Module 4: Plane Stress Constitutive Equations

## Lecture 15: Lamina Engineering Constants

**The Lecture Contains**:

## Introduction

In the previous lecture we have derived constitutive equation for planar state of stress in a lamina. We have derived these constitutive equations in principal material and global directions. In this lecture, we are going to see a practical application of the planar constitutive equations in industry. Although, the engineering constants in principal material directions are known, it is difficult to comment on the engineering constant for off axis lamina, instantly. When laminae are used for designing a structure, the engineering constants in global directions become very useful for a quick estimate of the behavior of the structure under certain loads. Thus, for practical application purpose, the various lamina engineering constants are obtained and their variation for fibre orientation between $-90^\circ \leq \theta \leq 90^\circ$ for a range of composite materials is given together. Thus, a designer can use the required lamina with appropriate fibre orientation and material.

Here we are going to obtain engineering constants for any off axis lamina as a function of engineering constants of that lamina in principal material directions and fibre orientation. This can be done with the help of lamina constitutive equation with appropriate one dimensional state of stress.

We have constitutive equation in global directions as given in Equation (4.5)

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}$$

Thus, for a given state of stress in global directions we can find the strains in global directions from this equation.

Previous Next

## Axial Modulus:

Consider a unidirectional off axis lamina. This lamina is subjected to the loading $\sigma_{xx} \neq 0$ and $\sigma_{yy} = \tau_{xy} = 0$ as shown in Figure 4.2.
Thus, from Equation (4.5) for this state of stress we can write the axial strain as

$$\varepsilon_{xx} = \bar{S}_{11}\sigma_{xx} \tag{4.61}$$

The Young's modulus in *x*-direction is now defined as

$$E_x = \frac{\sigma_{xx}}{\varepsilon_{xx}} \tag{4.62}$$

Thus, from Equation (4.61), we can write

$$E_x = \frac{1}{\bar{S}_{11}} \tag{4.63}$$

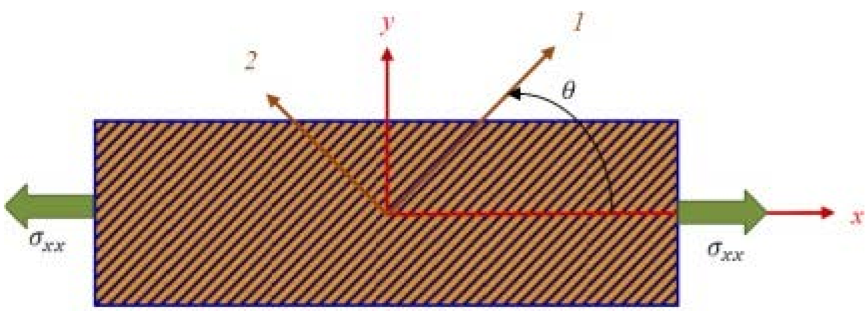


**Figure 4.2: Off axis lamina loaded in traction along *x* direction**

In the above equation, $\bar{S}_{11}$ is written using compliance terms in principal material directions as

$$E_x = \frac{1}{\bar{S}_{11}} = \frac{1}{m^4 S_{11} + m^2 n^2 (2S_{12} + S_{66}) + n^4 S_{22}} \tag{4.64}$$

Further, it can be writing compliance terms in principal directions using engineering constants doing some rearrangements as

$$E_x = \frac{E_1}{m^4 + m^2 n^2 \left(-2\nu_{12} + \frac{E_1}{G_{12}}\right) + n^4 \frac{E_1}{E_2}} \tag{4.65}$$

From this expression it is easy to see that the modulus $E_x = E_1$ when $\theta = 0°$ and $E_x = E_2$ when $\theta = 90°$. The variation of the modulus $E_x$ with fibre orientation for AS4/3501-6 Epoxy material is shown in Figure 4.3. The variation of the modulus $E_x$ for both positive and negative fibre orientations is identical in nature.

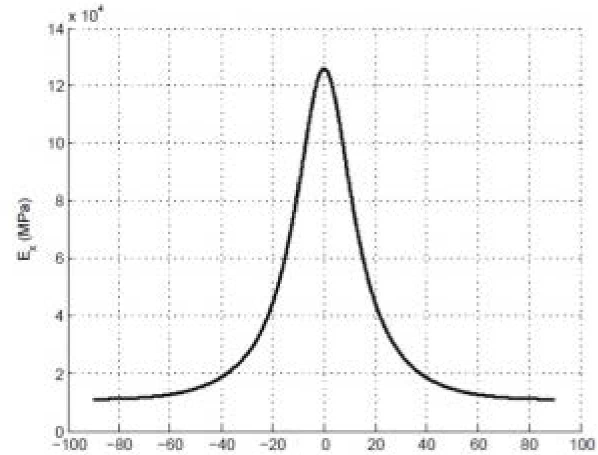


**Figure 4.3: Variation of axial modulus with fibre orientation for AS4/3501-6 Epoxy**

## Axial Poisson's Ratio:

The axial Poisson's ratio $\nu_{xy}$ can also be obtained for above loading condition. This Poisson's ratio is defined as

$$\nu_{xy} = \frac{-\varepsilon_{yy}}{\varepsilon_{xx}} \tag{4.66}$$

Using Equation (4.61) we can write

$$\nu_{xy} = \frac{-\bar{S}_{12}\sigma_{xx}}{\bar{S}_{11}\sigma_{xx}} = \frac{-\bar{S}_{12}}{\bar{S}_{11}} \tag{4.67}$$

Further, expressing the compliance terms in above equation in terms of engineering constants we can write

$$\nu_{xy} = \frac{-\left[m^2n^2\left(1+\frac{E_1}{E_2}-\frac{E_1}{G_{12}}\right)-\left(m^4+n^4\right)\nu_{12}\right]}{\left[m^4+m^2n^2\left(-2\nu_{12}+\frac{E_1}{G_{12}}\right)+n^4\frac{E_1}{E_2}\right]} \tag{4.68}$$

Previous Next

## Transverse Modulus:

Consider an off-axis lamina subjected to in-plane transverse loading as shown in Figure 4.4. Thus, for this loading condition we have $\sigma_{yy} \neq 0$ and $\sigma_{xx} = \tau_{xy} = 0$.

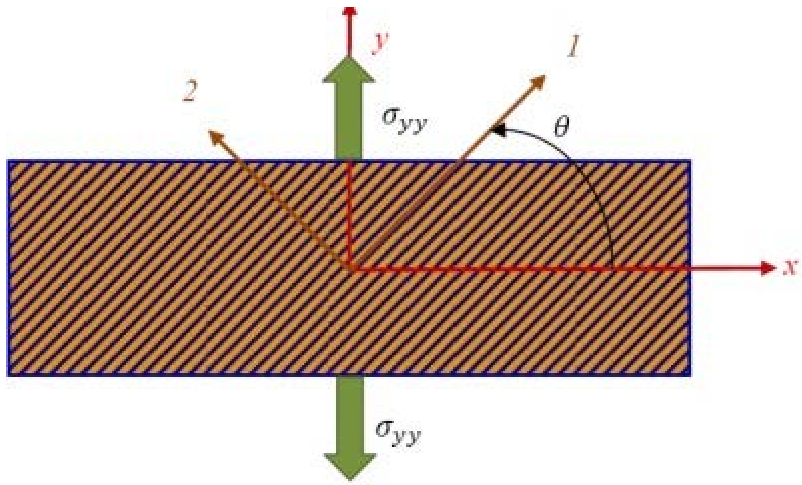


**Figure 4.4: Off axis lamina loaded in traction along *y* direction**

Let us define the transverse modulus as

$$E_y = \frac{\sigma_{yy}}{\varepsilon_{yy}} \tag{4.69}$$

Thus, from Equation (4.5) for the above loading, we can write

$$E_y = \frac{\sigma_{yy}}{\bar{S}_{22}\,\sigma_{yy}} = \frac{1}{\bar{S}_{22}} \tag{4.70}$$

If we express $\bar{S}_{22}$ using engineering constants, we get

$$E_y = \frac{E_1}{\left[n^4 + m^2 n^2\left(-2\nu_{12} + \frac{E_1}{G_{12}}\right) + m^4 \frac{E_1}{E_2}\right]} \tag{4.71}$$

From this expression, we can see that the modulus $E_y = E_2$ when $\theta = 0°$ and $E_y = E_1$ when $\theta = 90°$. The variation of the modulus $E_y$ with fibre orientation for AS4/3501-6 Epoxy material is shown in Figure 4.5. The variation of the modulus $E_y$, similar to the variation of $E_x$, for both positive and negative fibre orientations is identical in nature. Further, it can be observed that the curve for $E_y$ is shifted by $90°$ to that of $E_x$.

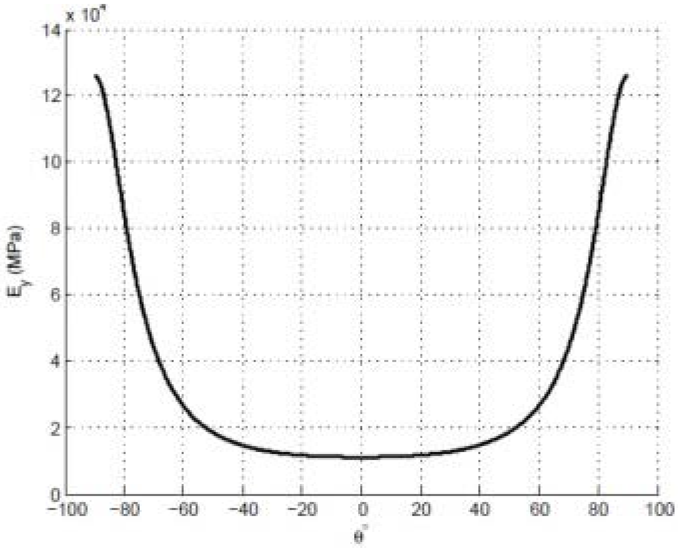


**Figure 4.5: Variation of transverse modulus with fibre orientation for AS4/3501-6 Epoxy**

### Other Poisson's Ratio:

The other Poisson's ratio $\nu_{yx}$ can be obtained from the loading condition given in Figure 4.4. Let us define this Poisson's ratio as

$$\nu_{yx} = \frac{-\varepsilon_{xx}}{\varepsilon_{yy}} \qquad (4.72)$$

Thus, using Equation (4.5) for this loading, it becomes

$$\nu_{yx} = \frac{-\varepsilon_{xx}}{\varepsilon_{yy}} = \frac{-\bar{S}_{12}\,\sigma_{yy}}{\bar{S}_{22}\,\sigma_{yy}} = \frac{-\bar{S}_{12}}{\bar{S}_{22}} \qquad (4.73)$$

which can be written using engineering constants as

$$\nu_{yx} = \frac{-\left[n^2 m^2\left(1+\frac{E_1}{E_2}-\frac{E_1}{G_{12}}\right)-\left(n^4+m^4\right)\nu_{12}\right]}{\left[n^4+m^2 n^2\left(-2\nu_{12}+\frac{E_1}{G_{12}}\right)+m^4\frac{E_1}{E_2}\right]} \qquad (4.74)$$

The fibre orientation dependence of axial Poisson's ratio and the other Poisson's ratio for AS4/3501-6 Epoxy is shown in Figure 4.6.

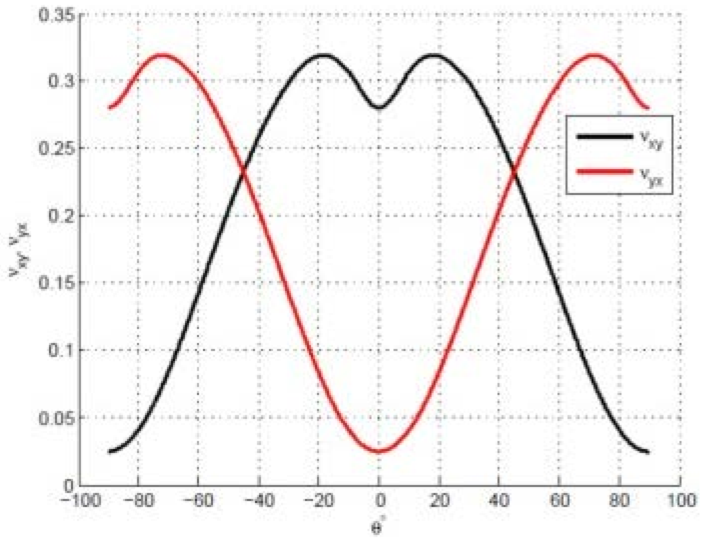


**Figure 4.6: Variation of Poisson's ratios with fibre orientation**

Previous Next

## In-plane Shear Modulus:

The in-plane or axial shear modulus for an off axis lamina can be obtained when it subjected to a pure shear loading as shown in Figure 4.7. Thus, for this loading condition we have $\tau_{xy} \neq 0$ and . $\sigma_{xx} = \sigma_{yy} = 0$.

For this loading, we define the in-plane shear modulus $G_{xy}$ as

$$G_{xy} = \frac{\tau_{xy}}{\gamma_{xy}} \tag{4.75}$$

With the help of Equation (4.5) we rewrite this equation as

$$G_{xy} = \frac{\tau_{xy}}{\gamma_{xy}} = \frac{\tau_{xy}}{\bar{S}_{66}\tau_{xy}} = \frac{1}{\bar{S}_{66}} \tag{4.76}$$

And in terms of engineering constants, it becomes

$$G_{xy} = \frac{E_1}{\left[4n^2\, m^2 \left(1+2\nu_{12}+\frac{E_1}{E_2}\right)+(n^2-m^2)^2\frac{E_1}{G_{12}}\right]} \tag{4.77}$$

The variation of $G_{xy}$ with fibre orientation between $-90°$ to $90°$ for AS4/3501-6 Epoxy material is shown in Figure 4.8. From this figure it can be seen that shear modulus is maximum when $\theta = \pm 45°$. At $\theta = \pm 45°$ the value of shear modulus is

$$G_{xy} = \frac{E_1}{\left(1+2\nu_{12}+\frac{E_1}{E_2}\right)} \tag{4.78}$$

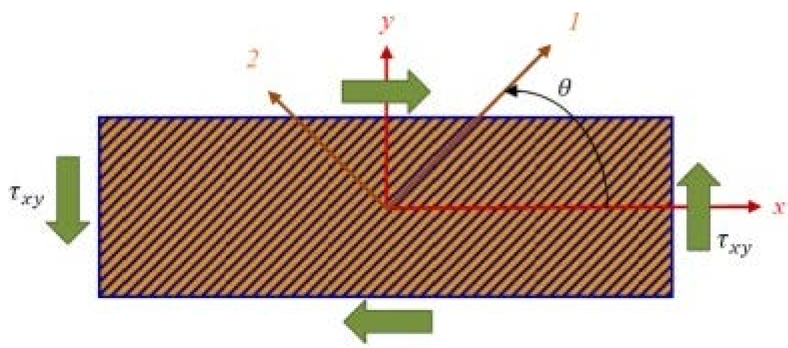


**Figure 4.7: Off axis lamina loaded in pure shear**

***Note***: When the material is isotropic, that is $E_1 = E_2$ and $\nu_{12} = \nu$, then the above expression reduces to the familiar relation

$$G = \frac{E}{(1+\nu)} \tag{4.79}$$

The minimum value of shear modulus is seen when the lamina is loaded in shear in principal material directions and its value becomes

$$G = \frac{E}{(1+\nu)} \quad (4.80)$$

***Note*:** It is very important to note that the shear modulus of the lamina is a minimum when lamina is in principal directions and a maximum when fibre orientation is $45°$ or $-45°$. Further, the behavior of a lamina under same pure shear for fibre orientation $45°$ is significantly different from that of lamina with fibre orientation of $-45°$. The physical significance of this phenomenon is explained in greater details in the later section.

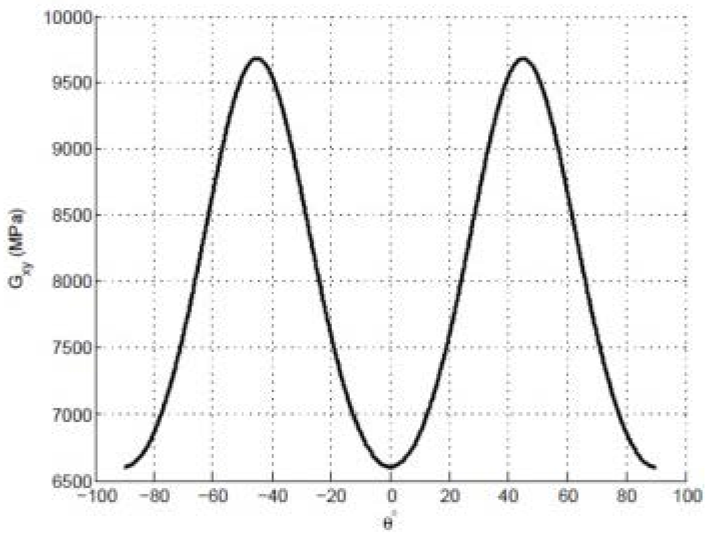


**Figure 4.8: Variation of shear modulus with fibre orientation for AS4/3501-6**

Previous Next

### Coefficients of Mutual Influence:

It is well understood by now that for an off-axis lamina there is normal-shear coupling. The terms $\overline{Q}_{26}$ $\overline{Q}_{16}$ and denote the normal-shear coupling. In other words, these terms express that when there is a normal stress there will be associated shear strain and in a similar way when there is a shear stress there will be associated normal strain or vice-a-versa.

The normal and shear coupling has been quantified by Lekhnitskii by coefficients of mutual influence. Two kinds of coefficients of mutual influence have been defined. The first one is defined for applied shear stress and the second one is defined for applied normal stress. These are defined as the ratio of an associated strain to the applied strain for the given state of stress. Thus, the coefficients of mutual influence of the first kind are defined as

$$\eta_{i,ij} = \frac{\varepsilon_i}{\gamma_{ij}} \tag{4.81}$$

where $\varepsilon_i$ denotes the axial normal strains, that is $\varepsilon_{xx}$ or $\varepsilon_{yy}$, and $\gamma_{ij} = \gamma_{xy}$ denotes the in-plane engineering shear strain. For this case, the state of stress would be $\tau_{xy} \neq 0$ and $\sigma_{xx} = \sigma_{yy} = 0$. Similarly, the coefficients of mutual influence of the second kind are defined as

$$\eta_{\,ij,i} = \frac{\gamma_{ij}}{\varepsilon_i} \tag{4.82}$$

The state of stress for this case could be either $\sigma_{xx} \neq 0$ and $\sigma_{yy} = \tau_{xy} = 0$ or $\sigma_{yy} \neq 0$ and $\sigma_{xx} = \tau_{xy} = 0$.

Now, let us obtain expressions for the coefficients of mutual influence of the first kind. We have

$$\eta_{\,x,xy} = \frac{\varepsilon_{xx}}{\gamma_{xy}} = \frac{S_{16}\tau_{xy}}{S_{66}\tau_{xy}} = \frac{mn\left[m^2\left(2+2\nu_{12}-\frac{E_1}{G_{12}}\right)+n^2\left(-2\nu_{12}-2\frac{E_1}{E_2}+\frac{E_1}{G_{12}}\right)\right]}{\left[4n^2\,m^2\left(1+2\nu_{12}+\frac{E_1}{E_2}\right)+(n^2-m^2)^2\frac{E_1}{G_{12}}\right]} \tag{4.83}$$

$$\eta_{\,y,xy} = \frac{\varepsilon_{yy}}{\gamma_{xy}} = \frac{S_{26}\tau_{xy}}{S_{66}\tau_{xy}} = \frac{mn\left[n^2\left(2+2\nu_{12}-\frac{E_1}{G_{12}}\right)+m^2\left(-2\nu_{12}-2\frac{E_1}{E_2}+\frac{E_1}{G_{12}}\right)\right]}{\left[4n^2\,m^2\left(1+2\nu_{12}+\frac{E_1}{E_2}\right)+(n^2-m^2)^2\frac{E_1}{G_{12}}\right]} \quad (4.84$$

Now, we will obtain expressions for the coefficients of mutual influence of the second kind. For the loading shown in Figure 4.2, we will get

$$\eta_{\,xy,x} = \frac{\gamma_{xy}}{\varepsilon_{xx}} = \frac{S_{16}\sigma_{xx}}{S_{11}\sigma_{xx}} \tag{4.85}$$

which will be simplified and expressed in terms of engineering constants in principle material directions and fibre orientation as

(4.86)

$$\eta_{xy,x} = \frac{S_{16}}{S_{11}} = \frac{mn\left[m^2\left(2+2\nu_{12}-\frac{E_1}{G_{12}}\right)+n^2\left(-2\nu_{12}-2\frac{E_1}{E_2}+\frac{E_1}{G_{12}}\right)\right]}{\left[m^4+m^2\ n^2\left(-2\nu_{12}+\frac{E_1}{G_{12}}\right)+n^4\ \frac{E_1}{E_2}\right]}$$

Similarly, for the loading shown in Figure 4.4, we get the remaining coefficient of mutual influence as

$$\eta_{xy,y} = \frac{\gamma_{xy}}{\varepsilon_{yy}} = \frac{S_{26}\,\sigma_{yy}}{S_{22}\,\sigma_{yy}} = \frac{mn\left[n^2\left(2+2\nu_{12}-\frac{E_1}{G_{12}}\right)+m^2\left(-2\nu_{12}-2\frac{E_1}{E_2}+\frac{E_1}{G_{12}}\right)\right]}{\left[n^4+m^2n^2\left(-2\nu_{12}+\frac{E_1}{G_{12}}\right)+m^4\frac{E_1}{E_2}\right]} \quad (4.87)$$

The variation of the coefficients of mutual influence of the first kind and second kind for AS4/3501-6 Epoxy material with fibre orientation between $-90^\circ$ to $90^\circ$ is shown in Figure 4.9.

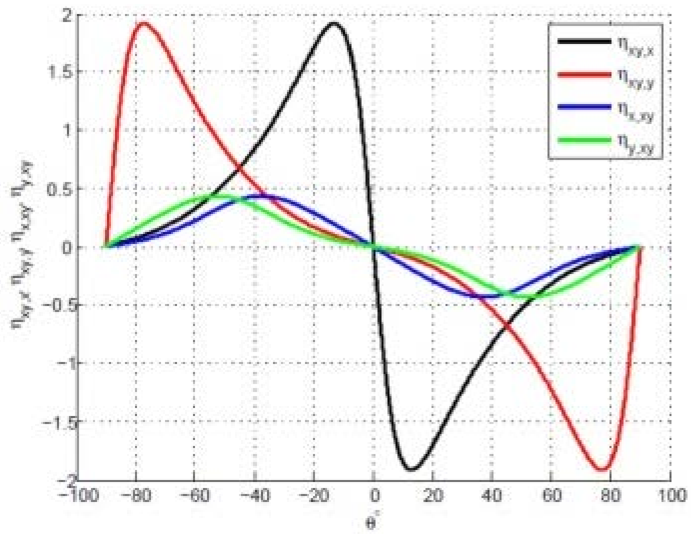


**Figure 4.9: Variation of coefficients of mutual influence with fibre orientation for AS4/3501-6 Epoxy**

Previous Next

### Significance of Shear Loading Directions in Off-Axis Lamina:

The direction of shear loads applied to a lamina, especially an off-axis lamina, is very important both from shear stiffness as well as strength point of view. This is explained with respect to an off-axis lamina. Here we have illustrated for $\theta = 45^\circ$.

The two cases of pure shear loading of a $\theta = 45^\circ$ lamina are shown in Figure 4.10. In these cases the direction of loading is reversed. The pure shear loading can be shown to be equivalent traction and compression loading along the $45^\circ$ diagonals of a square element. This is depicted in Figure 4.10 for both cases. For the first case, the fibres are subjected to tensile normal stress and matrix is subjected to compressive normal stress, whereas for the second case, the fibres are subjected to compressive normal stress and matrix is subjected tensile normal stress. The first case of shear loading shown in Figure 4.10 is called ***Positive Shear*** and the second case is called ***Negative Shear***.

In the case when fibres are oriented at $\pm 45^\circ$, either tensile or compressive normal stress is aligned along the fibres, thus resulting in higher shear stiffness at $\pm 45^\circ$. However, when the lamina is loaded in pure shear in principal material directions (as shown in Figure 4.11), the equivalent stress in fibre is neither pure normal tensile stress nor pure normal pure compressive stress. Thus, it results in lower shear stiffness, that is $\tau_{xy} = \tau_{12}$

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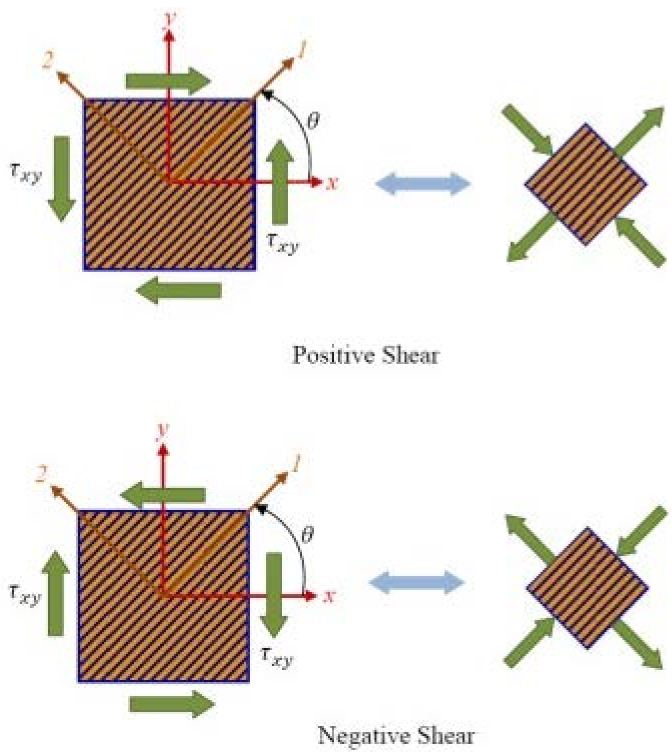


**Figure 4.10: Off-axis lamina loaded in pure shear**

It is well known that fibres are good in traction and weak in compressive loading. Thus, it is desirable from designing point of view that the shear loading should results in an equivalent loading in which the fibres are subjected to tensile normal stress. This kind of shear loading of an off-axis lamina will ensure the higher shear strength of the lamina. In case of $\pm 45^\circ$ off-axis lamina the fibres are in pure tensile for their positive shear loading. Thus, it results into the highest shear strength.

The loading of an off-axis lamina in pure shear should be, in general, positive shear. This is one of the important design consideration.

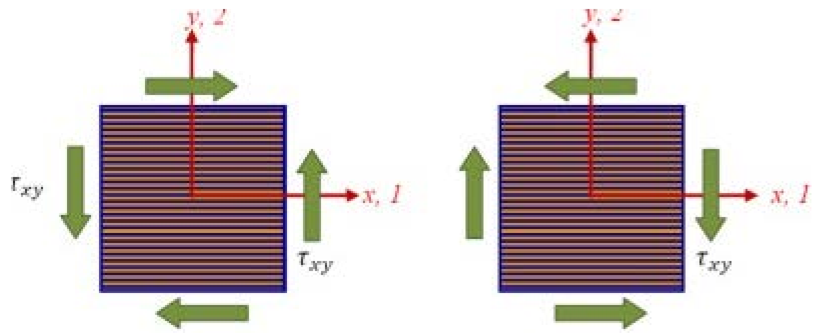


**Figure 4.11: Unidirectional lamina loaded in pure shear**

Previous Next

## Examples:

***Example 4.4 :*** Calculate all the lamina engineering constants for an off-axis lamina of AS4/3501-6 Epoxy with fibre orientation of $\theta = 22.5°$.

***Solution:***

We know that all the lamina engineering constants are either reciprocal of ratio of two compliance terms in global material directions. So, we obtain compliance terms in principal material directions and then we transform it into global directions with $\theta = 22.5°$.

$$S_{11} = \frac{1}{E_1} = 0.00793 \times 10^{-3} \qquad S_{12} = \frac{-\nu_{21}}{E_2} = -0.00222 \times 10^{-3}$$

$$S_{22} = \frac{1}{E_2} = 0.09091 \times 10^{-3} \qquad S_{66} = \frac{1}{G_{12}} = 0.15151 \times 10^{-3}$$

**Unit of all terms is 1/MPa.**

$$[\bar{S}] = [T_2]^{-1}[S][T_1]$$

Now, for $\theta = 22.5°$

$$[T_1] = \begin{bmatrix} 0.85355 & 0.14644 & 0.70710 \\ 0.14644 & 0.85355 & -0.70710 \\ -0.35355 & 0.35355 & 0.70710 \end{bmatrix} \text{ and } [T_2]^{-1} = \begin{bmatrix} 0.85355 & 0.14644 & -0.35355 \\ 0.14644 & 0.85355 & 0.35355 \\ 0.70710 & -0.70710 & 0.70710 \end{bmatrix}$$

Thus, carrying out matrix multiplications, we get

$$[\bar{S}] = \begin{bmatrix} 0.02611 & -0.00825 & -0.04139 \\ -0.00825 & 0.08478 & -0.01727 \\ -0.04139 & -0.01727 & 0.12740 \end{bmatrix} \times 10^{-3} \text{ 1/MPa}$$

Now, we calculate the lamina engineering constants as

$$E_x = \frac{1}{\bar{S}_{11}} = 38291\ MPa$$

$$E_y = \frac{1}{\bar{S}_{22}} = 11794\ MPa$$

$$G_{xy} = \frac{1}{\bar{S}_{66}} = 7849\ MPa$$

$$\nu_{xy} = \frac{-\bar{S}_{12}}{\bar{S}_{11}} = 0.3159$$

$$\nu_{yx} = \frac{-\bar{S}_{12}}{\bar{S}_{22}} = 0.0973$$

$$\eta_{x,xy} = \frac{\bar{S}_{16}}{\bar{S}_{66}} = -0.3248$$

$$\eta_{y,xy} = \frac{\bar{S}_{26}}{\bar{S}_{66}} = -0.1356$$

Using direct equations given above can also be used but this should be done only when one is confident of remembering these relations in terms engineering constants in principal material directions.

Previous Next

Module 4: Plane Stress Constitutive Equations

Lecture 15: Lamina Engineering Constants

**Homework:**

1. Obtain the lamina engineering constants for materials given in Table 3.1 for fibre orientation of $\theta = 30°, 60°, 67.5°$

2. Write a computer code to plot the variation of all lamina engineering constants and coefficients of mutual influence against the fibre orientation from $-90° < \theta < 90°$ . Further, plot the variations for the materials given in Table 3.1 simultaneously and compare their behaviour and comment on key observations.

Previous Next

Module 4: Plane Stress Constitutive Equations

Lecture 15: Lamina Engineering Constants

## References:


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- *PD Soden, MJ Hinton, AS Kaddour. Lamina properties, lay-up configurations and loading conditions for a range of fibre-reinforced composite laminates. Composite Science and Technology, 1998;58:1011-1022.*


Previous Next