Module-6

Lecture-26

6 DOF equations of motion

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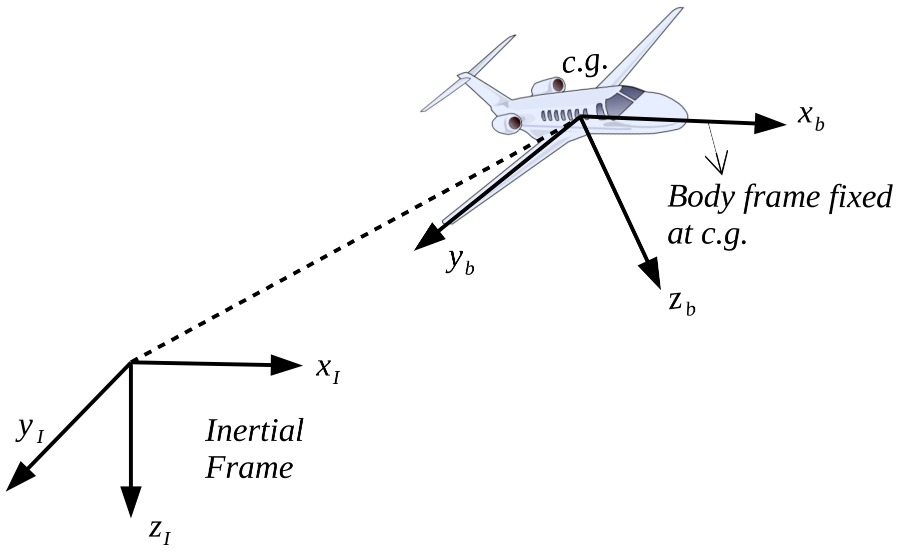

Figure 1: Inertial frame and body fixed frame

- Newtons laws, valid only in Inertial Frame

$$\sum \overline{F} = \frac{d}{dt}\left(m\overline{V}\right) \ ; \ \sum \overline{M} = \frac{d}{dt}\left(\overline{H}\right)$$

$$\overline{H} : \text{ Angular Momentum}$$

$$m\overline{V} : \text{ Linear Momentum}$$

- Consider an airplane as shown in Figure 2

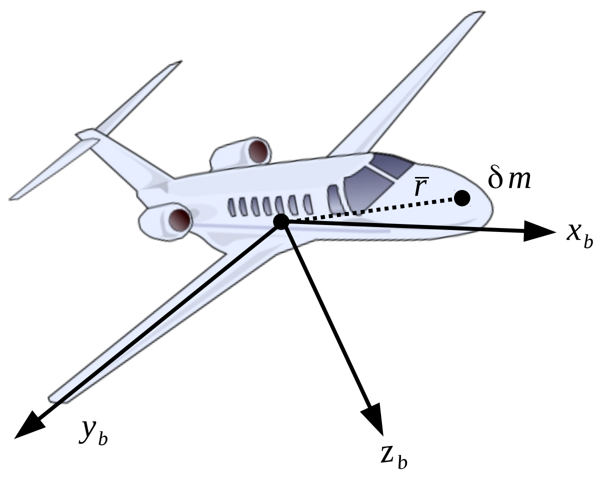

Figure 2: δm element

- Let V be the velocity of the elemental mass, δm of the airplane. V is the velocity of δm with respect to inertial frame.

- Let $\delta\overline{F}$ be the resulting force acting on the mass δm. By Newton's second law (assume mass is constant; not changing with time).

$$\delta\overline{F} = \delta m \frac{dV}{dt}$$

also

$$\sum \delta\overline{F} = \overline{F}$$

The velocity of δm with respect to inertial frame can be written as

$$V = V_c + \frac{d\overline{r}}{dt}$$

V_c is the velocity of the center of mass

Then

$$\sum d\overline{F} = \overline{F} = \frac{d}{dt}\sum\left(V_c + \frac{dr}{dt}\right)\delta m$$

$$\overline{F} = m\frac{dV_c}{dt} + \frac{d}{dt}\sum\frac{dr}{dt}\delta m$$

$$\overline{F} = m\frac{dV_c}{dt} + \frac{d^2}{dt^2}\sum r\delta m$$

- Since $\overline{r}$ is measured from the center of mass, so

$$\sum r\delta m = 0$$

- The force equation becomes

$$\overline{F} = m\frac{dV_c}{dt}$$

- Moment equation:

$$\delta M = \frac{d}{dt}\left(\delta\overline{H}\right) = \frac{d}{dt}\frac{d}{dt}\left(\overline{r} \times V\right)\delta m$$

$$V = V_c + \frac{d\overline{r}}{dt} = V_c + \overline{\omega} \times \overline{r}$$

$\overline{\omega}$ is the angular velocity of the vehicle

$\overline{r}$ is the position of the mean element measured from center of mass

$$H = \sum \delta H = \sum (\overline{r} \times V_c)\, \delta m + \sum \left[\overline{r} \times (\overline{\omega} \times \overline{r})\, \delta m\right]$$

V_c is constant with respect to the summation and can be taken outside the summation sign.

$$\overline{H} = \sum (\overline{r}\delta m) \times V_c + \sum \overline{r} \times (\overline{\omega} \times \overline{r})\, \delta m$$

$$\sum \overline{r}\delta m = 0 \text{ (Definition of } c.g.)$$

$$\overline{H} = \sum \left[\overline{r} \times \left(\overline{\omega} \times \overline{V}\right)\right] \delta m$$

Let

$$\overline{\omega} = p\hat{i} + q\hat{j} + r\hat{k}$$

$$\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Solving for $\overline{H}$, with $\overline{\omega}$ and $\overline{r}$ yields

$$H_x = p \sum \left(y^2 + z^2\right) \delta m - q \sum xy\delta m - r \sum xz\delta m$$

$$H_y = -p \sum xy\delta m + q \sum \left(x^2 + z^2\right) \delta m - r \sum yz\delta m$$

$$H_z = -p \sum xz\delta m - q \sum yz\delta m + r \sum \left(x^2 + y^2\right) \delta m$$

- $$I_x = \int\int\int \left(y^2 + z^2\right) \delta m \;;\; I_y = \int\int\int \left(x^2 + z^2\right) \delta m$$
  $$I_{xy} = \int\int\int xy\delta m \;;\; I_{xz} = \int\int\int xz\delta m$$
  $$I_z = \int\int\int \left(x^2 + y^2\right) \delta m \;;\; I_{yz} = \int\int\int yz\delta m$$

- $$H_x = pI_x - qI_{xy} - rI_{xz}$$
  $$H_y = -pI_{xy} + qI_y - rI_{yz}$$
  $$H_z = -pI_{xz} - qI_{yz} + rI_z$$

- If the reference frame (in this case inertial frame) is not rotating, then as the airplane rotates the moments and the product of inertia will vary with time.

- To avoid this difficulty, we fix the axis system to the aircraft (in this case at *c.g.* body axis system)

- Since the body fixed axis is not inertial frame, we need to use the following definition of derivative of a vector

$$\left.\frac{d\overline{A}}{dt}\right|_{Inertial} = \left.\frac{d\overline{A}}{dt}\right|_{Body} + \overline{\omega} \times \overline{A}$$

So

$$\overline{F} = m\left.\frac{d\overline{V_c}}{dt}\right|_{B} + \overline{\omega} \times \overline{V_c}$$

$$\overline{M} = \left.\frac{d\overline{H}}{dt}\right|_{B} + \overline{\omega} \times \overline{H}$$

Solving these two equations, we get

$$F_x = m\left(\dot{u} + qw - rv\right)$$

$$F_y = m\left(\dot{v} + ru - pw\right)$$

$$F_z = m\left(\dot{w} + pv - qu\right)$$

$$L = I_x\dot{p} - I_{xz}\dot{r} + qr(I_z - I_y) - I_{xz}pq$$

$$M = I_y\dot{q} + rp\left(I_x - I_z\right) + I_{xz}\left(p^2 - r^2\right)$$

$$N = I_z\dot{r} - I_{xz}\dot{p} + I_z\dot{r} + pq\left(I_y - I_x\right) + I_{xz}qr$$

- Since the airplane is choosen to be symmetric about xz plane;

$$I_{yz} = I_{xy} = 0$$

- $$F_x = F_{x_{gravity}} + F_{x_{propulsion}} + F_{x_{aerodynamic}}$$

  $$F_y = F_{y_{gravity}} + F_{y_{propulsion}} + F_{y_{aerodynamic}}$$

  $$F_z = F_{z_{gravity}} + F_{z_{propulsion}} + F_{z_{aerodynamic}}$$

- $$F_{x_{gravity}} = -mg\sin\theta$$

  $$F_{y_{gravity}} = mg\cos\theta\sin\phi$$

  $$F_{z_{gravity}} = mg\cos\theta\cos\phi$$