The Lecture Contains:

Introduction:

In this lecture we are going to introduce the concept of laminate and its analysis based on Classical Laminate Theory. Further, we will introduce the notations to designate a laminate and will explain in detail the development of the classical laminate theory.

As we have studied earlier, laminate is defined as stacking of two or more laminae with same or different fibre orientation with respect to global direction. The laminae may be made of same or different material and have individual thicknesses.

Stacking Sequence Notation:

A laminate is designated by using a special nomenclature. In this nomenclature, the fibre orientation of all layers stacked in the laminate is given. In the following the main steps are given to designate a laminate.

1. The stacking of layers starts from the top of the laminate.
2. The stacking sequence gives the orientation of fibres with respect to global axis in degrees.
3. The stacking sequence is enclosed in square brackets symbol, $[\square]$
4. The distinct layers or groups of layers are separated with a slash symbol, /.
5. For repeated groups or layers, subscript *n* is used to designate.
6. The symmetric laminate is designated by subscript *S* on the square bracket, that is, by $[\square]_S$.
7. The total stacking sequence is designated by subscript *T*, that is, by $[\square]_T$. However, in general, this is not used for denoting a complete stacking sequence.

To help the readers to understand the designation of stacking sequence of laminates, in the following Table 5.1, some laminate sequences, their description and total number of laminae in that laminate are given. A laminate with coordinate system and ply numbering is shown in Figure 5.1(a).

Note: In some of the books on composites and research articles the coordinate systems used have *z* direction positive in upward direction. In that case the stacking of layers in a laminate starts from the bottom. Accordingly, the ply top and bottom coordinate designation also changes. However, the end results remain unchanged.

Table 5.1: Sample laminate stacking sequence notations and their description

Laminate	Description	Layers
$[45/30/0]$	One layer each of $45°, 30°$ and $0°$	3
$45°, 30°$	One layer of $30°$ and $-30°$	2
$[45_2]$	Two layer of $45°$	2
$[(30)_2/0]$	Two layers of $30°$ (in a group of two layers) and one layer of $0°$	3

$[45/30]_S$	Symmetric with 45° and 30° layers	4
$[\pm 45/\pm 30]_S$	Symmetric with $+45^\circ, -45^\circ$and $+30^\circ$ and -30° layers	8
$[(\pm 45)_2/(\pm 30)_2]_S$	Symmetric with two groups of $+45^\circ, -45^\circ$ and two groups of $+30^\circ$ and -30° layers	16
$[\pm \theta_1]_S$	Symmetric with one layer of $+\theta_1$ and one layer of $-\theta_1$	4
$v_0(x,y)$	Symmetric with $+\theta_1, -\theta_1, +\theta_2, -\theta_2, +\theta_3$ and $-\theta_3$ layers	12

Laminate Coordinate System:

The coordinate systems for global and principal material directions for laminae are same as given earlier. Here, we introduce the coordinates in the thickness direction to get the *z* coordinate of the top and bottom of each ply. For example, the bottom coordinate of the *k*th ply is and the top coordinate of the ply is . Thus, the bottom coordinate of the first ply is and the top coordinate of top ply is . The total thickness of the laminate is taken as 2*H*. Thus, the bottom most coordinate of the laminate is –*H* and top most coordinate is *H*. The lamina thickness coordinate notations are shown in Figure 5.1(b).

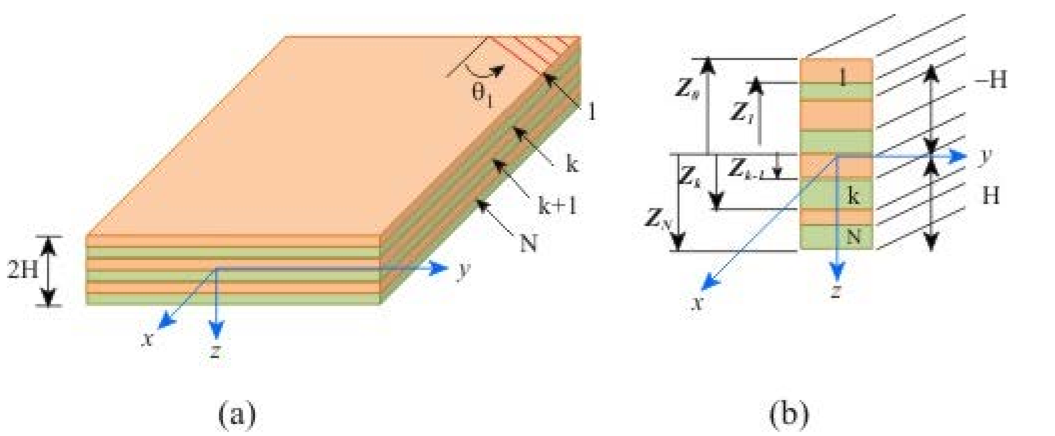

Figure 5.1: (a) Stacking of laminae in a laminate and (b) coordinate designation for laminate

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Classical Laminate Theory:

The classical laminate theory is a direct extension of the classical plate theory for isotropic and homogeneous material as proposed by Kirchhoff –Love (see [1, 2] for details). However, the extension of this theory to laminates requires some modifications to take into account the inhomogeneity in thickness direction. In the following, the assumptions made in this theory along with the assumptions made for classical plate theory are given.

Assumptions of Classical Lamination Theory:

1. The laminate consists of perfectly bonded layers. There is no slip between the adjacent layers. In other words, it is equivalent to saying that the displacement components are continuous through the thickness.
2. Each lamina is considered to be a homogeneous layer such that its effective properties are known.
3. Each lamina is in a state of plane stress.
4. The individual lamina can be isotropic, orthotropic or transversely isotropic.
5. The laminate deforms according to the Kirchhoff - Love assumptions for bending and stretching of thin plates (as assumed in classical plate theory). The assumptions are:
    a. The normals to the mid-plane remain straight and normal to the midplane even after deformation.
    b. The normals to the mid-plane do not change their lengths.

The classical laminate theory is abbreviated as CLT. This theory is known as the classical laminated plate theory and abbreviated as CLPT.

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***Displacement Field*:**

The strain-displacement field is derived using two approaches. In the first approach the deformation of the laminate according to the Kirchhoff - Love assumptions for bending and stretching is used. The undeformed and deformed geometries of laminate are used to develop the displacement field. In the second approach the transverse strain components resulting from the above assumptions are used. Further, using mathematical definitions of these strain components the displacement field is obtained. Thus, from this displacement field all strain components are obtained.

***First Approach*:**

The Figure 5.2(a) shows the geometry of a laminate in undeformed configuration and Figure 5.2(b) shows the deformed geometry according to Kirchhoff-Love assumptions in *xz* plane. Any generic normal to the undeformed mid-plane remains normal to the deformed mid-plane. This assumption results in zero transverse shear strains, that is, . However, due to stretching action the point of intersection of midplane and a normal moves by a distance along *x* axis. Further, the same point moves by distance in *z* direction due to bending action. The second assumption that the normal to the mid-plane does not change in length requires that the transverse normal strain, that is, . This holds true when the transverse deflection of any point in the laminate is independent of z location, that is, it is a function of *x* and *y* only and a constant for a given *x* and *y* location. So, we can write

$$w(x,y,z) = w_0(x,y) \tag{5.1}$$

Now, from the figure it is easy to find the slope of the deformed mid-plane as

$$\tan\alpha = \frac{\Delta W}{\Delta x} \tag{5.2}$$

Since, the deformations in this theory considered are very small, we can write

$$\tan\alpha \approx \alpha = \frac{\partial W}{\partial x} \tag{5.3}$$

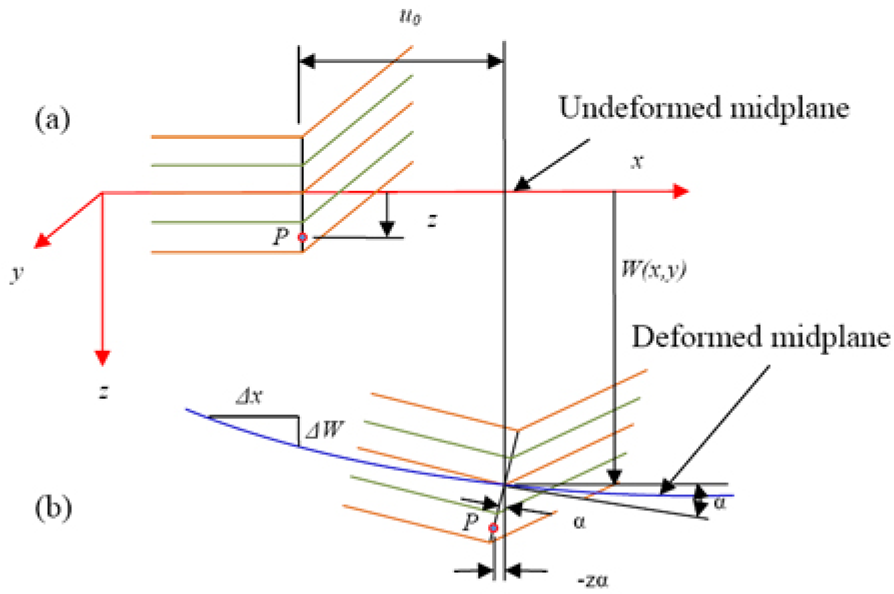

Figure 5.2: Laminate geometry (a) undeformed and (b) deformed

Now consider a generic point *P* on the mid-plane which is located at distance *z* from the mid-plane. After deformation, the displacement of this point along *x* direction can be given from the Figure 5.2 as

$$u(x,y,z) = u_0(x,y) - z\tan\alpha = u_0(x,y) - z\,\alpha = u_0(x,y) - z\,\frac{\partial w}{\partial x} \qquad (5.4)$$

Similarly, for the deformation in *yz* plane we can express the slope of the deformed mid-plane as . Thus, the displacement of a generic point along *y* axis can be given as

$$v(x,y,z) = v_0(x,y) - z\,\frac{\partial w}{\partial y} \qquad (5.5)$$

Thus, the complete displacement field for a generic point in the laminate according to the classical laminate theory is given below:

$$\begin{aligned} u(x,y,z) &= u_0(x,y) - z\,\frac{\partial w}{\partial x} \\ v(x,y,z) &= v_0(x,y) - z\,\frac{\partial w}{\partial y} \\ w(x,y,z) &= w_0(x,y) \end{aligned} \qquad (5.6)$$

Note that the displacements $u_0(x,y)$, $v_0(x,y)$ and $w_0(x,y)$ correspond to the mid-plane. Hence, they are called mid-plane displacements.

***Second Approach*:**

The second assumption is that the length of the normal to the mid-plane does not change even after deformation results into zero transverse normal strain. Thus,

$$\epsilon_{zz} = \frac{\partial w}{\partial z} = 0 \qquad (5.7)$$

Thus, from this expression it is clear that w is a function of x and y coordinates only. Thus, for any given location (x, y, z) we can write the transverse deflection component as

$$w(x, y, z) = w_0(x, y) \tag{5.8}$$

From the first assumption of the Kirchhoff-Love theory that the normals remain straight and normal to mid-plane even after deformation, results into zero transverse shear strains. Thus, $\gamma_{xz} = \gamma_{yz} = 0$.

Using the definitions of small strain, we can write the above equation as

$$\begin{aligned} \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \end{aligned} \tag{5.9}$$

From the first of the above equation we can write

$$\frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x}$$

Integrating this with respect to *z*, we get

$$u(x, y, z) = -z\frac{\partial w}{\partial x} + u_0(x, y) \tag{5.10}$$

where $u_0(x, y)$ is a constant of integration which is function of *x* and *y* alone. Similarly, from the second of Equation (5.9), we can get

$$v(x, y, z) = -z\frac{\partial w}{\partial y} + v_0(x, y) \tag{5.11}$$

Thus, Equations (5.8), (5.10) and (5.11) lead to the displacement field as in Equation (5.6).

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Strain Displacements Relations:

The strain displacement relations for infinitesimal strains using the displacement field as in Equation (5.6) can be given as

$$\begin{aligned}\epsilon_{xx} &= \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\\ \epsilon_{yy} &= \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w}{\partial y^2}\\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y}\end{aligned} \tag{5.12}$$

The above equation can be written as

$$\begin{Bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \gamma_{xy}\end{Bmatrix} = \begin{Bmatrix}\epsilon_{xx}^{(0)}\\ \epsilon_{yy}^{(0)}\\ \gamma_{xy}^{(0)}\end{Bmatrix} + z\begin{Bmatrix}\kappa_{xx}\\ \kappa_{yy}\\ \kappa_{xy}\end{Bmatrix}$$
$$\text{or}$$
$$\{\epsilon\}_{xy} = \left\{\varepsilon^{(0)}\right\}_{xy} + z\,\{\kappa\}_{xy} \tag{5.13}$$

where $\left\{\epsilon^{(0)}\right\}_{xy} = \left\{\epsilon_{xx}^{(0)} \quad \epsilon_{yy}^{(0)} \quad \gamma_{xy}^{(0)}\right\}^T = \left\{\frac{\partial u_0}{\partial x} \quad \frac{\partial v_0}{\partial y} \quad \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right\}^T$ are the midplane strains and $\{\kappa\}_{xy} = \{\kappa_{xx} \quad \kappa_{yy} \quad \kappa_{xy}\}^T = \left\{-\frac{\partial^2 w}{\partial x^2} \quad -\frac{\partial^2 w}{\partial y^2} \quad -2\frac{\partial^2 w}{\partial x \partial y}\right\}^T$ represents the midplane curvatures. The terms κ_{xx} and κ_{yy} are the bending moment curvatures and κ_{xy} is the twisting moment curvature.

***Note*:** It is clear from Equation (5.6) and Equation (5.13) that the midplane strains $\left\{\epsilon^{(0)}\right\}_{xy}$ and the curvatures $\{\kappa\}_{xy}$ are independent of *z* location.

***Note*:** From Equation (5.13), we see that the strains are continuous through the thickness of laminate and they vary linearly.

State of Stress in a Laminate:

The stresses at any location can be calculated from the strains and lamina constitutive relations. It is assumed that the lamina properties are known. Hence, the constitutive equation for a *kth* lamina is known, that is, the reduced stiffness matrices (in principal material directions and global directions) are known. Thus, the stresses in k^{th} lamina can be given as

$$\{\sigma\}_{xy}^{k} = \left[\bar{Q}\right]^{k}\{\epsilon\}_{xy}^{k} \tag{5.14}$$

Now, using Equation (5.13), we can write the stresses as

(5.15)

$$\{\sigma\}_{xy}^{k} = \left[\overline{Q}\right]^{k} \left\{\epsilon^{(0)}\right\}_{xy} + \left[\overline{Q}\right]^{k} z \left\{\kappa\right\}_{xy}$$

In these equations, the strains are given at a z location where the stresses are required. It should be noted that the strains are continuous and vary linearly through the thickness. If we look at the stress distribution through the thickness it is clear that the stresses are not continuous through the thickness, because the stiffness is different for different laminae in thickness direction. In a lamina the stress varies linearly. The slope of this variation in a lamina depends upon its moduli. However, at the interface of two adjacent laminae there is a discontinuity in the stresses. The same thing is depicted in Figure 5.3 with three layers.

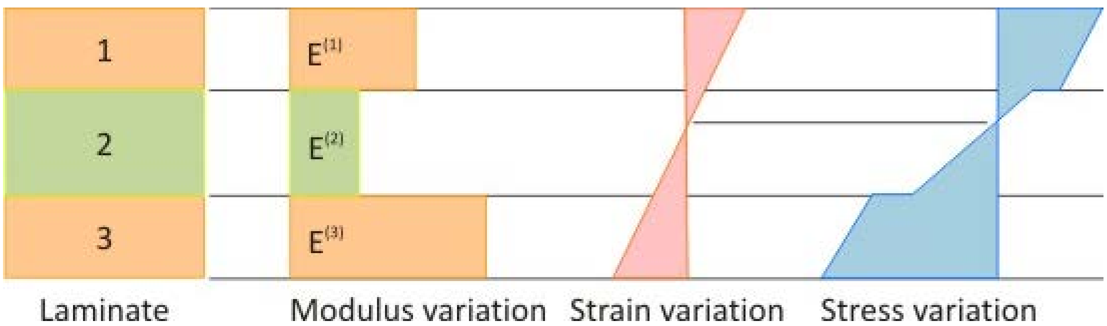

Figure 5.3: Elucidation of stress discontinuity at lamina interfaces in a laminate

***Note*:** The reduced transformed stiffness matrix $\left[\overline{Q}\right]^{k}$ for k^{th} lamina used in Equation (5.15) is the same as in the chapter on Planar Constitutive Equations. There we considered the state of stress as planar and the transverse normal strain was not zero. However, in this laminate theory we have plane stress assumption as well as all transverse strains are zero (plane strain conditions as well). Thus, we have an anomaly of transverse normal strains in using Equation (5.14). However, we will use this reduced transformed stiffness for a lamina. Inspite of this anomaly, the laminate theory works well (within its own scope). A detailed study on this issue can be seen in literature. However, this issue is out of scope of this course and will not be dealt with here.

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Inplane Resultant Forces:

The inplane forces per unit length are defined as

$$N_{xx} = \int_{-H}^{H} \sigma_{xx}\, dz, \qquad N_{yy} = \int_{-H}^{H} \sigma_{yy}\, dz, \qquad N_{xy} = \int_{-H}^{H} \tau_{xy}\, dz \quad (5.16)$$

Or these can be written as

$$\{N\}_{xy} = \int_{-H}^{H} \{\sigma\}_{xy}\, dz \quad (5.17)$$

Now, using Equation (5.15) we can write

$$\{N\}_{xy} = \sum_{k=1}^{N_{Lay}} \int_{-z_{k-1}}^{z_k} \left[\bar{Q}\right]^k \left\{\epsilon^{(0)}\right\}_{xy} dz + \sum_{k=1}^{N_{Lay}} \int_{-z_{k-1}}^{z_k} \left[\bar{Q}\right]^k \{\kappa\}_{xy}\, z\, dz \quad (5.18)$$

Now recall that the midplane strains $\left\{\epsilon^{(0)}\right\}_{xy}$ and the curvatures $\{\kappa\}_{xy}$ are independent of *z* location. The reduced transformed stiffness matrix $\left[\bar{Q}\right]$ is function of thickness and constant over a given lamina thickness. Now we can replace the integration over the laminate thickness as sum of the integrations over individual lamina thicknesses. Thus, Equation (5.18) can be written as

$$\{N\}_{xy} = \sum_{k=1}^{N_{Lay}} \int_{-z_{k-1}}^{z_k} \left[\bar{Q}\right]^k \left\{\epsilon^{(0)}\right\}_{xy} dz + \sum_{k=1}^{N_{Lay}} \int_{-z_{k-1}}^{z_k} \left[\bar{Q}\right]^k \{\kappa\}_{xy}\, z\, dz \quad (5.19)$$

Here, N_{Lay} is the total number of layers in the laminate. This equation can be written as

$$\{N\}_{xy} = [A]\left\{\epsilon^{(0)}\right\}_{xy} + [B]\{\kappa\}_{xy} \quad (5.20)$$

where

$$[A] = \sum_{k=1}^{N_{Lay}} \left[\bar{Q}\right]^k (z_k - z_{k-1}) \quad \text{and} \quad [B] = \frac{1}{2}\sum_{k=1}^{N_{Lay}} \left[\bar{Q}\right]^k (z_k^2 - z_{k-1}^2) \quad (5.21)$$

The matrix $[A]$ represents the in-plane stiffness, that is, it relates the in-plane forces with mid-plane strains and the matrix $[B]$ represents the bending stiffness coupling, that is, it relates the in-plane forces with mid-plane curvatures.

It should be noted that the matrices $[A]$ and $[B]$ are symmetric as the matrix $\left[\bar{Q}\right]$ is also symmetric for each lamina in the laminate.
The resultant in-plane forces are shown in Figure 5.4.

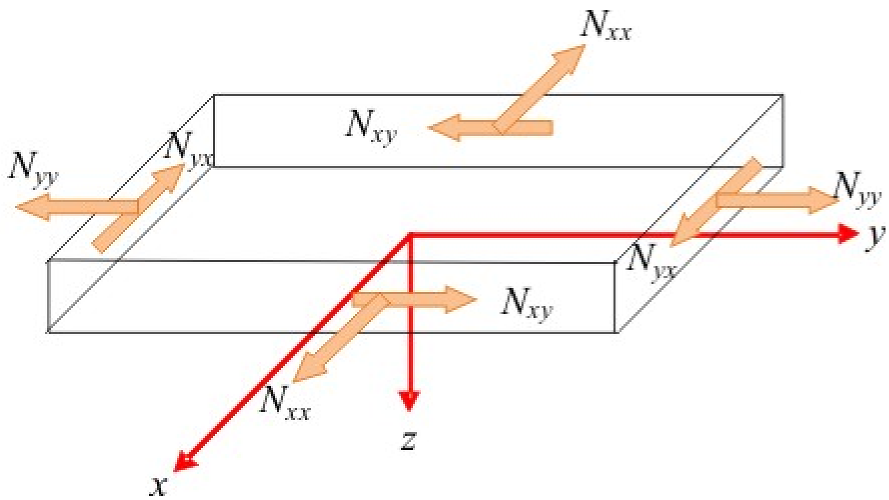

Figure 5.4: In plane resultant forces per unit length on a laminate

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Resultant Moments:

The resultant moments per unit length are defined as

$$M_{xx} = \int_{-H}^{H} \sigma_{xx}\, z\, dz, \qquad M_{yy} = \int_{-H}^{H} \sigma_{yy}\, z\, dz, \qquad M_{xy} = \int_{-H}^{H} \tau_{xy}\, z\, dz \quad (5.22)$$

Or these can be written as

$$\{M\}_{xy} = \int_{-H}^{H} \{\sigma\}_{xy}\, z\, dz \quad (5.23)$$

Now, using Equation (5.15) we can write,

$$\{M\}_{xy} = \int_{-H}^{H} \left[\overline{Q}\right]^k \left\{\epsilon^{(0)}\right\}_{xy} z\, dz + \int_{-H}^{H} \left[\overline{Q}\right]^k \{\kappa\}_{xy}\, z^2\, dz \quad (5.24)$$

Now, with the same justification as given for Equation (5.19), we can write the above equation as

$$\{M\}_{xy} = \sum_{k=1}^{N_{Lay}} \int_{-z_{k-1}}^{z_k} \left[\overline{Q}\right]^k \left\{\epsilon^{(0)}\right\}_{xy} z\, dz + \sum_{k=1}^{N_{Lay}} \int_{-z_{k-1}}^{z_k} \left[\overline{Q}\right]^k \{\kappa\}_{xy}\, z^2\, dz \quad (5.25)$$

This can be written as

$$\{M\}_{xy} = [B]\left\{\epsilon^{(0)}\right\}_{xy} + [D]\{\kappa\}_{xy} \quad (5.26)$$

where

$$[D] = \frac{1}{3}\sum_{k=1}^{N_{Lay}} \left[\overline{Q}\right]^k \left(z_k^3 - z_{k-1}^3\right) \quad (5.27)$$

The matrix $[D]$ represents the bending stiffness, that is, it relates resultant moments with mid-plane curvatures. Again, the matrix $[D]$ is also symmetric. Further, it is important to note that the matrix $[B]$ relates the resultant moments with mid-plane curvatures as well.

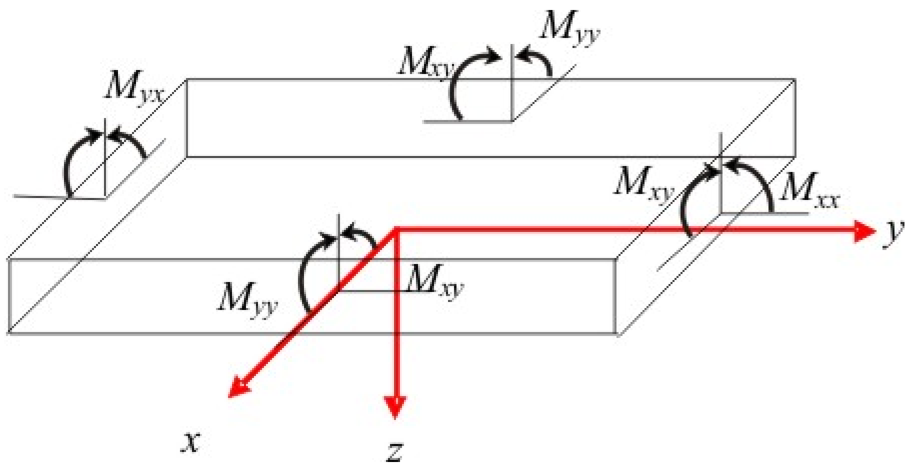

Figure 5.5: Resultant moments per unit length on a laminate

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Homework:

1. Write the key points in the designation of laminate sequence.
2. What are the assumptions in the classical laminate theory?
3. What are the assumptions in the classical laminate theory?
4. Why the stresses at the interface of two laminae are different according to the classical plate theory?
5. Derive the expressions for resultant inplane forces and bending moments for laminate

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Module 5: Laminate Theory

Lecture 16: Introduction to Classical Plate Theory

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