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Angular Momentum, Spherical Harmonics, Wigner Eckart Theorem

PCD_STiAP_P02

1. (a) Prove that $\frac{\vec{p}}{\hbar}$ is the generator of infinitesimal translation.

   (b) Prove that $\frac{\vec{L}}{\hbar}$ is the generator of infinitesimal rotation.

   (c) Write the commutation relations for the generators of infinitesimal translations.

   (d) Write the commutation relations for the generators of infinitesimal rotations.

   (e) Show that the angular momentum vector operator L commutes with the potential energy operator for a central field.

   (f) Show that:

   i. $J_z J_+ - J_+ J_z = \hbar J_+$

   ii. $J_- J_z - J_z J_- = \hbar J_-$

   iii. $J_+ J_- - J_- J_+ = 2\hbar J_z$

   iv. $J^2 J - J J^2 = 0$

   v. $J^2 - J_Z^2 \pm \hbar J_Z = J_\pm J_\mp$

   (e) Generate the matrix representation for $J^2, J_x, J_y and J_z$

   (i) $j = \frac{1}{2}$ (ii) j =1 (iii) $j = \frac{3}{2}$

2. (a) What quantities are conserved in the classical Kepler two body problem?

   (b) What are the corresponding symmetries?

   (c) What are the physical dimensions of Laplace- Runge-lenz vector $\vec{A}$?

   (d) What is the direction of the vector $\vec{A}$?

   (e) Write down the properties of SO (4) group.

   (f) Show that the group generated by L_{ij} $(i, j = 1,2,3,4)$ constitutes an SO (4) group.

3. (a) Obtain the matrix representation of the rotation operator $U_R(\delta\phi_x)$ which brings about a rotation of an arbitrary function $f(x, y, z)$ about the x axis through the infinitesimal angle $(\delta\phi_x)$.

   (b) Using the relation $U_R(\phi_x)|\gamma, jm\rangle = \sum_{m'} |\gamma, jm'\rangle D^{(j)}_{m'm}(\phi)$, obtain the new state function by the rotation of the p_z orbital about the x axis through the two angles (i) 60° and (ii) 90°. Of course, you should carry out the sum in the above expression explicitly.

   (c) Finally, ensure that the rotated function is normalized.

   (d) Show that $e^{\frac{-i\theta S_x}{\hbar}} = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$.

4. Let γ be the angle between two directions defined by spherical angles θ, φ and θ', φ'. Prove that $P_l(n.n') = \frac{4\pi}{2l+1} \sum_{m-l}^{l} Y_l^{m*}(n') Y_l^m(n)$ where n and n' are unit vectors and $Y_l^m(n)$ denotes the spherical harmonic function of the polar and azimuthal angle of the direction of n relative to a fixed coordinate system. This result is known as the Addition Theorem of Spherical harmonics.

5. (a) Prove that for a C.G. Coefficient to be non zero, m must be $m_1 + m_2$.
   (b) Prove that j is restricted to the range $j_1 - j_2 \le j \le j_1 + j_2$.
   (c) Also, prove that $j_2 - j_1 \le j \le j_1 + j_2$.
   (d) Obtain the condition $|j_1 - j_2| \le j \le j_1 + j_2$.
   (e) Prove that the C.G. coefficients satisfy the condition
      (i) $\sum_{m_1 m_2} \langle j_1 j_2 m_1 m_2 \mid j_1 j_2 jm \rangle \langle j_1 j_2 m_1 m_2 \mid j_1 j_2 j' m' \rangle = \delta_{mm'} \delta_{jj'}$
      (ii) $\sum_{m_1 m_2} \langle j_1 j_2 m_1 m_2 \mid j_1 j_2 jm \rangle \langle j_1 j_2 m_1 m_2 \mid j_1 j_2 j' m' \rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$
      [Remarks: These relations are known as Clebsh-Gordan Orthogonality relations]

6. Obtain the recursion relations for the Clebsch-Gordan coefficients

   (a) $\sqrt{(j+m)(j-m+1)}\langle m_1 m_2 \mid jm+1 \rangle = \sqrt{(j_1 - m_1)(j_1 + m_1 + 1)}\langle m_1 + 1, m_2 \mid jm \rangle + \sqrt{(j_2 - m_2)(j_2 + m_2 + 1)}\langle m_1, m_2 + 1 \mid jm \rangle$

   (b) $\sqrt{(j-m)(j+m+1)}\langle m_1 m_2 \mid jm+1 \rangle = \sqrt{(j_1 + m_1)(j_1 - m_1 + 1)}\langle m_1 - 1, m_2 \mid jm \rangle + \sqrt{(j_2 + m_2)(j_2 - m_2 + 1)}\langle m_1, m_2 - 1 \mid jm \rangle$

   (c) Prove that $\sqrt{2j}\langle j_1, j - j_1 \mid j, j \rangle = \sqrt{2j_1}\langle j_1 - 1, j - j_1 \mid j, j-1 \rangle + \sqrt{(j_2 + j - j_1)(j_2 - j + j_1 + 1)}\langle j_1, j - j_1 - 1 \mid j, j-1 \rangle$

   (d) From the condition $\sqrt{2j}\langle j_1, j - j_1 - 1 \mid j, j-1 \rangle = \sqrt{(j_2 - j + j_1 + 1)(j_2 + j - j_1)}\langle j_1, j - j_1 \mid j' j \rangle$,
   Determine $\langle j_1, j - j_1 - 1 \mid j, j-1 \rangle$, if $\langle j_1, j - j_1 \mid j, j \rangle$ is known.

   (e) Prove that

   (i) $\left\langle j_1 \frac{1}{2}, m - \frac{1}{2}, \frac{1}{2} \middle| j_1 \frac{1}{2}, j_1 \pm \frac{1}{2}, m \right\rangle = \pm\sqrt{\frac{j_1 \pm m + \frac{1}{2}}{2j_1 + 1}}$

   (ii) $\left\langle j_1 \frac{1}{2}, m + \frac{1}{2}, -\frac{1}{2} \middle| j_1 \frac{1}{2}, j_1 \pm \frac{1}{2}, m \right\rangle = \pm\sqrt{\frac{j_1 \mp m + \frac{1}{2}}{2j_1 + 1}}$

7. (a) Generate the C.G. Coefficients table corresponding to $j_1 = 1$ and $j_2 = \frac{1}{2}$

(b) Show that:

(i) $\left|(1,1)2,-1\right\rangle = \frac{1}{\sqrt{2}}\left[\left|10,1-1\right\rangle + \left|1-1,10\right\rangle\right]$

(ii) $\left|(1,1)2,-2\right\rangle = \left|1-1,1-1\right\rangle$

(c) The 3j symbol is defined as $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m \end{pmatrix} = \frac{\langle m_1 m_2 \mid jm\rangle}{(-1)^{j_1-j_2+m}\left(\sqrt{2j+1}\right)}$. Show that

(i) $\begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$

(ii) $\begin{pmatrix} j_2 & j_1 & j_3 \\ -m_2 & -m_1 & -m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$

(iii) $\begin{pmatrix} j & 1 & j \\ -j & 0 & j \end{pmatrix} = \left(\frac{j}{(j+1)(2j+1)}\right)^{\frac{1}{2}}$

(d) Prove that

(i) $(2j+1)\sum_{m_1,m_2}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & -m' \end{pmatrix} = \delta_{jj'}\delta_{mm'}$

(ii) $\sum_{j}(2j+1)\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j \\ m'_1 & m'_2 & -m \end{pmatrix} = \delta_{m_1m_1'}\delta_{m_2m_2'}$

8. The Clebsh-Gordan coefficients are related to the Wigner 3j-symbols by the following relation:

$$\langle m_1 m_2 \mid jm\rangle = (-1)^{j_1-j_2+m}\sqrt{2j+1}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}$$

The explicit form of the Wigner 3j-symbols is given by eq (106.4) [page 436] of Ref.2. Use the formulae to write a numerical algorithm to evaluate the Wigner 3j-symbols and Clebsh-Gordan coefficients.

9. Using the Wigner-Eckart Theorem (cf. Ref. 4):

$$\langle N'j'm'|T_\mu^\omega|Njm\rangle\, A_{m\mu}^{j\omega j'}\delta_{m',m+\mu}\langle N'j'\|T^\omega\|Nj\rangle$$

(a) $|\Delta j| = |j - j'| \le \omega$

(b) $\Delta m = m' - m = \mu \le \omega$

(c) Establish the dipole selection rules $\Delta j = \pm 1, 0$. For circularly polarized radiation about z-axis, $\mu = \pm 1 = \Delta m$. For radiation polarized along the quantization axis, $\mu = 0 = \Delta m$

(d) Prove that the transition in which j goes from 0 to 0 is forbidden even if $\Delta j = 0$.

(e) Show that dipole transition can take place only between states of opposite parity (Laporte rule).

10. (a) A sodium atom having its outer electron excited to a state with $n = 3$ and $l = 1$ is placed in a weak magnetic field. Spectral line A is the result of transition from $\left|1, \frac{1}{2}, \frac{3}{2}, -\frac{1}{2}\right\rangle$ to $\left|0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\rangle$ and line B is the result of the transition from $\left|1, \frac{1}{2}, \frac{3}{2}, -\frac{3}{2}\right\rangle$ to $\left|0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle$ wherin the coupled states are designated as $\left|L.S, J, M_J\right\rangle$. Find the ratio of the intensity of line A to that of line B using Wigner-Eckart theorem and the table of CG coefficients.

(b) A sodium atom having its outer electron excited to a state with $n = 3$ and $l = 1$ is placed in a weak magnetic field. Spectral line A is the result of transition from $\left|1, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}\right\rangle$ to $\left|0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\rangle$ and line B is the result of the transition from $\left|1, \frac{1}{2}, \frac{3}{2}, -\frac{1}{2}\right\rangle$ to $\left|0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle$ wherin the coupled states are designated as $\left|L.S, J, M_J\right\rangle$. Find the ratio of the intensity of line A to that of line B using Wigner-Eckart theorem and the table of CG coefficients.

11. (a) Prove that for an operator to be a scalar it must commute with all components of the total angular momentum operator J.
(b) If χ is a scalar operator then prove that $\left\langle \Phi_{\alpha jm} \middle| \chi \middle| \Phi_{\alpha' j' m'} \right\rangle = \lambda \delta_{\alpha\alpha'} \delta jj' \delta mm'$, where λ is the eigen value of χ.

12. Consider spherical component of a vector operator given by

$$V_1^1 = -\frac{1}{\sqrt{2}}\left(V_x + iV_y\right), V_0^1 = V_x, V_{-1}^1 = \frac{1}{\sqrt{2}}\left(V_x - iV_y\right)$$

If the vector operator is the total angular momentum J itself then show that:

(a) $\left\langle \alpha' j' m' \middle| J_q \middle| \alpha jm \right\rangle = \frac{1}{\sqrt{2j'+1}} \left\langle j1mq \middle|\middle| j'm' \right\rangle \left\langle \alpha' j' \middle|\middle| J_q \middle|\middle| \alpha j \right\rangle$

(b) $\left\langle \alpha' j' \middle| J \middle| \alpha j \right\rangle = \sqrt{j(j+1)}\hbar \delta_{\alpha\alpha'} \delta jj'$

(c) Prove that for any vector operator V $\left\langle \alpha' j' m' \middle| V \middle| \alpha jm \right\rangle = C \left\langle \alpha' j' m' \middle| J \middle| \alpha jm \right\rangle$, where C is independent of m and m'

(d) By setting $j' = j$ and $\alpha' = \alpha$ prove that the constant C is given by $C = \frac{\left\langle \alpha jm \middle| V.J \middle| \alpha jm \right\rangle}{j(j+1)\hbar^2}$.

(e) Prove that $j(j+1)\hbar^2 \left\langle \alpha jm' \middle| V \middle| \alpha jm \right\rangle = \left\langle \alpha jm \middle| V.J \middle| \alpha jm \right\rangle \left\langle \alpha jm' \middle| J \middle| \alpha jm \right\rangle$

(f) Prove that $j(j+1)\hbar^2 \left\langle \alpha jm' \middle| V \middle| \alpha jm \right\rangle = \left\langle \alpha jm' \middle| (V.J) J \middle| \alpha jm \right\rangle$.

Useful References:

1. Merzbacher, E., Quantum Mechanics, Wiley International Edition, 1970
2. L.D. Landau and E.M. Lifshitz, Quantum Mechanics' Third Edition. (Pergamom Press, 1985)
3. J.J.Sakurai, 'Modern Quantum Mechanics' Pearson Education.
4. Tinkham, 'Group theory and Quantum Mechanics', Tata McGraw-Hill Publishing Company Ltd, 1974.
5. Grenier, W. and Muller,B.; Quantum Mechanics Symmetries; Springer- Verlag Berlin, 2^{nd} edition (1989).
6. Bransden B.H and Joachin, C.J.: ‚Physics of atome and molecules', Longman Group Limited, 1983
7. Schiff, L, 'Quantum Mechanics', McGraw-Hill Book Company
8. John J Powell, Bernd Ceaseman, 'Quantum Mechanics' Addison Wesely Publishing company, 1961.