Introduction

In the previous lectures we have introduced the concept of CCA model. Further, we have determined the effective axial modulus and Poisson’s ratio using the concepts of mechanics and equivalence of strain energy approach.

In the present lecture we will derive the expressions for effective plane strain bulk modulus.

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Effective Plane Strain Bulk Modulus

We need to find the effective plane strain bulk modulus such that the axial strain is zero and there is same strain in two transverse directions. This state of strain can be applied as follows:

$$\varepsilon_{11} = 0, \varepsilon_{22} = \varepsilon_{33} = \varepsilon \qquad (7.219)$$

For this state of strain, the displacement field in cylindrical coordinates system on the outer boundary of the cylinders becomes

$$u(r,\theta) = 0, v(b,\theta) = 0, w(b,\theta) = \varepsilon b \qquad (7.220)$$

The displacement field at the outer boundary of the cylinders reflects a pure radial stress tractions, that is, $\sigma_{rr}(b) = \sigma$ will give us the same boundary conditions of a homogeneous cylinder. If these tractions are transferred to the Cartesian coordinate system, then we get

$$\sigma_{22} = \sigma_{33} = \sigma_{rr}(b) = \sigma \qquad (7.221)$$

Alternatively, for the displacement field in Equation (7.220) on the outer surface of the cylinder, the tractions in cylindrical coordinates are

$$\sigma_{\theta\theta} = \sigma_{rr} = \sigma_{rr}(b) = \sigma \qquad (7.222)$$

and the strains can be given in terms of stresses as

$$\varepsilon_{\theta\theta} = \varepsilon_{rr} = \frac{\sigma}{C_{23}^{*}} \qquad (7.223)$$

However, one can give the strains from the displacement field in Equation (7.220).

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It should be noted that the axisymmetric problem given here has the same form of solution as given previously. However, the constants involved in this problem will take different values depending upon the boundary conditions of this problem. For this problem in hand, we have the following boundary conditions:

The radial displacement at the fibre and matrix interface should be continuous as in the first of Equation (7.180).

The second condition is continuity of radial stresses at the fibre and matrix interface as in Equation (7.182). With transversely isotropic fibre, this condition leads to

$$\left(C^f_{22} + C^f_{23}\right)A^f = \left(C^m_{11} + C^m_{12}\right)A^m + \left(C^m_{12} - C^m_{11}\right)\frac{B^m}{a^2} \qquad (7.224)$$

and with isotropic fibre, this condition from Equation (7.202) with $\varepsilon_{xx} = 0$ becomes

$$K_f A^f a^2 - K_m A^m a^2 + \mu_m B^m = 0 \qquad (7.225)$$

The third condition is that the outer boundary of the cylinder has the radial stress equal to the σ as in Equation (7.221) or Equation (7.222). In case of fibre with transversely isotropic material, this leads to

$$\left(C^m_{11} + C^n_{12}\right)A^m + \left(C^m_{12} - C^m_{11}\right)\frac{B^m}{b^2} = \sigma \qquad (7.226)$$

and in case of fibre with isotropic material, from Equation (7.200) with $\varepsilon_{xx} = 0$ it becomes

$$K_m b^2 A^m - \mu_m B_m = \frac{\sigma}{2} b^2 \qquad (7.227)$$

For the case of transversely isotropic fibre the constants A^m, A^f and B^m are determined by solving the Equations. (7.180), (7.224) and (7.226). These constants are

$$
\begin{aligned}
A^f &= \frac{2b^2 C^m_{11}}{-a^2\left(C^m_{11} - C^m_{12}\right)\left(C^m_{11} + C^m_{12} - C^f_{22} - C^f_{23}\right) + b^2\left(C^m_{11} + C^m_{12}\right)\left(C^m_{11} - C^m_{12} + C^f_{22} + C^f_{23}\right)}\sigma \\
A^m &= \frac{b^2\left(C^m_{11} - C^m_{12} + C^f_{22} + C^f_{23}\right)}{-a^2\left(C^m_{11} - C^m_{12}\right)\left(C^m_{11} + C^m_{12} - C^f_{22} - C^f_{23}\right) + b^2\left(C^m_{11} + C^m_{12}\right)\left(C^m_{11} - C^m_{12} + C^f_{22} + C^f_{23}\right)}\sigma \qquad (7.228) \\
B^m &= \frac{a^2 b^2\left(C^m_{11} + C^m_{12} - C^f_{22} - C^f_{23}\right)}{-a^2\left(C^m_{11} - C^m_{12}\right)\left(C^m_{11} + C^m_{12} - C^f_{22} - C^f_{23}\right) + b^2\left(C^m_{11} + C^m_{12}\right)\left(C^m_{11} - C^m_{12} + C^f_{22} + C^f_{23}\right)}\sigma
\end{aligned}
$$

For the case of isotropic fibre the the constants A^m, A^f and B^m are determined by solving the Equations (7.180), (7.225) and (7.227). These constants are given as

(7.229)

$$A^f = \frac{b^2(k_m + \mu_m)}{2[a^2(k_m + \mu_m)\mu_m + b^2 k_m(k_m + \mu_m)]}\sigma$$
$$A^m = \frac{b^2(k_f + \mu_m)}{2[a^2(k_f - k_m)\mu_m + b^2 k_m(k_f - \mu_m)]}\sigma$$
$$B^m = \frac{b^2(k_f + \mu_m)}{2[a^2(k_f - k_m)\mu_m + b^2 k_m(k_f + \mu_m)]}\sigma$$

Thus, all stresses and strains can be given in terms of unknown A^m, A^f and B^m constants. The effective plane strain bulk modulus of the equivalent homogeneous material is then defined as

$$K_{23}^* = \frac{\sigma}{\varepsilon_{22} + \varepsilon_{33}} = \frac{\sigma_{rr}(b)}{2\varepsilon} \tag{7.230}$$

From Equation (7.220), we can write

$$\varepsilon = \frac{w(b)}{b} = A^m + \frac{B^m}{b^2} \tag{7.231}$$

Thus, combining Equation (7.230) and Equation (7.231) we get

$$K_{23}^* = \frac{\sigma}{\left(A^m + \frac{B^m}{b^2}\right)} \tag{7.232}$$

The effective plane strain bulk modulus can be also defined as

$$K_{23}^* = \frac{\sigma}{\left(\frac{\Delta V}{V}\right)} \tag{7.233}$$

where, $\frac{\Delta V}{V}$ is the ratio of change in volume per unit volume of the concentric cylinders. It is given as

$$\frac{\Delta V}{V} = \frac{\pi\left(b + w^{(m)}(b)\right)^2}{\pi b^2} = \frac{2bw^{(m)}(b)}{b^2} = 2\left(A^m + \frac{B^m}{b^2}\right) \tag{7.234}$$

In this the second order terms are ignored. Thus, the effective plane strain bulk modulus for this definition gives us same expression for plane strain bulk modulus as in Equation (7.232).

Thus, for the transversely isotropic fibre, using the constants as given in Equation (7.228) the effective plane strain bulk modulus becomes

$$k_{23}^* = \frac{-a^2(C_{11}^m - C_{12}^m)(C_{11}^m + C_{12}^m - C_{22}^f - C_{23}^f) + b^2(C_{11}^m + C_{12}^m)(C_{11}^m - C_{12}^m + C_{22}^f + C_{23}^f)}{2\left[a^2(C_{11}^m + C_{12}^m - C_{22}^f - C_{23}^f) + b^2(C_{11}^m - C_{12}^m + C_{22}^f + C_{23}^f)\right]} \tag{7.235}$$

Dividing the numerator and denominator by b^2 , we get

$$k_{23}^{*} = \frac{-V_f\left(C_{11}^{m} - C_{12}^{m}\right)\left(C_{11}^{m} + C_{12}^{m} - C_{22}^{f} - C_{23}^{f}\right) + \left(C_{11}^{m} + C_{12}^{m}\right)\left(C_{11}^{m} - C_{12}^{m} + C_{22}^{f} + C_{23}^{f}\right)}{2\left[V_f\left(C_{11}^{m} + C_{12}^{m} - C_{22}^{f} - C_{23}^{f}\right) + \left(C_{11}^{m} - C_{12}^{m} + C_{22}^{f} + C_{23}^{f}\right)\right]} \quad (7.236)$$

For isotropic fibre, using the constants from Equation (7.229) the effective plane strain bulk modulus is given as

$$K_{23}^{*} = K_m + \frac{\mu_m}{3} + \frac{V_f}{\frac{1}{k_f - k_m} + \frac{1}{3}\left(\mu_f - \mu_m\right)} + \frac{\left(1 - V_f\right)}{\left(k_m + \frac{4}{3}\mu_m\right)} \quad (7.237)$$

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Equivalence of Strain Energy Approach:

We can derive the effective bulk modulus using the concept of equivalence of strain energy. The strain energy of the equivalent homogeneous single cylinder is given as

$$U = \frac{1}{2}\int_V (\sigma_{xx}\varepsilon_{xx} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + \sigma_{rr}\varepsilon_{rr})dV$$
$$= \frac{1}{2}\int_{x=0}^{1}\int_{\theta=0}^{2\pi}\int_{r=0}^{b}((\sigma_{xx}\varepsilon_{xx} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + \sigma_{rr}\varepsilon_{rr})dx\, d\theta\, r dr) \tag{7.238}$$

The stresses and strains are known from Equation (7.219), Equation (7.222) and Equation (7.223). When these are substituted in above equation, for a unit length, it gives

$$U = \frac{\pi b^2\sigma^2}{(C_{22}^* + C_{23}^*)} = \frac{\pi b^2\sigma^2}{2K_{23}^*} \tag{7.239}$$

Now this strain energy is compared with that obtained for concentric cylinders as given in Equation (7.212). The stresses and strains in fibre and matrix for this expression can be obtained as the constants A^m, A^f and B^m are known.

Module 7: Micromechanics

Lecture 31: CCA Model: Effective Plane Strain Bulk Modulus

Home Work:

1. Write a short note on deformations/loads to be imposed on the concentric cylinders to obtain the effective plane strain bulk modulus.
2. Derive an expression for the effective plane strain bulk modulus using CCA model.

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