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Relativistic Quantum Mechanics, Foldy-Wouthuysen transformations

PCD_STiAP_P03

1.

a) Prove that moving objects are shortened by a factor $\gamma = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$ compared to their lengths in rest frames.

b) Prove that moving particle last longer compare to their life spans (average life time) in rest frames.

c) How is "proper velocity" defined in special theory of relativity? Explain why it is called as "proper"?

d) What is "proper momentum"?

e) Which of the following two equations is correct according to special theory of relativity?

$$E = mc^2$$

and $E = \gamma mc^2$ and explain why?

f) What is the relation between energy and momentum for a massless particle?

2.

a) Show that quantization in relativistic quantum mechanics would require the dynamical variable momentum to be replaced by the operator $p^\mu = i\hbar\left(\dfrac{\partial}{\partial ct}, -\vec{\nabla}\right)$ and $p^\mu = i\hbar\left(\dfrac{\partial}{\partial ct}, +\vec{\nabla}\right)$.

b) Show that the special theory of relativity invariant $p^\mu p_\mu = m^2c^2$ can be factored into two first order equations if the three-vector $\vec{p} = 0$.

c) If three-vector $\vec{p} \neq 0$, show that the electromagnetic relationship $p^\mu p_\mu - m^2c^2 = 0$ can be factored as $\left(\gamma^\mu p_\mu - mc\right)\left(\gamma^\nu p_\nu + mc\right) = 0$ wherein γ^μ and γ^ν must be matrices.

d) Quantize the factor $\gamma^\mu p_\mu - mc$ and construct the following quantum equation $\left(i\hbar\gamma^\mu \partial_\mu - mc\right)\Psi = 0_{4\times1}$. This is the famous Dirac equation.

3.

a) Quantize the equation $\left[p^\mu p_\mu - m^2c^2 = 0\right]$ and arrive at the Klein-Gordon equation.

b) What are the demerits of Klein-Gordon equation?

4.

a) Show that the Dirac equation is very often written in the following equivalent forms $(P\bullet mc)\Psi = 0$ i.e $(i\hbar\nabla\bullet mc)\Psi = 0$ and $i\hbar\frac{\partial\Psi}{\partial t} = \left[c\vec{\alpha}\cdot\left(\vec{p}-\frac{e}{c}\vec{A}\right)+e\phi+\beta mc^2\right]\Psi$ when the electron is in the presence of electromagnetic field described by potentials $(\phi,\vec{A})$.

b) Demonstrate that the non-relativistic Hamiltonian which describes the coupling of an electron with electromagnetic field given by $H_{nr}^{'} = -\frac{e}{c}\frac{\vec{p}}{m}\bullet\vec{A}+e\phi$ must be replaced by relativistic form $H_{rel}^{'} = -e\vec{\alpha}\bullet\vec{A}+e\phi$

5.

a) Starting with the four-component Dirac equation, $\frac{\partial}{\partial t}\begin{bmatrix}\tilde{\Phi}\\ \tilde{\chi}\end{bmatrix} = \left[c\vec{\alpha}\cdot\left(\vec{p}-\frac{e}{c}\vec{A}\right)+e\phi+\beta mc^2\right]\begin{bmatrix}\tilde{\Phi}\\ \tilde{\chi}\end{bmatrix}$ obtain the equations

(i) $[E-E_0-e\phi]\tilde{\Phi} = [c\vec{\sigma}\bullet\vec{\pi}]\tilde{\chi}$

(ii) $[E+E_0-e\phi]\tilde{\chi} = [c\vec{\sigma}\bullet\vec{\pi}]\tilde{\Phi}$ where $\begin{bmatrix}\tilde{\Phi}\\ \tilde{\chi}\end{bmatrix} = e^{\left(-i\frac{mc^2}{h}\right)}\begin{bmatrix}\Phi\\ \chi\end{bmatrix}$

b) Comment on the large and small part of the wavefunction and arrive at the relation between them $\chi \approx \left[\frac{\vec{\sigma}\bullet\vec{\pi}}{2mc}\right]\Phi$

c) Obtain the non-relativistic limit express by the Pauli equation $i\hbar\frac{\partial\Phi}{\partial t} = \left[\frac{\vec{p}^2}{2m}-\frac{e}{2mc}\left(\vec{l}+2\vec{s}\right)\bullet\vec{B}\right]\Phi$

d) Define the g factor; what is the value of electron g-factor according to Dirac-Pauli equation?

e) Refer to B.Odom, D.Hanneke, B.D'Urso and G.Gabrielse, Phys.Rev.Lett. 97, 030801 (2006) and acquaint yourself with the corrections to the g-factor.

f) Refer to G.Gabrielse, D.Hanneke, T.Kinoshita, M.Nio and B.Odom, Phys.Rev.Lett. 97, 030802(2006) and acquaint yourself with how corrections to the g-factor affect the value of the fine structure 'constant'.

g) Google the present best values of (1) g and (2) α, the fine structure constant.

6. Consider the Dirac Hamiltonian for an electron in an electromagnetic field $A^\mu(=\phi, A): H = \beta m + \alpha\bullet(p-eA)+e\phi$ (we use units in which $\hbar = 1$ and $c = 1$). For the sake of brevity we shall use the following notation $\Theta = \alpha\bullet(p-eA)$ and $\varepsilon = e\phi$. We search for a unitary transformation through an operator $U = e^{iS}$ (where S is a Hermition operator) such that the result of this transformation will eliminate the effect of all "odd" operators, such as the operator α, which couples the large and the small component of the four-component bispinors. <u>Show that</u> as $\Psi^{'} = e^{iS}\Psi$, the Dirac Hamiltonian H transforms to H', where

$$H^{'} = e^{iS}\left(H - i\frac{\partial}{\partial t}\right)e^{-iS}$$

7. Prove that

$$e^{iS}He^{-iS} = H + i[S,H] + \frac{i^2}{2!}[S,[S,H]] + \frac{i^3}{3!}[S,[S,[S,H]]] + ...$$

8. Prove that

$$-ie^{iS}\left(\frac{\partial}{\partial t}e^{-iS}\right) = -\dot{S} - \frac{i}{2}[S,\dot{S}] + \frac{1}{6}[S,[S,\dot{S}]] + \frac{i}{24}[S,[S,[S,\dot{S}]]] + ...$$

9. Show that

$$H^{'} = H + i[S,H] - \frac{1}{2}[S,[S,H]] - \frac{i}{6}[S,[S,[S,H]]] + \frac{1}{24}[S,[S,[S,[S,H]]]] + ...$$

$$-\dot{S} - \frac{i}{2}[S,\dot{S}] + \frac{1}{6}[S,[S,\dot{S}]] + ...$$

wherein we consider the transformation operator S to be of $O\left(\frac{1}{m}\right)$

10. We choose the transformation operator $S\left[ofO\left(\frac{1}{m}\right)\right]$ to be given by $-\frac{i\beta\Theta}{2m}$. This is the first Foldy-Wouthuysen transformation, and it will be denoted by S_1. Accordingly $S_1 = -\frac{i\beta\Theta}{2m} = -\frac{i\beta\{\alpha\bullet(p-eA)\}}{2m}$. Prove that, as a result of the first Foldy-Wouthuysen transformation, the transformed Dirac Hamiltonian becomes

$$H^{'} = \beta m + \varepsilon^{'} + \Theta^{'},$$

Where $$\Theta^{'} = \frac{\beta}{2m}[\Theta,\varepsilon] - \frac{\Theta^3}{3m^2} + \frac{i\beta\Theta}{2m}$$

And $$\varepsilon^{'} = e^{iS}\varepsilon e^{-iS} = \varepsilon + \frac{\beta\Theta^2}{2m} - \frac{\beta\Theta^4}{8m^3} - \frac{1}{8m^2}[\Theta[\Theta,\varepsilon]] - \frac{i}{8m^2}[\Theta,\dot{\Theta}]$$

Note that the terms $\varepsilon^{'}$ and $\Theta^{'}$ contain odd operators to $O\left(\frac{1}{m}\right)$.

11. Now consider second Foldy-Wouthuysen transformation through an operator

$$S^{'} = -\frac{i\beta\Theta^{'}}{2m} = -\frac{i\beta}{2m}\left\{\frac{\beta}{2m}[\Theta,\varepsilon] - \frac{\Theta^3}{3m^2} + \frac{i\beta\dot{\Theta}}{2m}\right\}$$

Show that, as a result of this transformation, we get

$$H^{''} = \beta m + \varepsilon^{'} + \Theta^{''}$$

Where $\Theta^{''} \approx O\left(\frac{1}{m^2}\right)$

12. Consider now third Foldy-Wouthuysen transformation through an operator

$$S'' = -\frac{i\beta\Theta''}{2m}$$

Show that, as a result of this transformation, we get

$$H''' = e^{iS''}\left(H'' - i\frac{\partial}{\partial t}\right)e^{-iS''} = \beta\left(m + \frac{\Theta^2}{2m} - \frac{\Theta^4}{8m^3}\right) + \varepsilon - \frac{1}{8m^2}\left[\Theta,\left[\Theta,\varepsilon\right]\right] - \frac{i}{8m^2}\left[\Theta,\dot{\Theta}\right]$$

Observe that as a result of the third Foldy-Wouthuysen transformation, the Dirac equation acquires the following form

$$H'''\Psi''' = \frac{i\partial\Psi'''}{\partial t}$$

in which we have terms which couple the large part and the small part of

$$\Psi''' = \begin{pmatrix}\Phi''' \\ \chi'''\end{pmatrix}$$

with the coupling terms restricted to $O\left(\frac{1}{m^3}\right)$.

13. By retaining terms of $O\left(\frac{1}{m^3}\right)$ show that

$$H''' = \beta\left\{m + \frac{(p-eA)^2}{2m} - \frac{p^4}{8m^3}\right\} + e\phi - \frac{e}{2m}\beta\sigma\bullet B - \frac{ie}{8m^2}\sigma\bullet\nabla\times E - \frac{e}{4m^2}\sigma\bullet E\times p - \frac{e}{8m^2}\nabla\bullet E$$

14. Show that the term $-\frac{e}{4m^2}\sigma\bullet E\times p$ in H''' reduces to

$$\frac{e}{4m^2}\frac{1}{r}\frac{\partial V}{dr}\sigma\bullet L$$

References:

1. David J. Griffiths, Introduction to Electrodynamics, Third edition, ch.12
2. (a) Motion of charged particles in Electromagnetic fields and Special theory of relativity, P Chaitanya Das, G.S Murthy, P.C. Deshmukh, K Satish Kumar and T.A. Venkatesh, Resonance July 2004
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4. Quantum Mechanics, A. Messiah, Volume II
5. Principles of Quantum Mechanics, P. A. M. Dirac, fourth Edition.
6. L.L Foldy and S.A Wouthuysen Phys Rev. 78 29 (1950)