

## Module 7: Micromechanics






### Lecture 26: Concepts of Equivalent Homogeneity, Volumetric Averaging and Standard Mechanics

#### Introduction

In the previous lectures, we have seen the use of RVE along with strength of materials approach in determining the effective elastic, thermal and hygral properties of the composite in terms of properties of individual constituents.

In this lecture and the following lectures we are going to see some more concepts and approaches to determine the effective properties of the composite. In this lecture and the subsequent lectures we will use the continuum approaches.

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## The Concept of Equivalent Homogeneity

As we have understood, at sufficiently small scale all materials are heterogeneous in nature. In such a situation, one would like to start at atomistic or molecular level. This will lead to an intractable situation for engineering materials. Hence, the continuum hypothesis is invoked in such situations. In this hypothesis, a statistical averaging process is considered. Further, the actual constituents and their structures are idealized in such a way that resulting material is considered to be a continuum. Once we establish the existence of continuum hypothesis then the concept of equivalent homogeneity can be developed based on the structure of the material.

In general, the heterogeneity can be divided into two types. In the first type, the heterogeneity occurs as an idealized continuous variation of properties with the position and in the second type there is an abrupt change in properties across the interfaces of the constituents. In unidirectional fibrous composites, in cross sectional planes, we get the second type of inhomogeneity. Hence, the second type of inhomogeneity is of our concern in this micromechanical study. However, within the constituent we assume that the constituents are homogeneous and orthotropic, transversely isotropic or isotropic in nature.

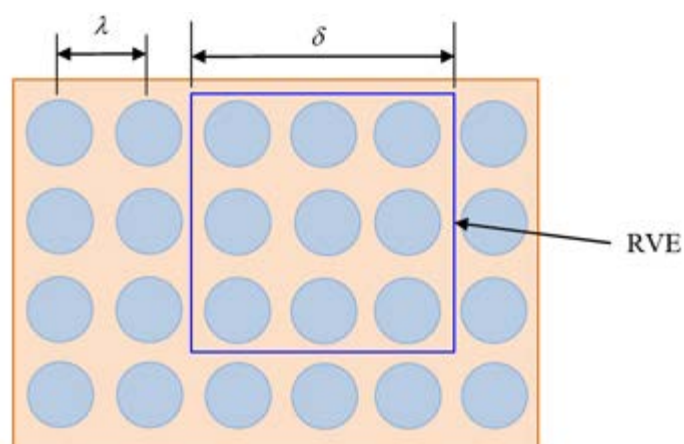
Now we will introduce the characteristic dimension of inhomogeneity based on the constituent arrangement and nature. We will consider an idealized system of fibres and matrix in a composite as shown in Figure 7.7. For this system the characteristic dimension of inhomogeneity is the mean distance between fibres,  $\lambda$  as shown. Now there also exists a length scale over which the properties can be averaged in some meaningful way, that is,  $\delta$ . The length scale of averaging must be a dimension much larger than that of the characteristic dimension of the inhomogeneity, that is,  $\delta \geq \lambda$ . When this condition is satisfied the material can be idealized as being effectively homogeneous and the analysis of such a body can be done using the average properties associated with length scale  $\delta$ .

The condition mentioned above is called the effective or equivalent homogeneity. The other terms used are macroscopic homogeneity and statistical homogeneity.

## Concept of Volumetric Averaging

The effective or the average properties of the composite can be given through the relations between the average stress and average strain in the composite. Here, we introduce the concept of volumetric averaging.

Let us consider an RVE with dimension of inhomogeneity of  $\lambda$  and averaging dimension of  $\delta$  (see Figure 7.7). Let  $V_{RVE}$  be the volume of the RVE. Let us further assume that this RVE is subjected to macroscopically homogeneous stress or deformation field. Let us define the average stress as



**Figure 7.7: Scale of inhomogeneity and averaging of properties along with an RVE**

$$\bar{\sigma}_{ij} = \frac{1}{V_{RVE}} \int_{V_{RVE}} \sigma_{ij}(\mathbf{x}) DV_{RVE} \quad (7.74)$$

and the average strain as

$$\bar{\varepsilon}_{ij} = \frac{1}{V_{RVE}} \int_{V_{RVE}} \varepsilon_{ij}(\mathbf{x}) DV_{RVE} \quad (7.75)$$

where,  $\varepsilon_{ij}$  is the infinitesimal strain tensor at  $\mathbf{x}$ . Now, the effective, linear stiffness tensor  $C_{ijkl}^*$  is given by the relation

$$\bar{\sigma}_{ij} = C_{ijkl}^* \bar{\varepsilon}_{kl} \quad (7.76)$$

The process of volume averaging may look very simple at the first sight. However, the averaging of stresses and strains involves a significant amount of task. The exact stress and strain fields, that is,  $\sigma_{ij}(\mathbf{x})$  and  $\varepsilon_{ij}(\mathbf{x})$  in heterogeneous material are needed.

The inverse of the relation in Equation (7.76) can be given as

$$\bar{\varepsilon}_{ij} = S_{ijkl}^* \bar{\sigma}_{kl} \quad (7.77)$$

where,  $S_{ijkl}^* = (C_{ijkl}^*)^{-1}$  is the effective compliance tensor.

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### Concept of Energy Equivalence:

The effective properties of composite are defined through Equation (7.76) using the average stress and strain in the RVE. One can propose the alternate definition of the effective properties using the concept of energy equivalence.

Let us write the Equation (7.76) by contracting it with average strain tensor  $\bar{\epsilon}_{ij}$  as

$$\bar{\sigma}_{ij} \bar{\epsilon}_{ij} = C_{ijkl}^* \bar{\epsilon}_{kl} \bar{\epsilon}_{ij} \quad (7.78)$$

It should be noted that the average stress and strain field used in Equation (7.78) are obtained by solving the problem with RVE applied by appropriate boundary conditions. The stresses and the strains in RVE are macroscopically homogeneous. Hence, the averages can also be obtained by boundary values instead of volume integrations. Thus, the left hand side of Equation (7.78) can be given in terms of surface tractions and displacements as

$$\int_{S_{RVE}} T_i u_i dS_{RVE} = C_{ijkl}^* \bar{\epsilon}_{kl} \bar{\epsilon}_{ij} \quad (7.79)$$

where,  $S_{RVE}$  denotes the RVE boundary,  $T_i$  and  $u_i$  denote the traction and displacement vector on RVE boundary. Using the divergence theorem and equilibrium equations for the body without anybody forces,  $\sigma_{ij,j} = 0$ , we get

$$\int_{V_{RVE}} \sigma_{ij} \epsilon_{ij} dV_{RVE} = C_{ijkl}^* \bar{\epsilon}_{kl} \bar{\epsilon}_{ij} \quad (7.80)$$

Thus, the above relation gives the effective properties  $C_{ijkl}^*$  through the equivalence of the strain energy stored in the heterogeneous material to that stored in the equivalent homogeneous material.

In the following example, we explain concept of averaging. Consider the Figure 7.8(a). This figure represents the alternate repetition of two different materials with Young's modulus  $E_1$  and  $E_2$  and  $h_1$  and  $h_2$  be their respective axial lengths. Further, let  $A$  be the cross sectional area of the bar. This is equivalent to the alternate repetition of fibre and matrix in cross section. The bar made of such a material is subjected to axial force  $F$ . From the equilibrium consideration between the interface of fibre and matrix, it is clear that the force is uniform throughout the bar as shown in Figure 7.8(b). Thus, the axial force in any section  $F(x) = F$ . The axial strain in each of the segment of the bar is given as

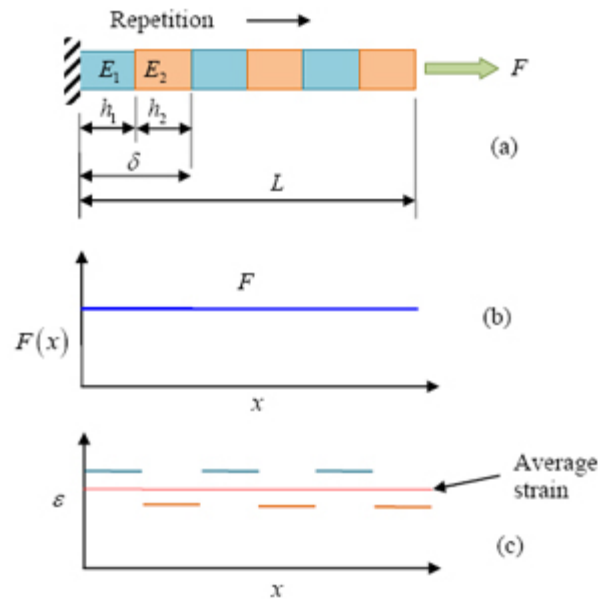
$$\epsilon_i = \frac{F}{E_i A} \quad (7.81)$$

From the above equation it is clear that the axial strains in fibre or matrix element are constants but there is a jump between interface of adjacent fibre and matrix. This is shown in Figure 7.8(c).

Now replace the material by an equivalent homogeneous material. Let  $E_H$  be the equivalent or homogeneous modulus of this material. Then, the force in equivalent material can be given as

$$(7.82)$$

$$F = E_H \varepsilon_H A$$



**Figure 7.8: (a) A beam representing alternate fibre and matrix (b) force distribution and (c) strain distribution along the length of the beam**

where,  $\varepsilon_H$  is the average or equivalent axial strain in the homogeneous material. Now, for a fibre and matrix element and its equivalent element the axial deformation is same. Thus,

$$\varepsilon_1 h_1 + \varepsilon_2 h_2 (h_1 + h_2) \quad (7.83)$$

Thus, from this equation we can write the average axial strain in the element as

$$\varepsilon_H = \frac{\varepsilon_1 h_1 + \varepsilon_2 h_2}{h_1 + h_2} \quad (7.84)$$

Now, from Equation (7.82) and Equation (7.84) we can give the equivalent axial Young's modulus as

$$E_H = \frac{F}{A \left[ \frac{\varepsilon_1 h_1 + \varepsilon_2 h_2}{h_1 + h_2} \right]} = \frac{h_1 + h_2}{\left[ \frac{h_1}{E_1} + \frac{h_2}{E_2} \right]} = \frac{E_1 E_2 (h_1 + h_2)}{E_2 h_1 + E_1 h_2} \quad (7.85)$$

In the above derivation Equation (7.81) has been used. Thus, Equation (7.85) gives the equivalent axial Young's modulus based on equilibrium considerations.

Now we will use the equivalence of strain energy approach to derive the equivalent axial Young's modulus. Let us consider that the fibre and matrix materials are linear elastic in behaviour. The strain energies of the non-homogeneous and homogeneous materials are equated as follows:

$$\frac{1}{2} E_1 \varepsilon_1^2 A h_1 + \frac{1}{2} E_2 \varepsilon_2^2 A h_2 = \frac{1}{2} E_H \varepsilon_H^2 A (h_1 + h_2) \quad (7.86)$$

Using Equation (7.81), we get

$$E_1 \frac{F^2}{A^2 E_1^2} h_1 + E_2 \frac{F^2}{A^2 E_2^2} h_2 = E_H \frac{F^2}{E_H^2 A^2} (h_1 + h_2) \quad (7.87)$$

which upon simplification gives us

$$E_H = \frac{E_1 E_2 (h_1 + h_2)}{E_2 h_1 + E_1 h_2} \quad (7.88)$$

Thus, Equation (7.88) gives the equivalent axial Young's modulus based on energy equivalence approach.

**Note:** In the above example, the equilibrium approach and energy equivalence approach gave the same expression for effective axial Young's modulus. However, for other examples and geometric details the effective properties obtained at the end may be different.

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### Standard Mechanics Approach:

In this approach the standard mechanics based problems imposed on an RVE are solved. There can be two types of loads: displacements or tractions. The boundary conditions that are imposed on RVE are such that in case of displacement loading an average strain is produced in homogeneous material of same size as the RVE. While in case of traction loading, the boundary conditions are chosen such that an average stress is produced in homogeneous material of same size as the RVE.

The average strain as defined in Equation (7.75) is further written in terms of displacements using divergence theorem as

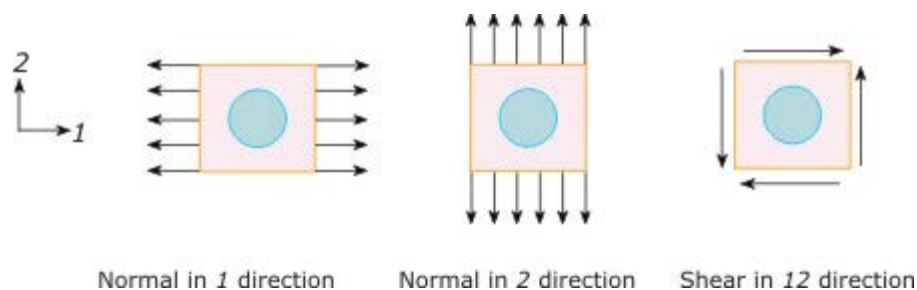
$$\bar{\varepsilon}_{ij} = \frac{1}{V_{RVE}} \int_{V_{RVE}} \varepsilon_{ij}(\mathbf{x}) dV = \frac{1}{V_{RVE}} \int_{V_{RVE}} \frac{1}{2} (u_i n_j + u_j n_i) dS_{RVE} \quad (7.89)$$

where,  $n_i$  denotes the unit normal to the RVE boundary and other quantities are as defined earlier. Now, let us write the average stress in Equation (7.74) as

$$\bar{\sigma}_{ij} = \frac{1}{V_{RVE}} \int_{V_{RVE}} \sigma_{ij}(\mathbf{x}) dV = \frac{1}{V_{RVE}} \int_{V_{RVE}} \frac{1}{2} (T_i x_j + T_j x_i) dS_{RVE} \quad (7.90)$$

where,  $x_i$  denotes the local boundary coordinates. For more details of the above derivation one can see work due to Hill [4].

Now the important task is to choose the boundary conditions that will give us either average strain or stress for displacement or traction loading. However, there is no unique relationship either between average strain and displacements or average stress and tractions. **For example**, a number of different combinations of displacements can produce the same average strain. Similarly, for the average stress in the material case there can be different combinations of tractions. Thus, in general, a uniform displacement or traction loading is chosen as boundary condition. These boundary conditions are shown in Figure 7.9. It should be noted that the applied boundary condition shown in this figure is displacement when average strain is desired and is traction if average stress is desired. Further, it should be noted that the boundary conditions shown in this figure are for a planar problem.



**Figure 7.9: Uniform boundary conditions on an RVE**

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The standard weak form of the equilibrium equation is solved to calculate the local RVE strain. In case of the applied displacements, the weak form of the RVE equilibrium equations is

$$\int_{V_{RVE}} C_{ijkl} \varepsilon_{ij}(\mathbf{v}) \varepsilon_{kl}^{mn}(\mathbf{u}) dV_{RVE} - \int_{S_{RVE}} \lambda v_i (u_i^{mn} - d_i^{mn}) dS_{RVE} \quad (7.91)$$

where,  $C_{ijkl}$  is the point-wise stiffness tensor in the RVE materials,  $\varepsilon_{ij}(\mathbf{v})$  are the virtual strains,  $\varepsilon_{kl}^{mn}(\mathbf{u})$  are the microstructural strains due to the applied displacements,  $u_i^{mn}$  is the applied displacement,  $d_i^{mn}$  is the specified boundary displacement which produces the desired uniform average strain  $\bar{\varepsilon}_{ij}$  in a homogeneous material and  $v_i$  is the virtual displacement. Here, the boundary displacements are implemented using a penalty method with  $\lambda$  as the penalty parameter.

In case of applied tractions the weak form the equilibrium equation to be solved for each applied  $mn^{\text{th}}$  stress component is

$$\int_{V_{RVE}} C_{ijkl} \varepsilon_{ij}(\mathbf{v}) \varepsilon_{kl}^{mn}(\mathbf{u}) dV_{RVE} = \int_{S_{RVE}} T_i^{mn} v_i dS_{RVE} \quad (7.92)$$

where,  $T_i^{mn}$  is the applied traction which produces the average stress  $\bar{\sigma}_{ij}$  in a homogeneous material. Further,  $\varepsilon_{kl}^{mn}(\mathbf{u})$  are the microstructural strains due to the applied  $mn^{\text{th}}$  traction. The other terms in above equation are as defined earlier.

Equation (7.91) and Equation (7.92) need to be solved only for three times. In these two equations we solve for local or microstructural strains. In this case we are considering symmetry of stress and strain tensors. In case of three dimensional problems we need to solve for six problems.

The relation between the average strain and local or microstructural strains is given as

$$\varepsilon_{ij}^{mn} = A_{ijkl} \bar{\varepsilon}_{kl}^{mn} \quad (7.93)$$

where,  $A_{ijkl}$  is called as local structure tensor or strain concentration factors. It should be noted that this tensor has minor symmetries like  $A_{ijkl} = A_{jikl} = A_{ijlk}$ . However, it does not have the major symmetry  $A_{ijkl} = A_{klij}$ . Thus, if the local structure tensor is known then local strain at any point can be given as

$$\varepsilon_{ij}(\mathbf{x}) = A_{ijkl}(\mathbf{x}) \bar{\varepsilon}_{kl} \quad (7.94)$$

Now the effective stiffness tensor  $C_{ijkl}^*$  is as defined by Equation (7.76). This tensor can be calculated from  $A_{ijkl}$ . Using the Hooke's law at microscopic level, we can write

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} \quad (7.95)$$



Now, using the definition of average stress as in Equation (7.74), the above equation can be written as

$$\frac{1}{V_{RVE}} \int_{V_{RVE}} \sigma_{ij} dV_{RVE} = \frac{1}{V_{RVE}} \int_{V_{RVE}} C_{ijkl} \varepsilon_{kl} dV_{RVE} \quad (7.96)$$

Now using Equation (7.94) for  $\varepsilon_{kl}$  the above equation becomes

$$\bar{\sigma}_{ij} = \frac{1}{V_{RVE}} \int_{V_{RVE}} C_{ijkl} A_{klpm} dV_{RVE} \bar{\varepsilon}_{pm} \quad (7.97)$$

Now, it should be noted that  $\bar{\varepsilon}_{pm}$  is the average strain.

Thus, we can define the effective stiffness tensor as

$$C_{ijkl}^* = \frac{1}{V_{RVE}} \int_{V_{RVE}} C_{ijpm}(\mathbf{x}) A_{pmkl}(\mathbf{x}) dV_{RVE} \quad (7.98)$$

The tensors  $C_{ijpm}(\mathbf{x})$  and  $A_{pmkl}(\mathbf{x})$  are the local tensors and can be obtained in individual constituents of the RVE. Once, the effective stiffness tensor is obtained the inverse of this gives the effective compliance tensor. Now the relation between the individual elements of this tensor and engineering constants can be used to determine the effective engineering constants.

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#### Home Work:

1. What is statistical homogeneity?
2. Write a short note on volumetric averaging.
3. Write a short note on energy equivalence approach in averaging.
4. Explain in detail the standard mechanics approach.
5. For the one dimensional problem as shown in Figure 7.8, show that the effective Young's modulus determined using standard mechanics approach is same as given in Equation (7.85) (which is same as in Equation (7.88)).

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