Module 3: 3D Constitutive Equations

Lecture 10: Constitutive Relations: Generally Anisotropy to Orthotropy

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Introduction

In this lecture, we are going to develop the *3D* constitutive equations. We will start with the the generalized Hooke's law for a material, that is, material is generally anisotropic in nature. Finally, we will derive the constitutive equation for isotropic material, with which the readers are very familiar. The journey for constitutive equation from anisotropic to isotropic material is very interesting and will use most of the concepts that we have learnt in earlier Lecture 9.

The generalized Hooke's law for a material is given as

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} \quad i,j,k,l = 1,2,3 \qquad (3.1)$$

where, σ_{ij} is a second order tensor known as **stress tensor** and its individual elements are the stress components. ε_{ij} is another second order tensor known as **strain tensor** and its individual elements are the strain components. C_{ijkl} is a fourth order tensor known as **stiffness tensor**. In the remaining section we will call it as **stiffness matrix**, as popularly known. The individual elements of this tensor are the stiffness coefficients for this linear stress-strain relationship. Thus, stress and $(3)^4$ strain tensor has ($3 \times 3 =$) 9 components each and the stiffness tensor has (=) 81 independent elements. The individual elements =) 81 are referred by various names as **elastic constants, moduli and stiffness coefficients**. The reduction in the number of these elastic constants can be sought with the following symmetries.

Stress Symmetry:

The stress components are symmetric under this symmetry condition, that is, $\sigma_{ij} = \sigma_{ji}$. Thus, there are six independent stress components. Hence, from Equation. (3.1) we write

$$\sigma_{ji} = C_{jikl}\,\varepsilon_{kl} \qquad (3.2)$$

Subtracting Equation (3.2) from Equation (3.1) leads to the following equation

$$0 = \left(C_{ijkl} - C_{jikl}\right)\varepsilon_{kl} \quad \Rightarrow \quad C_{ijkl} = C_{jikl} \qquad (3.3)$$

There are six independent ways to express *i* and *j* taken together and still nine independent ways to express *k* and *l* taken together. Thus, with stress symmetry the number of independent elastic constants reduce to ($6 \times 9 =$) 54 from 81.

Strain Symmetry:

The strain components are symmetric under this symmetry condition, that is, $\varepsilon_{ij} = \varepsilon_{ji}$. Hence, from Equation (3.1) we write

$$\sigma_{ij} = C_{ijlk}\,\varepsilon_{lk}$$

Subtracting Equation (3.3) from Equation (3.2) we get the following equation

$$0 = \left(C_{ijkl} - C_{ijlk}\right)\varepsilon_{kl} \quad \Rightarrow \quad C_{ijkl} = C_{ijlk} \qquad (3.4)$$

It can be seen from Equation (3.4) that there are six independent ways of expressing *i* and *j* taken together when *k* and *l* are fixed. Similarly, there are six independent ways of expressing *k* and *l* taken together when *i* and *j* are fixed in Equation (3.4). Thus, there are $6 \times 6 = 36$ independent constants for this linear elastic material with stress and strain symmetry.

With this reduced stress and strain components and reduced number of stiffness coefficients, we can write Hooke's law in a contracted form as

$$\sigma_i = C_{ij}\varepsilon_j \quad (i, j = 1, 2, \cdots, 6) \qquad (3.5)$$

where

$$\begin{aligned}
\sigma_1 &= \sigma_{11} & \varepsilon_1 &= \varepsilon_{11} \\
\sigma_2 &= \sigma_{22} & \varepsilon_2 &= \varepsilon_{22} \\
\sigma_3 &= \sigma_{33} & \varepsilon_3 &= \varepsilon_{33} \\
\sigma_4 &= \sigma_{23} & \varepsilon_4 &= 2\varepsilon_{23} \\
\sigma_5 &= \sigma_{13} & \varepsilon_5 &= 2\varepsilon_{13} \\
\sigma_6 &= \sigma_{12} & \varepsilon_6 &= 2\varepsilon_{12}
\end{aligned} \qquad (3.6)$$

Note: The shear strains are the engineering shear strains.

For Equation (3.5) to be solvable for strains in terms of stresses, the determinant of the stiffness matrix must be nonzero, that is $\left|C_{ij}\right| \neq 0$.
The number of independent elastic constants can be reduced further, if there exists strain energy density function W, given as below.

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Strain Energy Density Function (*W*):

The strain energy density function *W* is given as

$$W = \frac{1}{2} C_{ji} \varepsilon_j \varepsilon_i \tag{3.7}$$

with the property that

$$\sigma_i = \frac{\partial W}{\partial \varepsilon_i} \tag{3.8}$$

It is seen that W is a quadratic function of strain. A material with the existence of W with property in Equation (3.8) is called as Hyperelastic Material.

The W can also be written as

$$W = \frac{1}{2} C_{ji} \varepsilon_j \varepsilon_i \tag{3.9}$$

Subtracting Equation (3.9) from Equation (3.7) we get

$$0 = \left(C_{ij} - C_{ji}\right) \varepsilon_i \varepsilon_j \tag{3.10}$$

which leads to the identity $C_{ij} = C_{ji}$. Thus, the stiffness matrix is symmetric. This symmetric matrix has 21 independent elastic constants. The stiffness matrix is given as follows:

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & \text{Symmetric} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \tag{3.11}$$

The existence of the function W is based upon the first and second law of thermodynamics. Further, it should be noted that this function is positive definite. Also, the function *W* is an invariant (An invariant is a quantity which is independent of change of reference).

The material with 21 independent elastic constants is called Anisotropic or Aelotropic Material.
Further reduction in the number of independent elastic constants can be obtained with the use of planes of material symmetry as follows.

Material Symmetry:

It should be recalled that both the stress and strain tensor follow the transformation rule and so does the stiffness tensor. The transformation rule for these quantities (as given in Equation (3.1)) is known as follows

$$\begin{aligned} \sigma'_{ij} &= a_{ki}\, a_{lj}\, \sigma_{kl} \\ \varepsilon'_{ij} &= a_{ki}\, a_{lj}\, \varepsilon_{kl} \\ C'_{ijkl} &= a_{pi}\, a_{qj}\, a_{rk}\, a_{sl}\, C_{pqrs} \end{aligned} \tag{3.12}$$

where a_{ij} are the direction cosines from *i* to *j* coordinate system. The prime indicates the quantity in new coordinate system.

When the function W given in Equation (3.9) is expanded using the contracted notations for strains and elastic constants given in Equation (3.11) W has the following form:

$$W = \frac{1}{2}\begin{bmatrix} C_{11}\varepsilon_1^2 + 2C_{12}\varepsilon_1\varepsilon_2 + 2C_{13}\varepsilon_1\varepsilon_3 + 2C_{14}\varepsilon_1\varepsilon_4 + 2C_{15}\varepsilon_1\varepsilon_5 + 2C_{16}\varepsilon_1\varepsilon_6 + \\ C_{22}\varepsilon_2^2 + 2C_{23}\varepsilon_2\varepsilon_3 + 2C_{24}\varepsilon_2\varepsilon_4 + 2C_{25}\varepsilon_2\varepsilon_5 + 2C_{26}\varepsilon_2\varepsilon_6 + \\ C_{33}\varepsilon_3^2 + 2C_{34}\varepsilon_3\varepsilon_4 + 2C_{35}\varepsilon_3\varepsilon_5 + 2C_{36}\varepsilon_3\varepsilon_6 + \\ C_{44}\varepsilon_4^2 + 2C_{45}\varepsilon_4\varepsilon_5 + 2C_{46}\varepsilon_4\varepsilon_6 + \\ C_{55}\varepsilon_5^2 + 2C_{56}\varepsilon_5\varepsilon_6 + \\ C_{66}\varepsilon_6^2 \end{bmatrix} \tag{3.13}$$

Thus, from Equation (3.13) it can be said that the function *W* has the following form in terms of strain components:

$$W = W\begin{bmatrix} \varepsilon_1^2, \varepsilon_2^2, \varepsilon_3^2, \varepsilon_4^2, \varepsilon_5^2, \varepsilon_6^2, \\ \varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_3, \varepsilon_1\varepsilon_4, \varepsilon_1\varepsilon_5, \varepsilon_1\varepsilon_6, \\ \varepsilon_2\varepsilon_3, \varepsilon_2\varepsilon_4, \varepsilon_2\varepsilon_5, \varepsilon_2\varepsilon_6, \\ \varepsilon_3\varepsilon_4, \varepsilon_3\varepsilon_5, \varepsilon_3\varepsilon_6, \\ \varepsilon_4\varepsilon_5, \varepsilon_4\varepsilon_6, \\ \varepsilon_5\varepsilon_6 \end{bmatrix} \tag{3.14}$$

With these concepts we proceed to consider the planes of material symmetry. The planes of the material, also called **elastic symmetry** are due to the symmetry of the structure of anisotropic body. In the following, we consider some special cases of material symmetry.

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(A) Symmetry with respect to a Plane:

Let us assume that the anisotropic material has only one plane of material symmetry. A material with one plane of material symmetry is called Monoclinic Material.
Let us consider the x_1- x_2 (x_3= 0) plane as the plane of material symmetry. This is shown in Figure 3.1. This symmetry can be formulated with the change of axes as follows

$$x_1^{'} = x_1, \quad x_2^{'} = x_2, \quad x_3^{'} = -x_3 \tag{3.15}$$

With this change of axes,

$$a_{ij} = \frac{\partial x_i^{'}}{\partial x_j} \text{ and } \frac{\partial x_i^{'}}{\partial x_j} = \delta_{ij} \text{ for } j = 1, 2 \text{ and } \frac{\partial x_i^{'}}{\partial x_3} = -\delta_{i3} \tag{3.16}$$

This gives us along with the use of the second of Equation (3.12)

$$\varepsilon_{11}^{'} = \varepsilon_{11}, \varepsilon_{22}^{'} = \varepsilon_{22}, \varepsilon_{33}^{'} = \varepsilon_{33}, \varepsilon_{23}^{'} = -\varepsilon_{23}, \varepsilon_{13}^{'} = -\varepsilon_{13}, \varepsilon_{12}^{'} = \varepsilon_{12} \tag{3.17}$$

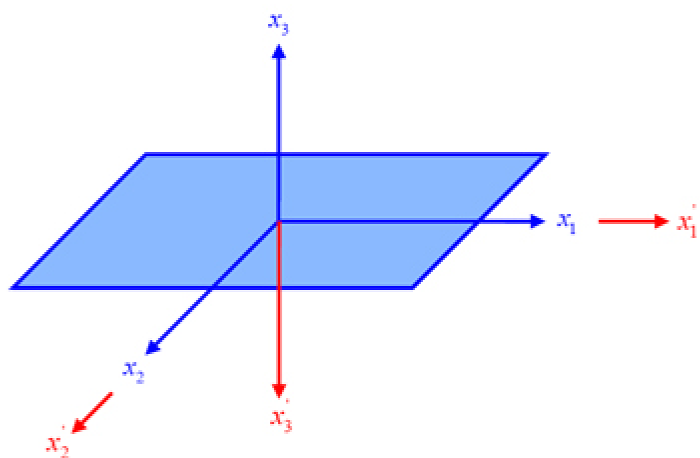

Figure 3.1: Material symmetry about *x1-x2* plane

First Approach: Invariance Approach

Now, the function ***W*** can be expressed in terms of the strain components $\varepsilon_{ij}^{'}$. If ***W*** is to be invariant, then it must be of the form

$$W = W\left[\varepsilon_1^2, \varepsilon_2^2, \varepsilon_3^2, \varepsilon_4^2, \varepsilon_5^2, \varepsilon_6^2, \varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_3, \varepsilon_1\varepsilon_6, \varepsilon_2\varepsilon_3, \varepsilon_2\varepsilon_6, \varepsilon_3\varepsilon_6, \varepsilon_4\varepsilon_5\right] \tag{3.18}$$

Comparing this with Equation (3.13) it is easy to conclude that

$$C_{14} = C_{15} = C_{24} = C_{25} = C_{34} = C_{35} = C_{46} = C_{56} = 0 \tag{3.19}$$

Thus, for the monoclinic materials the number of independent constants are 13. With this reduction of number of independent elastic constants the stiffness matrix is given as

$$C_{ij}=\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ & C_{22} & C_{23} & 0 & 0 & C_{26} \\ & & C_{33} & 0 & 0 & C_{36} \\ & & & C_{44} & C_{45} & 0 \\ & \text{Symmetric} & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \qquad (3.20)$$

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Second Approach: Stress Strain Equivalence Approach

The same reduction of number of elastic constants can be derived from the stress strain equivalence approach. From Equation (3.12) and Equation (3.16) we have

$$\sigma_{11}^{'} = \sigma_{11}, \sigma_{22}^{'} = \sigma_{22}, \sigma_{33}^{'} = \sigma_{33}, \; \sigma_{23}^{'} = -\sigma_{23}, \sigma_{13}^{'} = -\sigma_{13}, \sigma_{12}^{'} = \sigma_{12} \qquad (3.21)$$

The same can be seen from the stresses on a cube inside such a body with the coordinate systems shown in Figure 3.1. Figure 3.2 (a) shows the stresses on a cube with the coordinate system x_1, x_2,x_3 and Figure 3.2 (b) shows stresses on the same cube with the coordinate system $x_1^{'}, x_2^{'}, x_3^{'}$. Comparing the stresses we get the relation as in Equation (3.21).

Now using the stiffness matrix as given in Equation (3.11), strain term relations as given in Equation (3.17) and comparing the stress terms in Equation (3.21) as follows:

$$\sigma_{11} = \sigma_{11}^{'}$$
$$C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3 + C_{14}\varepsilon_4 + C_{15}\varepsilon_5 + C_{16}\varepsilon_6 = C_{11}^{'}\varepsilon_1^{'} + C_{12}^{'}\varepsilon_2^{'} + C_{13}^{'}\varepsilon_3^{'} + C_{14}^{'}\varepsilon_4^{'} + C_{15}^{'}\varepsilon_5^{'} + C_{16}^{'}\varepsilon_6^{'}$$

Using the relations from Equation (3.17), the above equations reduce to

$$C_{14}\varepsilon_4 + C_{15}\varepsilon_5 = C_{14}^{'}\varepsilon_4^{'} + C_{15}^{'}\varepsilon_5^{'}$$

Noting that $C_{ij} = C_{ij}^{'}$, this holds true only when $C_{14} = C_{15} = 0$

Similarly,

$$\sigma_{22} = \sigma_{22}^{'} \text{ gives } C_{24} = C_{25} = 0$$
$$\sigma_{33} = \sigma_{33}^{'} \text{ gives } C_{34} = C_{35} = 0$$
$$\sigma_{23} = \sigma_{23}^{'} \text{ gives } C_{46} = 0$$
$$\sigma_{13} = \sigma_{13}^{'} \text{ gives } C_{56} = 0$$

This gives us the C_{ij} matrix as in Equation (3.20).

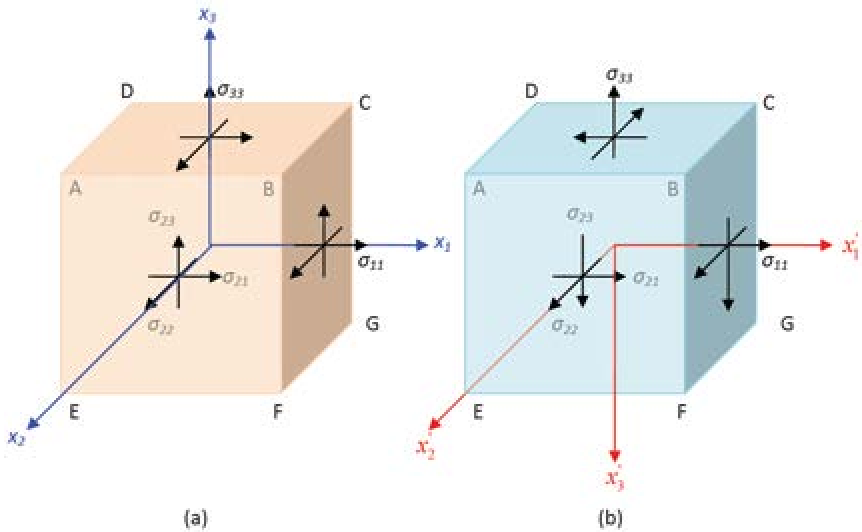

Figure 3.2: State of stress (a) in x_1, x_2, x_3 system

(b) with x_1-x_3 plane of symmetry

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Symmetry with respect to two Orthogonal Planes:

Let us assume that the material under consideration has one more plane, say x_2-x_3 is plane of material symmetry along with x_1-x_2 as in (A). These two planes are orthogonal to each other. This transformation is shown in Figure 3.3.
This can be mathematically formulated by the change of axes as

$$x_1' = -x_1,\ x_2' = x_2,\ x_3' = -x_3 \qquad (3.22)$$

And

$$a_{ij} = \frac{\partial x_i'}{\partial x_j} \quad \text{and} \quad \frac{\partial x_i'}{\partial x_j} = -\delta_{ij} \ \text{for}\ j = 1, 3 \ \text{and}\ \frac{\partial x_i'}{\partial x_2} = \delta_{i2} \qquad (3.23)$$

This gives us the required strain relations as (from Equation (3.12)).

$$\varepsilon_{11}' = \varepsilon_{11},\ \varepsilon_{22}' = \varepsilon_{22},\ \varepsilon_{33}' = \varepsilon_{33},\ \varepsilon_{23}' = -\varepsilon_{23},\ \varepsilon_{13}' = \varepsilon_{13},\ \varepsilon_{12}' = -\varepsilon_{12}$$

or using contracted notations, we can write,

$$\varepsilon_1' = \varepsilon_1,\ \varepsilon_2' = \varepsilon_2,\ \varepsilon_3' = \varepsilon_3,\ \varepsilon_4' = -\varepsilon_4,\ \varepsilon_5' = \varepsilon_5,\ \varepsilon_6' = -\varepsilon_6 \qquad (3.24)$$

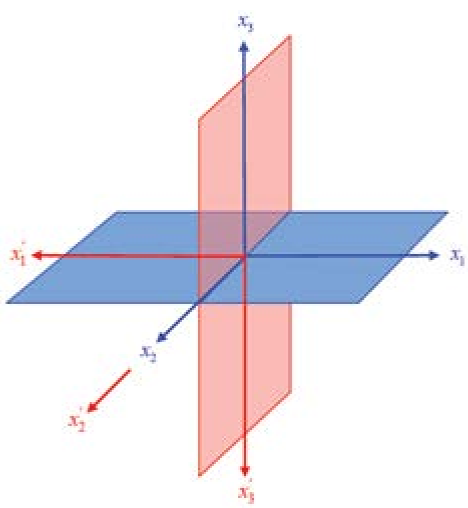

Figure 3.3: Material symmetry about x_1-x_2 and x_2-x_3 planes

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First Approach: Invariance Approach

We can get the function ***W*** simply by substituting ε'_{ij} in place of ε_{ij} and using contracted notations for the strains in Equation (3.18). Noting that ***W*** is invariant, its form in Equation (3.18) must now be restricted to functional form

$$W = W\left[\varepsilon_1^2, \varepsilon_2^2, \varepsilon_3^2, \varepsilon_4^2, \varepsilon_5^2, \varepsilon_6^2, \varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_3, \varepsilon_2\varepsilon_3\right] \tag{3.25}$$

From this it is easy to see that

$$C_{16} = C_{26} = C_{36} = C_{45} = 0$$

Thus, the number of independent constants reduces to 9. The resulting stiffness matrix is given as

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & \text{Symmetric} & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \tag{3.26}$$

When a material has (any) two orthogonal planes as planes of material symmetry then that material is known as Orthotropic Material. It is easy to see that when two orthogonal planes are planes of material symmetry, the third mutually orthogonal plane is also plane of material symmetry and Equation (3.26) holds true for this case also.

Note: Unidirectional fibrous composites are an example of orthotropic materials.

Second Approach: Stress Strain Equivalence Approach

The same reduction of number of elastic constants can be derived from the stress strain equivalence approach. From the first of Equation (3.12) and Equation (3.23) we have

$$\sigma'_{11} = \sigma_{11},\ \sigma'_{22} = \sigma_{22}, \sigma'_{33} = \sigma_{33},\ \sigma'_{23} = -\sigma_{23}, \sigma'_{13} = \sigma_{13}, \sigma'_{12} = -\sigma_{12} \tag{3.27}$$

The same can be seen from the stresses on a cube inside such a body with the coordinate systems shown in Figure 3.3. Figure 3.4 (a) shows the stresses on a cube with the coordinate system x_1, x_2, x_3 and Figure 3.4 (b) shows stresses on the same cube with the coordinate system x'_1, x'_2, x'_3. Comparing the stresses we get the relation as in Equation (3.27).
Now using the stiffness matrix given in Equation (3.20) and comparing the stress equivalence of Equation (3.27) we get the following:

$$\sigma_{11} = \sigma'_{11}$$
$$C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3 + C_{16}\varepsilon_6 = C'_{11}\varepsilon'_1 + C'_{12}\varepsilon'_2 + C'_{13}\varepsilon'_3 + C'_{16}\varepsilon'_6$$

This holds true when $C_{16} = 0$. Similarly,

$$\sigma_{22} = \sigma'_{22} \text{ gives } C_{26} = 0$$
$$\sigma_{33} = \sigma'_{33} \text{ gives } C_{36} = 0$$
$$\sigma_{23} = -\sigma'_{23} \text{ (or } \sigma_{13} = \sigma'_{13}\text{) gives } C_{45} = 0$$

This gives us the C_{ij} matrix as in Equation (3.26).

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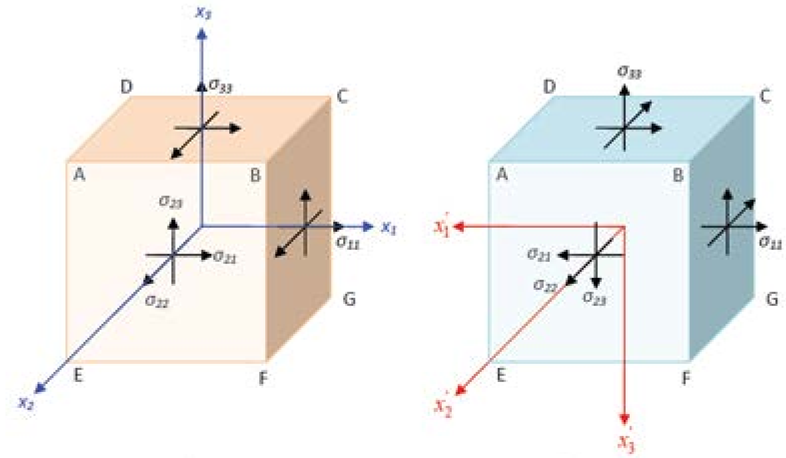

Figure 3.4: State of stress (a) in x_1, x_2, x_3 system
(b) with x_1-x_2 and x_2-x_3 planes of symmetry

Alternately, if we consider x_1-x_3 as the second plane of material symmetry along with x_1-x_2 as shown in Figure 3.5, then

$$x_1^{'} = x_1,\ x_2^{'} = -x_2,\ x_3^{'} = -x_3 \tag{3.28}$$

And

$$a_{ij} = \frac{\partial x_i^{'}}{\partial x_j} \quad \text{and} \ \frac{\partial x_i^{'}}{\partial x_j} = -\delta_{ij} \ \text{for}\ j = 2, 3 \ \text{and} \ \frac{\partial x_i^{'}}{\partial x_1} = \delta_{i1} \tag{3.29}$$

This gives us the required strain relations as (from Equation (3.12))

$$\varepsilon_{11}^{'} = \varepsilon_{11},\ \varepsilon_{22}^{'} = \varepsilon_{22},\ \varepsilon_{33}^{'} = \varepsilon_{33},\ \varepsilon_{23}^{'} = \varepsilon_{23},\ \varepsilon_{13}^{'} = -\varepsilon_{13},\ \varepsilon_{12}^{'} = -\varepsilon_{12}$$

or in contracted notations, we write

$$\varepsilon_1^{'} = \varepsilon_1,\ \varepsilon_2^{'} = \varepsilon_2,\ \varepsilon_3^{'} = \varepsilon_3,\ \varepsilon_4^{'} = \varepsilon_4,\ \varepsilon_5^{'} = -\varepsilon_5,\ \varepsilon_6^{'} = -\varepsilon_6$$

Substituting these in Equation (3.18) the function W reduces again to the form given in Equation (3.25) for W to be invariant. Finally, we get the reduced stiffness matrix as given in Equation (3.26).

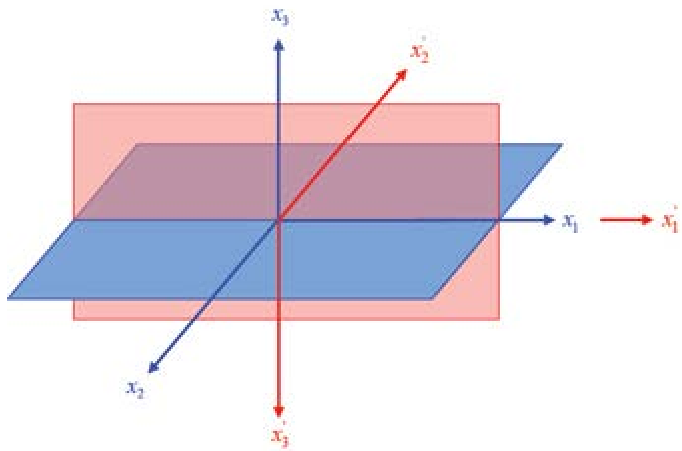

Figure 3.5: Material symmetry about *x_1-x_2* and *x_1-x_3* planes

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The stress transformations for this coordinate transformations are (from the first of Equation (3.12) and Equation (3.29))

$$\sigma_{11}^{'} = \sigma_{11}, \sigma_{22}^{'} = \sigma_{22}, \sigma_{33}^{'} = \sigma_{33}, \sigma_{23}^{'} = \sigma_{23}, \sigma_{13}^{'} = -\sigma_{13}, \sigma_{12}^{'} = -\sigma_{12}$$

The same can be seen from the stresses shown on the same cube in x_1, x_2, x_3 and $x_1^{'}, x_2^{'}, x_3^{'}$ coordinate systems in Figure 3.6 (a) and (b), respectively. The comparison of the stress terms leads to the stiffness matrix as given in Equation (3.26).

***Note*:** It is clear that if any two orthogonal planes are planes of material symmetry the third mutually orthogonal plane has to be plane to material symmetry. We have got the same stiffness matrix when we considered two sets of orthogonal planes. Further, if we proceed in this way considering three mutually orthogonal planes of symmetry then it is not difficult to see that the stiffness matrix remains the same as in Equation (3.26).

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Module 3: 3D Constitutive Equations

Lecture 10: Constitutive Relations: Generally Anisotropy to Orthotropy

Homework:

1. Starting with hyperelastic material, first take x_2-x_3 plane as plane of material symmetry and obtain the stiffness matrix. Is this matrix the same as in Equation (3.20) ? Justify your answer.

2. Starting with the stiffness matrix obtained in the above problem, take x_1-$x_3$as an additional plane of symmetry and obtain the stiffness matrix. Is this matrix the same as in Equation (3.26)? Justify your answer.

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References:

- *AE Green, W Zerna. Theoretical Elasticity. Oxford, Clarendon Press, 1963.*
- *SG Lekhnitskii. Theory of Elasticity of an Anisotropic Body. Mir Publishers, Moscow, 1981.*
- *IS Sokolnikoff. Mathematical Theory of Elasticity, First Edition, McGraw Hill Publications, New York.*
- *LE Malvern. Introduction to Mechanics of a Continuous Medium, Prentice-Hall, Inc. New Jersey, 1969.*
- *CT Herakovich. Mechanics of Fibrous Composites, John Wiley & Sons, Inc. New York, 1998.*

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