Introduction

In the earlier lectures we have used the effective properties of the unidirectional layers in the development of various lamina or laminate mechanics issues. However, we know that at microscale the fibrous composites are heterogeneous. A composite is made of two main phases - fiber and matrix. Further, we know that apart from these two phases, additional phase may be present in the composite. These phases may be fillers, zones formed due to reaction between fibre and matrix and the coatings applied to the fiber, if any. The properties of these constituents, their amounts present and their distribution affect the effective properties of the composite.

It is now well understood that to determine the effective properties of a composite one needs to consider the microscale, that is, the scale at which the fibre and matrix are present. Thus, the study of composites at the fiber and matrix level is referred to as micromechanics.

In the present lecture we will present various methods to determine the effective hygro-thermo-mechanical properties of the composite. It is assumed that the properties of constituents, their arrangements and amounts are known a-priori.

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Effective Elastic Constants:

The unidirectional lamina is interest of this course. The unidirectional lamina is orthotropic in nature. We know from the 3D constitutive equations that an orthotropic material has 9 independent constants. Further, for a transversely isotropic material there are 5 independent constants. The average or effective constitutive equation for transversely isotropic material is given as below. The transverse isotropy is in plane 2-3.

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11}^{*} & C_{12}^{*} & C_{12}^{*} & 0 & 0 & 0 \\ C_{12}^{*} & C_{22}^{*} & C_{23}^{*} & 0 & 0 & 0 \\ C_{12}^{*} & C_{23}^{*} & C_{22}^{*} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{C_{22}^{*} - C_{23}^{*}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66}^{*} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66}^{*} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \nu_{23} \\ \nu_{13} \\ \nu_{12} \end{Bmatrix}$$

where $C_{11}^{*}, C_{22}^{*}, C_{12}^{*}, C_{23}^{*}, C_{66}^{*}$, are the effective elastic constants of the equivalent homogeneous material.

In this chapter, we are going to see the micromechanical methods to obtain the effective engineering constants that define the above effective elastic constants of the equivalent homogeneous material.

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Idealization of Microstructure of Fibrous Composite:

As mentioned earlier, the micromechanics is a study at fibre and matrix level. Thus, the geometry of arrangement of the fibres and matrix in a composite is an essential requirement to develop a model for the study. Some of the methods do not use the geometry of arrangement. Most of the methods developed for micromechanical analysis assume that:

1. The fibers and matrix are perfectly bonded and there is no slip between them.
2. The fibres are continuous and parallel.
3. The fibres are assumed to be circular in cross section with a uniform diameter along its length.
4. The space between the fibres is uniform throughout the composite.
5. The elastic, thermal and hygral properties of fibre and matrix are known and uniform.
6. The fibres and matrix obey Hooke's law.
7. The fibres and the matrix are only two phases in the composite.
8. There are no voids in the composite.

There are many ways to idealize the cross section of a lamina. In Figure 7.1 are shown two popular idealizations. The most commonly preferred arrangements are square packed and hexagonal packed arrays of fibres in matrix. The square and hexagonal packed arrays can be as shown in Figure 7.1(a), and (b), respectively.

In these idealizations it is seen that due to symmetry and periodicity of these arrays one can consider only one array to analyze the lamina at micro scale. Further, if this one array represents the general arrangement of fibres with respect to matrix and the interactions of fibre and matrix phases, then such array is called ***Representative Volume Element* (RVE)**. Further, this RVE as a volume of material statistically represents a homogeneous material. In the analysis of an RVE the boundary conditions are chosen such that they reflect the periodicity. Thus, the arrays shown in Figure 7.1 are various RVEs. One should be able to see that the RVE also reflects the volume fractions. The term RVE was first coined by Hill in 1963.
For example, the square RVE represents a lower fibre volume fraction than a hexagonal RVE. Note that RVE is also called as ***Unit Cell*.**

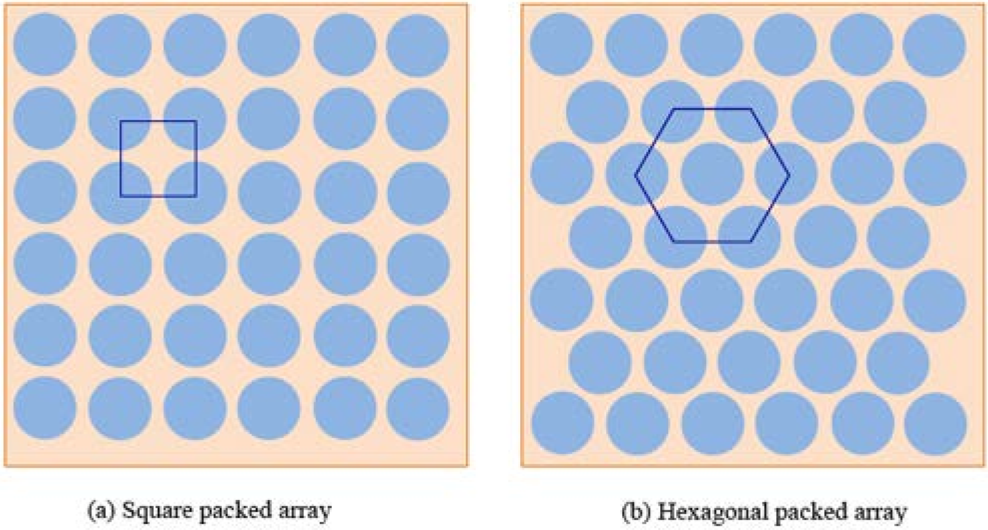

Figure 7.1: Idealization of cross section of a lamina

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Volume and Mass Fractions, Density and Void Content:

In the present section we are going to introduce some important concept of relative fraction of fibres and matrix by volume and mass. This is very important from the point that the most of the micromechanics based approaches use these fractions, along with the properties of individual phases, to express the properties of the equivalent homogeneous material.

In the present case, the effective properties of a composite are obtained with the assumption that the fibre is orthotropic or transversely isotropic and matrix is isotropic in behaviour. However, with appropriate changes, fibre can also be considered to be isotropic. In the following, the subscripts or superscripts (f) and (m) will denote fibre and matrix, respectively.

Volume Fractions:

As stated earlier, the fibre volume fraction is defined as the ratio of fibre volume to composite volume and matrix volume fraction is defined as the ratio of matrix volume fraction to composite volume. Let, v_f, v_m be the volume occupied by fibres and matrix, respectively. Let, v_c be the composite volume. We know that,

$$v_f + v_m = v_c \tag{7.1}$$

Thus, from these two definitions of volume fractions, we can write

$$\frac{\text{Fiber volume}}{\text{Total volume}} + \frac{\text{Matrix volume}}{\text{Total volume}} = 1 \quad \text{or}$$
$$\frac{v_f}{v_c} + \frac{v_m}{v_c} = 1 \tag{7.2}$$

Thus, in notations

$$V_f + V_m = 1 \tag{7.3}$$

where, V_f denotes the fibre volume fraction and V_m denotes the matrix volume fraction. Note that "total volume" and "composite volume" are used interchangeably.

Note: If the interphase is also present as a third phase then, Equation (7.2) is modified as

$$\frac{\text{Fiber volume}}{\text{Total volume}} + \frac{\text{Matrix volume}}{\text{Total volume}} + \frac{\text{Interphase volume}}{\text{Total volume}} = 1$$

or

$$V_f + V_m + V_i = 1 \tag{7.4}$$

where, $V_i = \dfrac{v_i}{v_c}$ denotes the interphase volume fraction and v_i denotes the interphase volume.

In case, there are voids present in composite, then the above equation becomes as

$$\frac{\text{Fiber volume}}{\text{Total volume}}+\frac{\text{Matrix volume}}{\text{Total volume}}+\frac{\text{Interphase volume}}{\text{Total volume}}+\frac{\text{Void volume}}{\text{Total volume}}=1$$

or

$$V_f+V_m+V_i+V_v=1 \qquad (7.5)$$

where, $V_v=\frac{v_v}{v_c}$ denotes the void volume fraction and v_v denotes the void volume. In the remaining, we will consider that there are only two phases and Equation (7.3) is used.

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Mass Fractions:

Let w_f, w_m and w_c be the mass of fibres, matrix and composite, respectively. We know that

$$w_f + w_m = w_c \tag{7.6}$$

The mass fractions, similar to volume fractions, are defined as the ratio of mass of respective phase to the mass of composite. Thus, we can write,

$$\begin{aligned} &\frac{w_f}{w_c} + \frac{w_m}{w_c} = 1 \text{ or} \\ &W_f + W_m = 1 \end{aligned} \tag{7.7}$$

where, W_f is fibre mass fraction and W_m is matrix mass fraction. Now, let us write the mass of each phase in terms of density and volume of respective phase as

$$\begin{aligned} w_c &= \rho_c v_c \\ w_f &= \rho_f v_f \\ w_m &= \rho_m v_m \end{aligned} \tag{7.8}$$

where, ρ_f, ρ_m and ρ_c are the densities of fibre, matrix and composite, respectively. Now, mass fractions can be written in terms of density and volume fractions as

$$\begin{aligned} W_f &= \frac{\rho_f}{\rho_c} V_f \\ W_m &= \frac{\rho_m}{\rho_c} V_m \end{aligned} \tag{7.9}$$

This relation between mass and volume fractions is given in terms of individual constituent properties (using Equations (7.6) and (7.8)) as

$$\begin{aligned} W_f &= \frac{\dfrac{\rho_f}{\rho_m}}{\dfrac{\rho_f}{\rho_m} V_f + V_m} V_f \\ W_m &= \frac{1}{\dfrac{\rho_f}{\rho_m}\left(1 - V_m\right) + V_m} V_m \end{aligned} \tag{7.10}$$

Thus, it is clear from the above equation that the volume and mass fractions are not the same. One should always state the basis for calculating the fibre content in a composite.

Density:

The density of composite is derived in terms of densities and volume fractions of the individual phases as follows. The mass of composite is given by Equation (7.6). We can write this in terms of respective volume fractions and densities (with rearrangement) as

$$\rho_c v_c = \rho_f v_f + \rho_m v_m \quad \text{or} \quad \rho_c = \rho_f \frac{v_f}{v_c} + \rho_m \frac{v_m}{v_c} \tag{7.11}$$

This is written using the definition of volume fraction for fibre and matrix as

$$\rho_c = \rho_f V_f + \rho_m V_m \tag{7.12}$$

We will write the density of composite in terms of mass fraction from Equation (7.9) as

$$\frac{1}{\rho_c} = \frac{W_f}{\rho_f} + \frac{W_m}{\rho_m} \tag{7.13}$$

Strength of Material Approximations:

In general, the laminates made are thin. Hence, for such laminates the analysis done using Kirchhoff and plane stress assumptions is reasonably good. For such analysis, one needs the engineering constants that occur in defining planar constitutive equations. These engineering constants are:

1. E_1^* - **the axial modulus**
2. $E_2^* = E_3^*$ - **transverse modulus**
3. $\nu_{12}^* = \nu_{13}^*$ - **axial Poisson's ratio (for loading in x_1- direction)**
4. $G_{12}^* = G_{13}^*$- **axial shear modulus (shear stress parallel to the fibers)**

Further, it is seen that for transversely isotropic composite, four out of five (the fifth one is G_{23}^*) properties can be developed from this approach. For the planar hygro-thermal analysis of such laminates, one can also obtain the in-plane coefficients of thermal expansions α_1^* and α_2^* and hygroscopic expansion β_1^* and β_2^* as well.

It is important to note that this approach involves assumptions which do not necessarily satisfy the requirements of an exact elasticity solution. In this approach the effective properties will be expressed in terms of the elastic properties and volume fractions of the fiber and matrix. The lamina is considered to be an alternate arrangement of fibres and matrix. The RVE chosen in these derivations is shown in Figure 7.2. The RVE here does not take into account the cross sectional arrangement of fibres and matrix, rather it represents volume of the material through the cross sectional area of fibre and matrix.

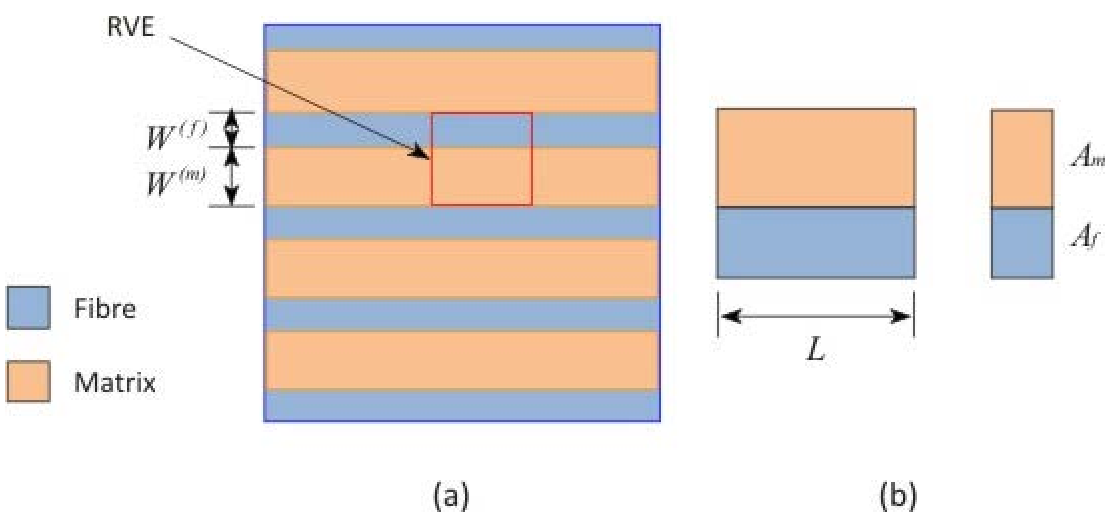

Figure 7.2: (a) Unidirectional lamina, (b) RVE for unidirectional composite for prediction of elastic properties

Let, A_f and A_m represent fibre area and matrix area, respectively. $W^{(f)}$ and $W^{(m)}$represent fibre and matrix widths, respectively. L be the length of the RVE.

Effective Axial Modulus E_1^*:

The unit cell as shown in Figure 7.2 is used to compute the effective axial modulus E_1^*. It should be noted that the thickness of the unit cell is not important in this computation. Further, the cross sectional shapes are not considered in this calculation. However, the cross sectional areas are important in this calculation. The thicknesses of the fibre and matrix constituents are same in the unit cell. Hence, the areas of the constituents represent the volume fractions of the constituents.

In the calculation of effective axial modulus, it is assumed that the axial strain in the composite is uniform such that the axial strains in the fibers and matrix are identical. This assumption is justified by the fact that the fibre and the matrix in the unit cell are perfectly bonded. Hence, the elongation in the axial direction of the fibre and matrix will also be identical. Thus, the strains in the fibre and matrix can be given as

$$\bar{\varepsilon}_1 = \varepsilon_1^{(f)} = \varepsilon_1^{(m)} = \frac{\Delta L}{L} \tag{7.14}$$

where, $\bar{\varepsilon}_1$ is the axial strain in the composite and $\varepsilon_1^{(f)}$ and $\varepsilon_1^{(m)}$ are the axial strains in fibre and matrix, respectively. Now, let $E_1^{(f)}$ and $E^{(m)}$ be the axial Young's moduli of the fibre and matrix, respectively. We can give the axial stress in the fibre, $\sigma_1^{(f)}$ and matrix, $\sigma_1^{(m)}$ as

$$\sigma_1^{(f)} = E_1^{(f)} \varepsilon_1^{(f)} \quad \text{and} \quad \sigma_1^{(m)} = E^{(m)} \varepsilon_1^{(m)} \tag{7.15}$$

Using the above equation and the cross section areas of the respective constituent in the unit cell, we can calculate the forces in them as

$$F_1^{(f)} = \sigma_1^{(f)} A_f \quad \text{and} \quad F_1^{(m)} = \sigma_1^{(m)} A_m \tag{7.16}$$

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The total axial force in the composite is sum of the axial forces in fibre and matrix. Thus, the total axial force in the composite substituting the expressions for axial strains in fibre and matrix from Equation (7.14) in above equation, can be given as

$$F_1 = F_1^{(f)} + F_1^{(m)} = \sigma_1^{(f)} A_f + \sigma_1^{(m)} A_m = \left(E_1^{(f)} A_f + E^{(m)} A_m\right) \frac{\Delta L}{L} \tag{7.17}$$

Now $\bar{\sigma}_1$ be the average axial stress in composite. The total cross sectional area of the composite is $A = A_f + A_m$. Thus, using the average axial stress and cross sectional area of the composite, the axial force is

$$F_1 = \bar{\sigma}_1 A \tag{7.18}$$

Thus, combining Equation (7.17) and Equation (7.18) and rearranging, we get

$$\bar{\sigma}_1 = \left(E_1^{(f)} \frac{A_f}{A} + E^{(m)} \frac{A_m}{A}\right) \frac{\Delta L}{L} \tag{7.19}$$

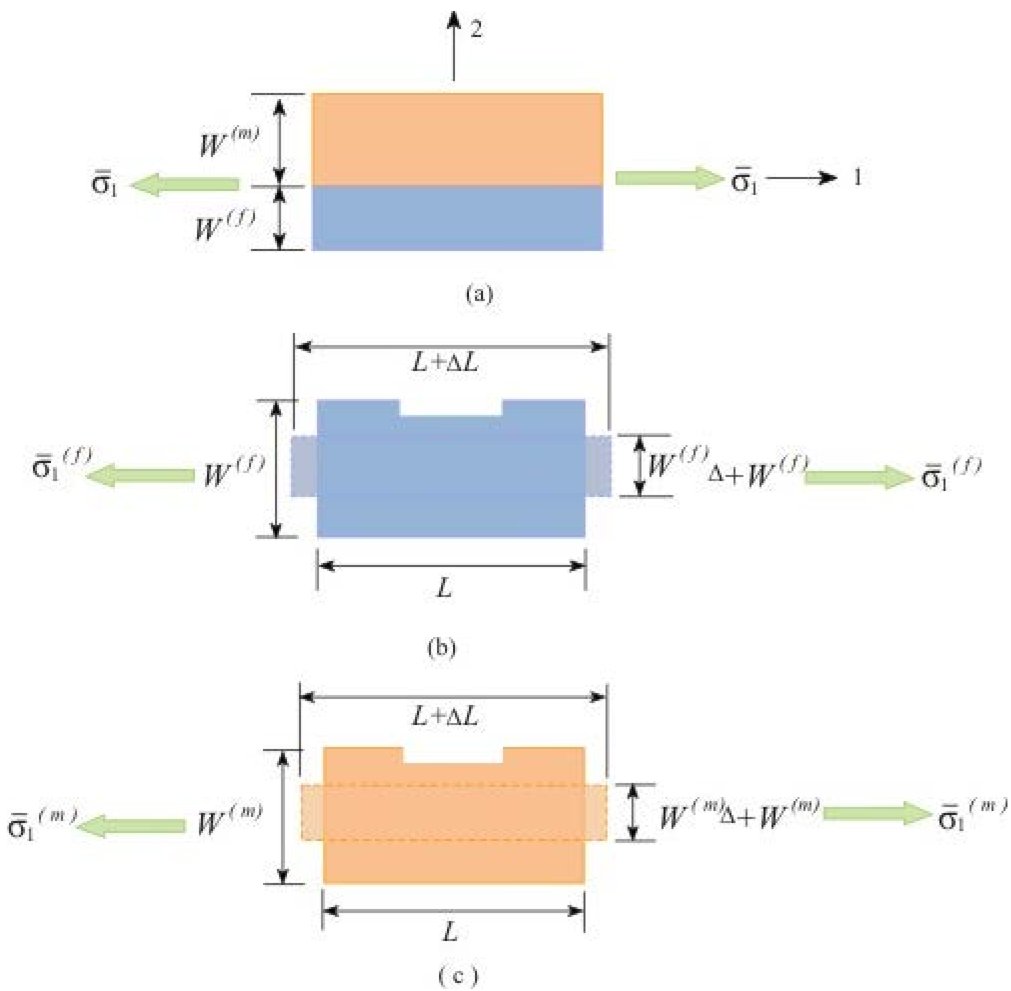

Figure 7.3: (a) Undeformed unit cell under $\bar{\sigma}_1$ (b) and (c) deformed individual constituents of the unit cell

Let us define

$$\bar{\sigma}_1 = E_1^* \bar{\varepsilon}_1 = E_1^* \frac{\Delta L}{L} \tag{7.20}$$

Further, noting that the ratios $\frac{A_f}{A}$ and $\frac{A_m}{A}$ for same length of fibre and matrix represent the fibre and matrix volume fractions, respectively. Thus, combining Equations (7.19) and (7.20), we get

$$E_1^* = E_1^{(f)} V_f + E^{(m)} V_m = E_1^{(f)} V_f + E^{(m)} \left(1 - V_f\right) \tag{7.21}$$

The above equation relates the axial modulus of the composite to the axial moduli of the fibre and matrix through their volume fractions. Thus, the effective axial modulus is a linear function of the fiber volume fraction. This equation is known as rule of mixtures equation. It should be noted that the effective properties are functions of the fiber volume fractions; hence it should always be quoted in reporting the effective properties of a composite.

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Effective Axial (Major) Poison's Ratio ν_{12}^{*}:

To determine the effective axial Poisson's ratio we consider the loading as in the case applied for determining the effective axial modulus. Here, for this loading we have $\bar{\sigma}_1 \neq 0$ and other stresses are zero. We define the effective axial Poisson's ratio as

$$\nu_{12}^{*} = -\frac{\bar{\varepsilon}_2}{\bar{\varepsilon}_1} \tag{7.22}$$

The effective strain in direction 2 from Figure 7.3(b) and (c) can be given as

$$\bar{\varepsilon}_2 = \frac{\Delta W}{W} = \frac{\Delta W^{(f)} + \Delta W^{(m)}}{W^{(f)} + W^{(m)}} \tag{7.23}$$

Now, the changes in $W^{(f)}$ and $W^{(m)}$ can be obtained using the Poisson's ratio of individual constituents. The axial Poisson's ratios for fibre and matrix are given as

$$\nu_{12}^{(f)} = -\frac{\varepsilon_2^{(f)}}{\varepsilon_1^{(f)}} = -\frac{\Delta W^{(f)} / W^{(f)}}{\Delta L / L} \quad \text{and} \quad \nu^{(m)} = -\frac{\varepsilon_2^{(m)}}{\varepsilon_1^{(m)}} = -\frac{\Delta W^{(m)} / W^{(m)}}{\Delta L / L} \tag{7.24}$$

Thus, the changes in $W^{(f)}$ and $W^{(m)}$ are given as

$$\Delta W^{(f)} = -\nu_{12}^{(f)} W^{(f)} \frac{\Delta L}{L} \quad \text{and} \quad \Delta W^{(m)} = -\nu^{(m)} W^{(m)} \frac{\Delta L}{L} \tag{7.25}$$

The total change in W is given as

$$\Delta W = \Delta W^{(f)} + \Delta W^{(m)} \tag{7.26}$$

The strain in direction 2 for the composite can be given using Equation (7.25) and Equation (7.26) as

$$\bar{\varepsilon}_2 = \frac{\Delta W}{W} = \frac{\Delta W^{(f)} + \Delta W^{(m)}}{W} = -\left(\nu_{12}^{(f)} \frac{W^{(f)}}{W} + \nu^{(m)} \frac{W^{(m)}}{W}\right) \frac{\Delta L}{L} \tag{7.27}$$

Here, $\frac{W^{(f)}}{W}$ and $\frac{W^{(m)}}{W}$ denote the fibre and matrix volume fractions for same length of fibre and matrix. Note that $\frac{\Delta L}{L}$ denotes the effective axial strain $\bar{\varepsilon}_1$. Thus, from Eq. (7.27) the effective axial Poisson's ratio is written as

$$\nu_{12}^{(*)} = \nu_{12}^{(f)} V_f + \nu^{(m)} V_m \tag{7.28}$$

The above equation is the rule of mixtures expression for composite axial Poisson's ratio.

Effective Transverse Modulus E_2^*:

Here, we are going to derive the effective transverse modulus by loading the RVE in direction 2 as shown in Figure 7.4(a). There are two considerations while deriving this effective modulus. The first approach considers that the deformation of the each constituent is independent of each other as shown in Figure 7.4(b) and (c) and the deformation in direction 1 is not considered. The second approach considers that deformations of the fibre and matrix in direction 1 are identical as they are perfectly bonded.

To calculate the effective modulus in direction 2, a stress $\bar{\sigma}_2$ is applied to the RVE as shown in Figure 7.4(a).

First Approach:

As mentioned, the fibre and matrix deform independently of each other. The resulting deformation in direction 1 is not considered here. This assumption is simplistic and was used by early researchers.

The fibre and matrix are subjected to same state of stress. The state of stress is unidirectional, that is, $\sigma_2^{(f)} = \sigma_2^{(m)} = \bar{\sigma}_2$. Now, using the individual moduli and deformations in direction 2, these stresses can be given as

$$\begin{aligned} \sigma_2^{(f)} &= E_2^{(f)} \varepsilon_2^{(f)} = E_2^{(f)} \frac{\Delta W^{(f)}}{W^{(f)}} \\ \sigma_2^{(m)} &= E^{(m)} \varepsilon_2^{(m)} = E^{(m)} \frac{\Delta W^{(m)}}{W^{(m)}} \end{aligned} \tag{7.29}$$

From this equation we can write the individual deformations, which give the total deformation in direction 2 as

$$\Delta W = \Delta W^{(f)} + \Delta W^{(m)} = \left(\frac{W^{(f)}}{E_2^{(f)}} + \frac{W^{(m)}}{E^{(m)}} \right) \bar{\sigma}_2 \tag{7.30}$$

Now, the composite strain in direction 2 can be calculated from the definition as

$$\bar{\varepsilon}_2 = \frac{\Delta W}{W} = \left(\frac{W^{(f)}}{W} \frac{1}{E_2^{(f)}} + \frac{W^{(m)}}{W} \frac{1}{E^{(m)}} \right) \bar{\sigma}_2 \tag{7.31}$$

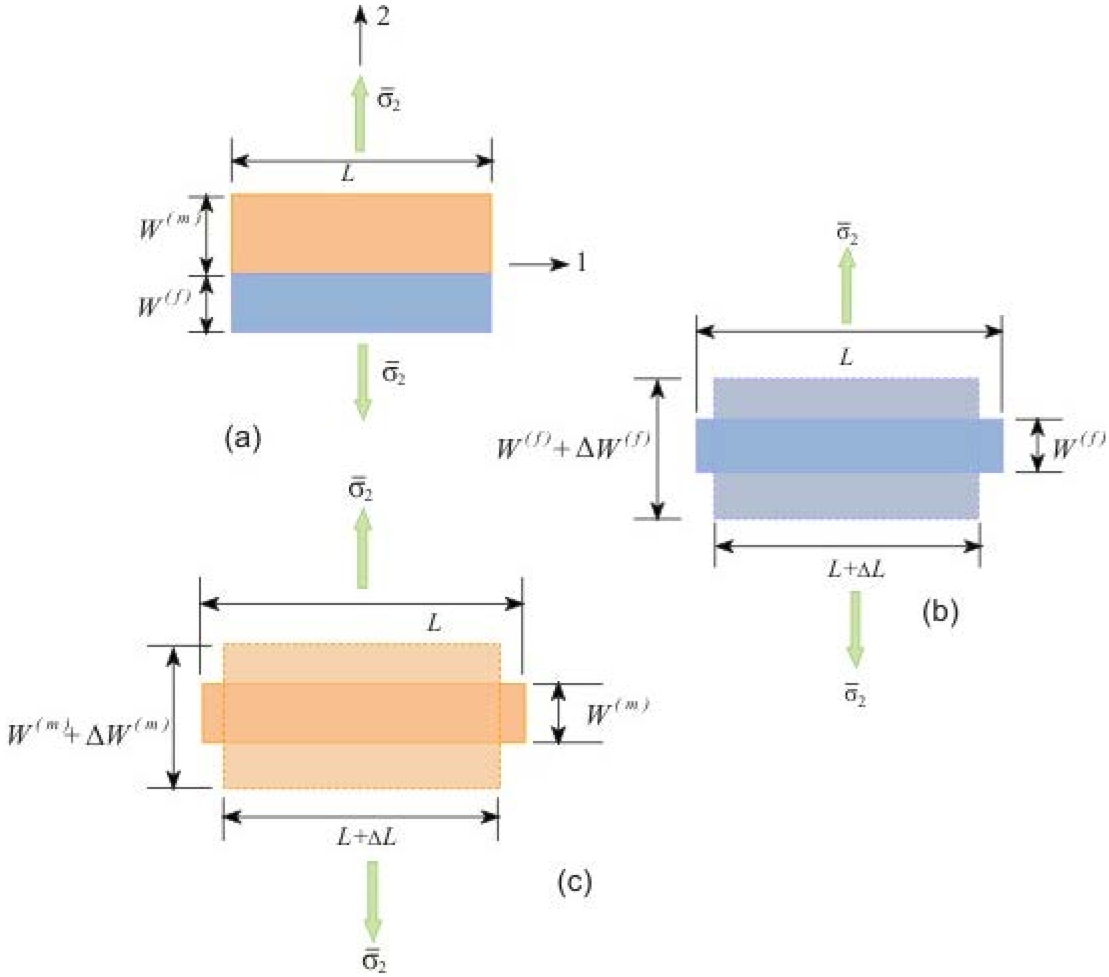

Figure 7.4: (a) Undeformed unit cell under uniform $\bar{\sigma}_2$ stress (b) and (c) deformed individual constituents of the unit cell

Introducing the volume fractions in the above equation,

$$\bar{\varepsilon}_2 = \left(V_f \frac{1}{E_2^{(f)}} + V_m \frac{1}{E^{(m)}} \right) \bar{\sigma}_2 \qquad (7.32)$$

Noting that $\frac{\bar{\sigma}_2}{\bar{\varepsilon}_2} = E_2^*$, from the above equation, we get

$$\frac{1}{E_2^*} = \frac{V_f}{E_2^{(f)}} + \frac{V_m}{E^{(m)}} = \frac{V_f}{E_2^{(f)}} + \frac{(1-V_f)}{E^{(m)}} \qquad (7.33)$$

This equation is the rule of mixtures equation for effective modulus E_2^*.

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Module 7: Micromechanics

Lecture 24: Strength of Materials Approach

Home Work:

1. What are the assumptions in a typical micromechanical analysis?
2. Write a short note on RVE/Unit Cell.
3. Define volume and mass fractions for fibre and matrix and derive expressions for them.
4. Derive an expression for density of a composite in terms of densities of its constituents.
5. Using strength of materials approach, derive expressions for effective axial modulus, Poisson's ratio and transverse modulus.

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