Module-6

Lecture-27

Euler angles & Kinematic equations

Euler Angles

- Formulas described in previous lecture provide linear and angular velocity w.r.t. (X, Y, Z) Body fixed coordinate system.

- Now analysis of the relative motion of body fixed reference frame and inertial reference frame is required.

- There are several methods of tracking the orientation of the X, Y, Z frame with respect to earth based inertial frame X', Y', Z'.

- The most common approach is based on Euler angles.

- The introduction of Euler angle is based on a rigorous sequence that involves the introduction of a number of reference frames based on successive rotations.

  - **Step1:** Introduce a reference frame X_1, Y_1, Z_1 that moves with the aircraft center of gravity while being parallel to the earth based frame X', Y', Z'.
  - **Step2:** Rotation around Z_1 of an angle Ψ from the frame X_1, Y_1, Z_1 to a new frame X_2, Y_2, Z_2 with $Z_1 = Z_2$.
  - **Step3:** Rotation around Y_2 of an angle Θ from the frame X_2, Y_2, Z_2 to a new frame X_3, Y_3, Z_3 with $Y_2 = Y_3$.
  - **Step4:** Rotation around X_3 of an angle Φ from the frame X_3, Y_3, Z_3 to the aircraft body frame X, Y, Z with $X_3 = X$.

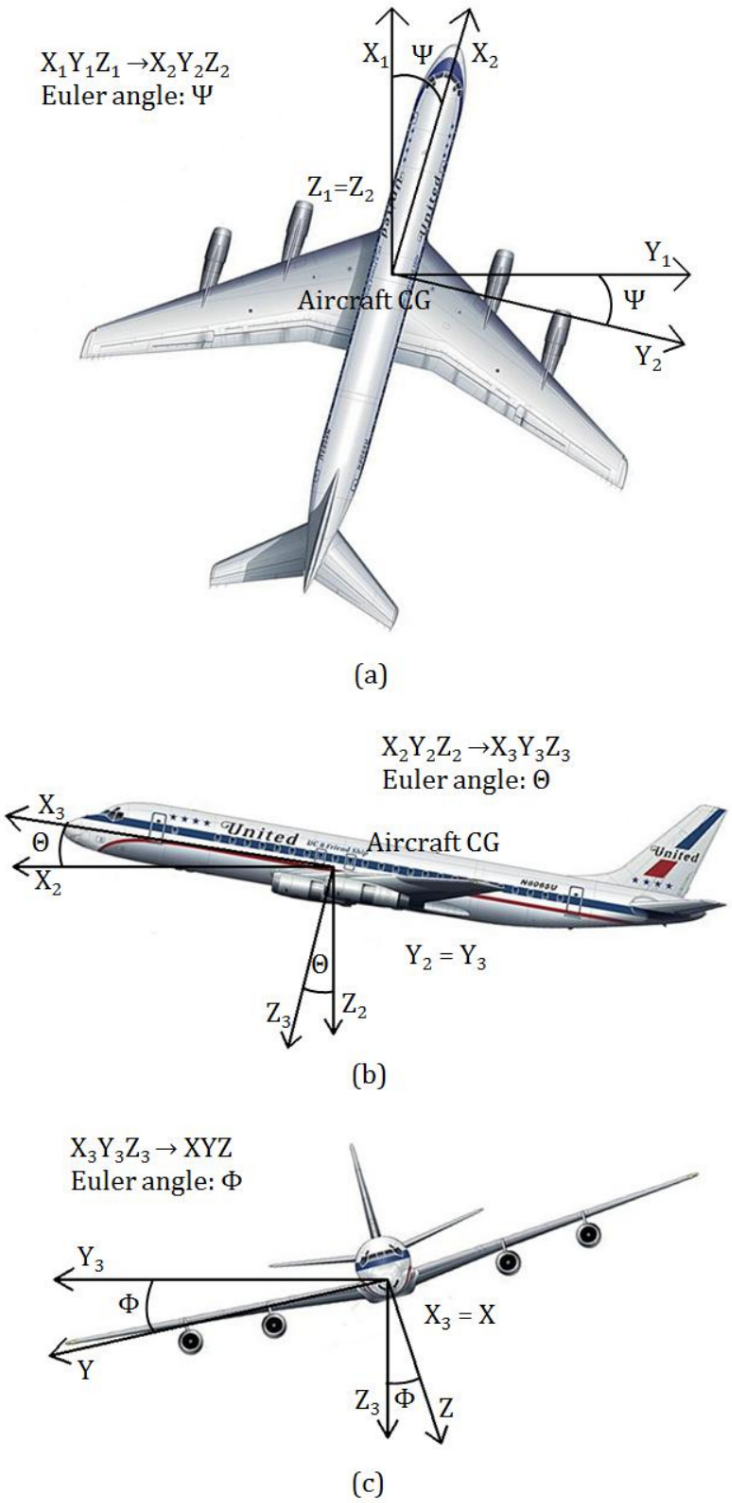

Figure 1: Introduction of the Euler Angles Ψ, Θ, Φ

Kinematic Equations

- Angular velocities in body frame can be expressed in terms of rate of change of the Euler angles.

$$\boldsymbol{\omega} = P\hat{\boldsymbol{i}} + Q\hat{\boldsymbol{j}} + R\hat{\boldsymbol{k}} = \dot{\boldsymbol{\Psi}} + \dot{\boldsymbol{\Theta}} + \dot{\boldsymbol{\Phi}}$$

- Starting with the transformation $X_1, Y_1, Z_1 \rightarrow X_2, Y_2, Z_2$, we have $Z_1 = Z_2$ which implies $\hat{\boldsymbol{k_1}} = \hat{\boldsymbol{k_2}}$. Therefore

$$\dot{\boldsymbol{\Psi}} = \dot{\Psi}\hat{\boldsymbol{k_1}} = \dot{\Psi}\hat{\boldsymbol{k_2}}$$

- Similarly, with the transformation $X_2, Y_2, Z_2 \rightarrow X_3, Y_3, Z_3$, we have $Y_2 = Y_3$ which implies $\hat{\boldsymbol{j_2}} = \hat{\boldsymbol{j_3}}$. Therefore

$$\dot{\boldsymbol{\Theta}} = \dot{\Theta}\hat{\boldsymbol{j_2}} = \dot{\Theta}\hat{\boldsymbol{j_3}}$$

- Similarly, with the transformation $X_3, Y_3, Z_3 \rightarrow X, Y, Z$, we have $X_3 = X$ which implies $\hat{\boldsymbol{i_3}} = \hat{\boldsymbol{i}}$. Therefore

$$\dot{\boldsymbol{\Phi}} = \dot{\Phi}\hat{\boldsymbol{i_3}} = \dot{\Phi}\hat{\boldsymbol{i}}$$

$$\boldsymbol{\omega} = P\hat{\boldsymbol{i}} + Q\hat{\boldsymbol{j}} + R\hat{\boldsymbol{k}} = \dot{\boldsymbol{\Psi}} + \dot{\boldsymbol{\Theta}} + \dot{\boldsymbol{\Phi}} = \dot{\Psi}\hat{\boldsymbol{k_2}} + \dot{\Theta}\hat{\boldsymbol{j_3}} + \dot{\Phi}\hat{\boldsymbol{i}}$$

- In transformation $X_2, Y_2, Z_2 \rightarrow X_3, Y_3, Z_3$,

$$\begin{Bmatrix} U_2 \\ V_2 \\ W_2 \end{Bmatrix} = \begin{bmatrix} \cos\Theta & 0 & \sin\Theta \\ 0 & 1 & 0 \\ -\sin\Theta & 0 & \cos\Theta \end{bmatrix} \begin{Bmatrix} U_3 \\ V_3 \\ W_3 \end{Bmatrix} \tag{1}$$

$$\begin{Bmatrix} \hat{\boldsymbol{i_2}} \\ \hat{\boldsymbol{j_2}} \\ \hat{\boldsymbol{k_2}} \end{Bmatrix} = \begin{bmatrix} \cos\Theta & 0 & \sin\Theta \\ 0 & 1 & 0 \\ -\sin\Theta & 0 & \cos\Theta \end{bmatrix} \begin{Bmatrix} \hat{\boldsymbol{i_3}} \\ \hat{\boldsymbol{j_3}} \\ \hat{\boldsymbol{k_3}} \end{Bmatrix} \tag{2}$$

Similarly, in the transformation $X_3, Y_3, Z_3 \rightarrow X, Y, Z$

$$\begin{Bmatrix} U_3 \\ V_3 \\ W_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & \sin\Phi & \cos\Phi \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \end{Bmatrix} \tag{3}$$

$$\begin{Bmatrix} \hat{\boldsymbol{i_3}} \\ \hat{\boldsymbol{j_3}} \\ \hat{\boldsymbol{k_3}} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & \sin\Phi & \cos\Phi \end{bmatrix} \begin{Bmatrix} \hat{\boldsymbol{i}} \\ \hat{\boldsymbol{j}} \\ \hat{\boldsymbol{k}} \end{Bmatrix} \tag{4}$$

- Expression for $\hat{\boldsymbol{k_2}}$ in $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$

$$\hat{\boldsymbol{k_2}} = -\sin\Theta\hat{\boldsymbol{i_3}} + \cos\Theta\hat{\boldsymbol{k_3}} = -\sin\Theta\hat{\boldsymbol{\imath}} + \cos\Theta\hat{\boldsymbol{k_3}}$$

where, $\hat{\boldsymbol{k_3}} = \sin\Phi\hat{\boldsymbol{\jmath}} + \cos\Phi\hat{\boldsymbol{k}}$, so

$$\hat{\boldsymbol{k_2}} = -\sin\Theta\hat{\boldsymbol{\imath}} + \cos\Theta\sin\Phi\hat{\boldsymbol{\jmath}} + \cos\Theta\cos\Phi\hat{\boldsymbol{k}}$$

Similarly, Expression for $\hat{\boldsymbol{j_3}}$ in $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$

$$\hat{\boldsymbol{j_3}} = \cos\Phi\hat{\boldsymbol{\jmath}} - \sin\Phi\hat{\boldsymbol{k}}$$

- Now using

$$\boldsymbol{\omega} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}} + R\hat{\boldsymbol{k}} = \dot{\boldsymbol{\Psi}} + \dot{\boldsymbol{\Theta}} + \dot{\boldsymbol{\Phi}} = \dot{\Psi}\hat{\boldsymbol{k_2}} + \dot{\Theta}\hat{\boldsymbol{j_3}} + \dot{\Phi}\hat{\boldsymbol{\imath}}$$

we have

$$= \dot{\Psi}\hat{\boldsymbol{k_2}}(-\sin\Theta\hat{\boldsymbol{i_3}} + \cos\Theta\sin\Phi\hat{\boldsymbol{\jmath}} + \cos\Theta\cos\Phi\hat{\boldsymbol{k}}) + \dot{\Theta}(\cos\Phi\hat{\boldsymbol{\jmath}} - \sin\Phi\hat{\boldsymbol{k}}) + \dot{\Phi}\hat{\boldsymbol{\imath}}$$

$$\boxed{\begin{aligned} P &= \dot{\Phi} - \sin\Theta\dot{\Psi} \\ Q &= \cos\Phi\dot{\Theta} + \cos\Theta\sin\Phi\dot{\Psi} \\ R &= \cos\Theta\cos\Phi\dot{\Psi} - \sin\Phi\dot{\Theta} \end{aligned}}$$

- Rearranging in matrix form

$$\begin{Bmatrix} P \\ Q \\ R \end{Bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\Theta \\ 0 & \cos\Phi & \cos\Theta\sin\Phi \\ 0 & -\sin\Phi & \cos\Theta\cos\Phi \end{bmatrix} \begin{Bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{Bmatrix}$$

or,

$$\begin{Bmatrix} \dot{\Phi} \\ \dot{\Theta} \\ \dot{\Psi} \end{Bmatrix} = \begin{bmatrix} 1 & \sin\Phi\tan\Theta & \cos\Phi\tan\Theta \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & \sin\Phi\sec\Theta & \cos\Phi\sec\Theta \end{bmatrix} \begin{Bmatrix} P \\ Q \\ R \end{Bmatrix}$$

Note:

There is a singularity associated with $\Theta = 90°$. This is one of the reason for using quaternion for large-scale simulation