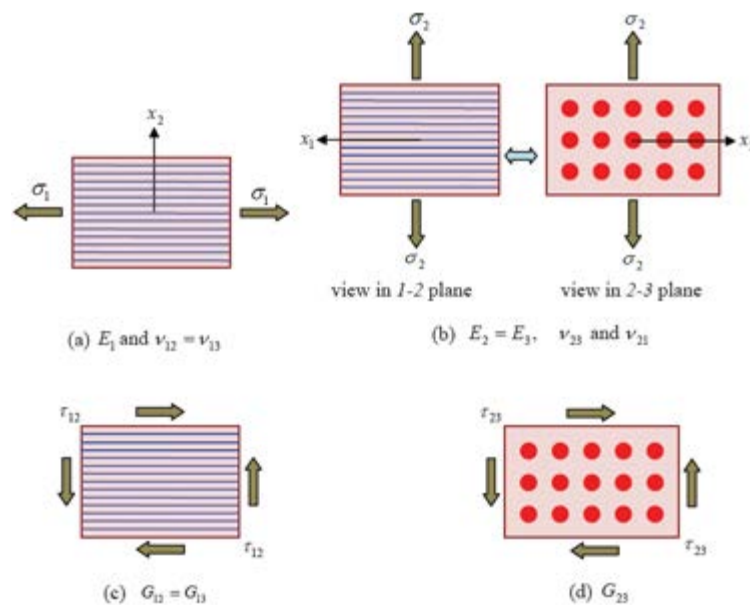
Experiment 4: The lamina is loaded in shear in *2-3* plane as shown in Figure 3.8(d). The corresponding shear stress and engineering shear strain curve yields the shear modulus G_{23}.

Note: We will see the experimental details to measure some of these engineering constants in a chapter on experimental characterization of lamina, laminates, fibres and matrix materials.

Constitutive Equation for an Orthotropic Material:

Now, let us assume that we have measured all the engineering constants of an orthotropic material along principal directions. With these engineering constants we know the relation between the strain and stress components as given in Equation (3.42) and Equation (3.43). Thus, it is easy to see that we can relate the strain components to stress components through compliance matrix. Let us recall from previous lecture the stiffness matrix for orthotropic material (Equation (3.26)). The inverse of this matrix (compliance) will have the same form as the stiffness matrix. Thus, we write the relationship between strain and stress components using compliance matrix as follows

(a) E_1 and $\nu_{12} = \nu_{13}$

(b) $E_2 = E_3$, ν_{23} and ν_{21}

view in 1-2 plane   view in 2-3 plane

(c) $G_{12} = G_{13}$

(d) G_{23}

Figure 3.8: Experiments to extract engineering constants for a transversely isotropic material

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} \qquad (3.45)$$

Now compare Equation (3.42) and Equation (3.43) with Equation (3.45). This gives us the compliance coefficients in terms of engineering constants. The coefficients are given in Equation (3.46).

$$S_{11} = \frac{1}{E_1}, \qquad S_{12} = \frac{-\nu_{21}}{E_2}, \qquad S_{13} = \frac{-\nu_{31}}{E_3}$$
$$S_{21} = \frac{-\nu_{12}}{E_1}, \qquad S_{22} = \frac{1}{E_2}, \qquad S_{23} = \frac{-\nu_{32}}{E_3}$$
$$S_{31} = \frac{-\nu_{13}}{E_1}, \qquad S_{32} = \frac{-\nu_{23}}{E_2}, \qquad S_{33} = \frac{1}{E_3}$$
$$S_{44} = \frac{1}{G_{23}}, \qquad S_{55} = \frac{1}{G_{13}}, \qquad S_{66} = \frac{1}{G_{12}}$$

(3.46)

It should be noted that like stiffness matrix, the compliance matrix is also symmetric. The compliance matrix given in Equation (3.45) is shown symmetric.

Note: It is known from our elementary knowledge of linear algebra that inverse of a symmetric matrix is also a symmetric matrix. Since, the stiffness matrix, which is the inverse of compliance matrix, is symmetric; the compliance matrix has to be symmetric.

Now, let us derive some more useful relations using the symmetry of compliance matrix. If we compare S_{12} and S_{21} we get $\dfrac{\nu_{21}}{E_2} = \dfrac{\nu_{12}}{E_1}$. Similarly, comparison of S_{13} and S_{31} and comparison of S_{23} and S_{32} give two more similar relations. All these relations are given in Equation (3.47).

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}, \qquad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}, \qquad \frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3}$$

(3.47)

or one can write this relation in index form as

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \qquad \left(\text{no sum over } i \text{ and } j\right)$$

(3.48)

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The relations in Equation (3.47) or Equation (3.45) are referred to as the reciprocal relations. These relations are also written as

$$E_1 \nu_{21} = E_2 \nu_{12}, \quad E_1 \nu_{31} = E_3 \nu_{13}, \quad E_3 \nu_{23} = E_2 \nu_{32} \qquad (3.49)$$

It should be noted that, in general $\nu_{ij} \neq \nu_{ji}$. From Equation (3.48) we can write for ν_{21} and ν_{32} as

$$\nu_{21} = \frac{E_2}{E_1} \nu_{12} \quad \text{and} \quad \nu_{32} = \frac{E_3}{E_2} \nu_{23} \qquad (3.50)$$

It is known that for transversely isotropic material (in *2-3* plane) E_1 is much greater than E_2 and $E_2 = E_3$ Thus, from the first of Equation (3.47) one can easily see that ν_{21} is much smaller than ν_{12}. Further, it is clear from the relation that $\nu_{23} = \nu_{32}$.

Note: Since value of ν_{21} (and may be of other Poisson's ratios) will be small, the readers are suggested to use appropriate precision level while calculating (in examinations and writing computer codes) any data involving these coefficients.

We will get the stiffness matrix by inversion of compliance matrix. Equation (3.46) is substituted in Equation. (3.45) and the resulting equation is inverted to give the stiffness matrix of an orthotropic material as

$$
\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} =
\begin{bmatrix}
\dfrac{1-\nu_{23}\,\nu_{32}}{E_2\,E_3\,\Delta} & \dfrac{\nu_{21}+\nu_{23}\,\nu_{31}}{E_2\,E_3\,\Delta} & \dfrac{\nu_{31}+\nu_{21}\,\nu_{32}}{E_2\,E_3\,\Delta} & 0 & 0 & 0 \\
 & \dfrac{1-\nu_{13}\,\nu_{31}}{E_1\,E_3\,\Delta} & \dfrac{\nu_{32}+\nu_{12}\,\nu_{31}}{E_1\,E_3\,\Delta} & 0 & 0 & 0 \\
 & & \dfrac{1-\nu_{12}\,\nu_{21}}{E_1\,E_2\,\Delta} & 0 & 0 & 0 \\
 & & & G_{23} & 0 & 0 \\
 & \text{symmetric} & & & G_{13} & 0 \\
 & & & & & G_{12}
\end{bmatrix}
\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}
\qquad (3.51)
$$

where

$$\Delta = \frac{\left(1 - \nu_{12}\,\nu_{21} - \nu_{23}\,\nu_{32} - \nu_{13}\,\nu_{31} - 2\,\nu_{21}\,\nu_{32}\,\nu_{13}\right)}{\left(E_1\,E_2\,E_3\right)} \qquad (3.52)$$

is the determinant of stiffness matrix in Equation (3.51). We can write the stiffness matrix for transversely isotropic material with the following substitutions in the stiffness matrix.

$$E_2 = E_3, \ \nu_{12} = \nu_{13}, \ G_{12} = G_{13} \ \text{and} \ G_{23} = \frac{E_2}{2(1+\nu_{23})}$$

Further, from the resulting, one can reduce the constitutive equation for isotropic material with following substitutions:

$$E = E_1 = E_2, \nu = \nu_{12} = \nu_{13} = \nu_{23}, G = G_{12} = G_{13} \ \text{and} \ G = \frac{E}{2(1+\nu)}$$

The readers should verify these results.

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Constraints on Engineering Constants in Orthotropic Materials

For orthotropic materials there are constraints on engineering constants. These constrains arise due to thermodynamic admissibility. For example, in case of isotropic materials it is well known that the Young's modulus and shear modulus are always positive. Further, the Poisson's ratio lager than half are not thermodynamically admissible. If these constrains are violated then it is possible to have a nonpositive strain energy for certain load conditions. However, for isotropic materials the strain energy must be a positive definite quantity.

In this section, based on the work done by Lempriere we are going to assess the implications of this thermodynamic requirement (positive definiteness of strain energy) for orthotropic materials.
The sum of work done by all stress components must be positive, otherwise energy will be created. This condition imposes a thermodynamic constraint on elastic constants. This condition requires that both compliance and stiffness matrices must be positive definite. In other words, the invariants of these matrices should be positive.

Let us look at this condition with physical arguments. For example, consider that only one normal stress component is applied. Then we can find the corresponding strain component from the corresponding diagonal entry of the compliance matrix. Thus, we can say that for the strain energy to be positive definite the diagonal entries of the compliance matrix must be positive. Thus,

$$E_1, E_2, E_3, G_{23}, G_{13}, G_{12} > 0 \tag{3.53}$$

In a similar way, it is possible under certain conditions to have a deformation which will give rise to only one normal strain component. We can find the corresponding stress using the corresponding diagonal entry in stiffness matrix. For the strain energy produced by this stress component to be positive the diagonal entry of the stiffness matrix must be positive. Thus, this condition reduces to

$$(1 - \nu_{23}\,\nu_{32}), (1 - \nu_{13}\,\nu_{31}), (1 - \nu_{23}\,\nu_{32}) > 0 \tag{3.54}$$

and the determinant of the compliance matrix must also be positive. That is,

$$\Delta = 1 - \nu_{12}\,\nu_{21} - \nu_{23}\,\nu_{32} - \nu_{13}\,\nu_{31} - 2\,\nu_{21}\,\nu_{32}\,\nu_{13} > 0 \tag{3.55}$$

Now, using the reciprocal relations given in Equation (3.49), the condition in Equation (3.54) can be expressed as

$$|\nu_{12}| < \left(\frac{E_1}{E_2}\right)^{\frac{1}{2}} \qquad |\nu_{21}| < \left(\frac{E_2}{E_1}\right)^{\frac{1}{2}}$$

$$|\nu_{13}| < \left(\frac{E_1}{E_3}\right)^{\frac{1}{2}} \qquad |\nu_{31}| < \left(\frac{E_3}{E_1}\right)^{\frac{1}{2}} \tag{3.56}$$

$$|\nu_{23}| < \left(\frac{E_2}{E_3}\right)^{\frac{1}{2}} \qquad |\nu_{32}| < \left(\frac{E_3}{E_2}\right)^{\frac{1}{2}}$$

This condition also justifies that the Poisson's ratio greater than unity is feasible for orthotropic lamina. Poisson's ratio greater than unity is sometimes observed in experiments.

The condition in Equation (3.55) can be written as

$$\nu_{21} \nu_{32} \nu_{13} < \frac{1 - \nu_{12} \nu_{21} - \nu_{23} \nu_{32} - \nu_{13} \nu_{31}}{2}$$

$$< \frac{1 - \nu_{21}^2 \left(\frac{E_1}{E_2}\right) - \nu_{32}^2 \left(\frac{E_2}{E_3}\right) - \nu_{13}^2 \left(\frac{E_3}{E_1}\right)}{2}$$

The terms inside the brackets are positive. Thus, we can write

$$\nu_{21} \nu_{32} \nu_{13} < \frac{1}{2} \qquad (3.57)$$

This condition shows that all three Poisson's ratios cannot have large positive values and that their product must be less than half. However, if one of them is negative no restriction is applied to remaining two ratios.

Let us consider the transverse isotropy as a special case. Let us consider transverse isotropy in 2-3 plane. Let

$$\begin{aligned} \nu_{12} = \nu_{13} = \nu' \qquad & E_2 = E_3 = E \\ \nu_{23} = \nu_{32} = \nu \qquad & E_1 = E' \end{aligned} \qquad (3.58)$$

Then, the conditions in Equation (3.56) reduce as

$$-1 < \nu < 1$$

$$-\left(\frac{E'}{E}\right)^{\frac{1}{2}} < \nu' < \left(\frac{E'}{E}\right)^{\frac{1}{2}} \qquad (3.59)$$

and Equation (3.57) (using reciprocal relations in Eq. (3.49)) becomes as

$$\nu < 1 - 2\left(\nu'\right)^2 \left(\frac{E}{E'}\right) \qquad (3.60)$$

The condition posed by above equation is more stringent than that posed in Equation (3.59). Note that the quantities $\left(\nu'\right)^2$ and $\left(\frac{E}{E'}\right)$ are both positive. Thus, the limits on Poisson's ratio in transverse plane are

$$-1 < \nu < 1 - 2\left(\nu'\right)^2 \left(\frac{E}{E'}\right) \qquad (3.61)$$

Further, consider a special case of isotropic material where $\nu = \nu'$ and $E' = E$. This simplifies Equation (3.61) to a well known condition

$$-1 < \nu < \frac{1}{2}$$

(3.62)

Stress and Strain Transformation about an Axis

Often it is required to transform the stress or strain tensor from one coordinate axes system to another. For example, if the fibres in a lamina are not oriented along direction *x*, then we may need to transform the stress and strain components from principal material directions 1-2-3 to global directions *xyz* or vice-a-versa. It should be noted that the stress and strain tensors are second order tensors. Hence, they follow tensor transformation rules.

In this section we are going to introduce two notations. The subscripts 123 will denote a quantity (like constitutive equation, engineering constants, etc.) in principal material directions, while subscripts *xyz* will denote the corresponding quantity in global coordinate directions.
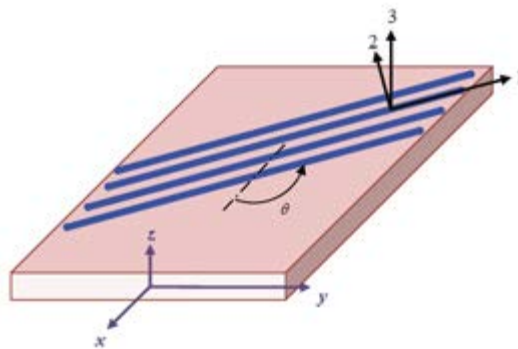
Figure 3.9: Unidirectional lamina with global *xyz* directions and principal material 1-2-3 directions

Let us transform the stress and strain components for the case shown in Figure 3.9. Here, *xy* plane is rotated about direction *z* to *1-2* plane. Here, direction *z* and direction 3 are in same directions, that is, along the thickness direction of lamina. The direction cosines for this axes transformation are as given in Equation (2.2). However, these are again given below.

From \ To	1	2	3
x	$a_{11} = \cos\theta$	$a_{12} = -\sin\theta$	$a_{13} = 0$
y	$a_{21} = \sin\theta$	$a_{22} = \cos\theta$	$a_{23} = 0$
z	$a_{31} = 0$	$a_{32} = 0$	$a_{33} = 1$

Stress Transformation:

Let us do the stress transformation as given in Equation (2.7). In this equation the primed stress components denote the component in 123 coordinate system. Using the expanded form of Equation (2.7) and stress symmetry, let us obtain σ_{11} component of stress

$$\sigma_{11} = a_{11} a_{11} \sigma_{xx} + a_{11} a_{21} \sigma_{xy} + a_{11} a_{31} \sigma_{xz}$$
$$+ a_{21} a_{11} \sigma_{xy} + a_{21} a_{21} \sigma_{yy} + a_{21} a_{31} \sigma_{yz}$$
$$+ a_{31} a_{11} \sigma_{xz} + a_{31} a_{21} \sigma_{yz} + a_{31} a_{31} \sigma_{zz}$$

Thus, substituting the values of direction cosines from above, we get

$$\sigma_{11} = \cos^2 \theta \; \sigma_{xx} + 2 \sin \theta \cos \theta \; \sigma_{xy} + \sin^2 \theta \; \sigma_{yy}$$

The remaining five stress terms (using stress symmetry) on the left hand side are also obtained in a similar way. Let us write the final form of the relation as

$$\{\sigma\}_{123} = [T_1] \{\sigma\}_{xyz} \tag{3.63}$$

Here, $\{\sigma\}_{123} = \{\sigma_{11} \; \sigma_{22} \; \sigma_{33} \; \sigma_{23} \; \sigma_{13} \; \sigma_{12}\}^T$ and $\{\sigma\}_{xyz} = \{\sigma_{xx} \; \sigma_{yy} \; \sigma_{zz} \; \sigma_{yz} \; \sigma_{xz} \; \sigma_{xy}\}^T$ and $[T_1]$ is the stress transformation matrix. Thus, comparing all the terms as in Equation (3.63), we can write $[T_1]$ as

$$[T_1] = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & 2mn \\ n^2 & m^2 & 0 & 0 & 0 & -2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -mn & mn & 0 & 0 & 0 & m^2 - n^2 \end{bmatrix} \tag{3.64}$$

where $m = \cos \theta$ and $n = \sin \theta$. It should be noted that $[T_1]$ is not symmetric.

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Strain Transformation:

In a similar way, we can transform the strain components from *xy* plane to *1-2* plane. In this transformation we will use engineering shear strains. Let us find the γ_{12} using the transformation equation similar to stress transformation and using strain symmetry as

$$\varepsilon_{12} = a_{11} a_{12} \varepsilon_{xx} + a_{11} a_{22} \varepsilon_{xy} + a_{11} a_{32} \varepsilon_{xz}$$
$$+ a_{21} a_{12} \varepsilon_{xy} + a_{21} a_{22} \varepsilon_{yy} + a_{21} a_{32} \varepsilon_{yz} \qquad (3.65)$$
$$+ a_{31} a_{12} \varepsilon_{xz} + a_{31} a_{22} \varepsilon_{yz} + a_{31} a_{32} \varepsilon_{zz}$$

Substituting the direction cosines and rearranging, we get

$$\varepsilon_{12} = -\sin\theta\cos\theta\ \varepsilon_{xx} + \sin\theta\cos\theta\ \varepsilon_{yy} + \left(\cos^2\theta - \sin^2\theta\right)\varepsilon_{xy} \quad (3.66)$$

We know from Equation (2.24) that the tensorial shear strains are half the engineering shear strains. Thus, in Equation (3.53) we substitute

$$\varepsilon_{12} = \frac{\gamma_{12}}{2} \quad \text{and} \quad \varepsilon_{xy} = \frac{\gamma_{xy}}{2}$$

On simplification and putting $m = \cos\theta$ and $n = \sin\theta$, we get

$$\gamma_{12} = -2mn\ \varepsilon_{xx} + 2mn\ \varepsilon_{yy} + \left(m^2 - n^2\right)\gamma_{xy} \qquad (3.67)$$

The other five strain terms (using strain symmetry) on the left hand side are also obtained in a similar way. Let us write the final form of the relation as

$$\{\varepsilon\}_{123} = [T_2]\{\varepsilon\}_{xyz} \qquad (3.68)$$

Here, $\{\varepsilon\}_{123} = \{\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ \gamma_{23}\ \gamma_{13}\ \gamma_{12}\}^T$ and $\{\varepsilon\}_{xyz} = \{\varepsilon_{xx}\ \varepsilon_{yy}\ \varepsilon_{zz}\ \gamma_{yz}\ \gamma_{xz}\ \gamma_{xy}\}^T$ and $[T_2]$ is the strain transformation matrix. Thus, comparing all the terms as in Equation (3.68), we can write $[T_2]$ as

$$[T_2] = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & mn \\ n^2 & m^2 & 0 & 0 & 0 & -mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -2mn & 2mn & 0 & 0 & 0 & m^2 - n^2 \end{bmatrix} \qquad (3.69)$$

Note: The transformation matrices, $[T_1]$ and $[T_2]$ differ by factors 2 in two terms.

Note: The transformation matrices, $\left[T_1^{'}\right]$ and $\left[T_2^{'}\right]$ are not symmetric.

Note: The order of stress and strain components in Equatjion (3.63) and Equation (3.68) is important. Some books and research articles follow different orders. The readers are cautioned to take a note of it.

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Examples:

Example 1: Calculate the stiffness and compliance coefficients for transversely isotropic material AS4/3501 Epoxy. The properties are as given below for a fibre volume fraction of 60%.

$$E_1 = 126\, GPa,\ E_2 = E_3 = 11\, GPa,\ G_{12} = G_{12} = 6.6\, GPa, \nu_{12} = \nu_{13} = 0.28,\ \nu_{23} = 0.4$$

Solution:

$$S_{11} = \frac{1}{E_1} = 0.00794 \qquad S_{23} = \frac{-\nu_{32}}{E_3} = \frac{-\nu_{23}}{E_3} = -0.03636$$

$$S_{12} = S_{13} = \frac{-\nu_{21}}{E_2} = \frac{-\nu_{12}}{E_1} = -0.00223 \quad S_{44} = \frac{1}{G_{23}} = \frac{2(1+\nu_{23})}{E_2} = 0.25455$$

$$S_{22} = S_{33} = \frac{1}{E_2} = 0.09091 \qquad S_{55} = S_{66} = \frac{1}{G_{12}} = 0.15152$$

Unit of all compliance coefficients is *1/GPa*.

The corresponding stiffness coefficients are calculated by inversion of the compliance matrix.

$$C_{11} = 128.942 \qquad C_{23} = 5.452$$

$$C_{12} = C_{13} = 5.253 \qquad C_{44} = 3.928$$

$$C_{22} = C_{33} = 13.309 \quad C_{55} = C_{66} = 6.6$$

Unit of all stiffness coefficients is *GPa*.

Note: Both stiffness and compliance matrices are symmetric.

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Homework:

1. Write the number of independent elastic constants for *3D* hyperelastic, monoclinic, orthotropic, transversely isotropic and isotropic materials.

2. Are the Poisson's ratio ν_{ij} and ν_{ji} independent of each other for an orthotropic unidirectional lamina?

3. Take the form of stiffness matrix for an orthotropic material as given in Equation (3.26). Using any symbolic calculation software like *Maple* or *Mathematica*, obtain the inverse of this matrix and confirm that the form of compliance matrix written in Equation (3.42) is correct. Further, confirm that this matrix is symmetric. (One should be able to do this using the concepts of linear algebra alone.)

4. Extend the Problem 3 to get the stiffness matrix given in Equation (3.51).

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