Mathematical Methods in Engineering and Science

IIT Kanpur

Assignment Policy of the Course

- Assignments of this course are of the traditional problem-solving type. Not only the result, but also the approach will be considered for evaluation.
- Submissions are invariably to be in the PDF format. PDF version of scanned copy of neat handwritten work is acceptable. Brevity and to the point solutions will be rewarded. Illegible writing, sloppy work and beating around the bush will be penalized.
- For many segments of the course, a programming background will be useful, though it is not essential.
- For some problems in the assignments, programming will be needed to get the complete solution. But, the programme does not need to be submitted.
  Those students who do not have a programming background may conduct the intial few (2 to 5) steps or iterations manually and indicate through what repititive and automated procedure a programme would need to proceed in order to get the final complete solution. Such solutions will be considered for partial credit.
- There may also be some problems, for which programming may not be needed as such, but may be useful in solving the problem systematically. Students having a programming background are encouraged to take advantage of this.

MMES (2017) Assignment 8

(Full marks = 100)

1. Using backward Euler's method, solve the IVP

$$y^{iv} + 9999y'' - 10^4 y = x \sin 2x, \quad y(0) = y'(0) = y''(0) = y'''(0) = 0$$

for $0 \le x \le 120$. [*Hint:* Try step sizes 1.2 and 2.4 to appreciate the issue of stability.]

2. The laminar boundary layer on a flat plate can be studied by solving the Blasius equation, $f''' + ff'' = 0$, in which $f(\eta)$ is the stream function, $f'(\eta) = u/U$ is the ratio of the local velocity to the free-stream velocity and η is a similarity variable that was used to reduce the original equation (which is a PDE) to this ODE. The 'no slip' condition at the plate and free-stream condition at large distances give the boundary conditions as $f(0) = f'(0) = 0$ and $f'(\infty) = 1$. Replacing the last condition with $f'(\eta_{free}) = 1$ for some large η_{free}, say 10, estimate $f''(0)$ by shooting and determine the solution.

3. Show that the solution of BVP, $y'' + P(x)y' + Q(x)y = R(x),\ y(a) = p_1,\ y(b) = p_2$, can be obtained by the superposition of the solutions of two IVP's. Generalize the theme to a higher dimensional state space for the BVP of a linear ODE system

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}(t),\ \mathbf{y}_1(t_i) = \mathbf{a},\ \mathbf{y}_2(t_f) = \mathbf{b},\ \mathbf{y}_1 \in R^{n_1},\ \mathbf{y}_2 \in R^{n_2},\ n_1 + n_2 = n,\ \mathbf{y} \in R^n.$$

[Here, $\mathbf{y}_1$ and $\mathbf{y}_2$ are just collections of state variables from $\mathbf{y}$, need not be disjoint sets.]

4. The deflection $y(x)$ of a cantilever beam with varying cross section has been modelled as $\frac{d^2}{dx^2}\left(k(x)\frac{d^2y}{dx^2}\right) = w(x)$, with $k(x) = (x+a)(L+a-x)$ and loading $w(x) = 10(L-x)$. The boundary conditions at the fixed end are obviously $y(0) = y'(0) = 0$. The absence of bending moment and shear force at the free end give the other boundary conditions as $y''(L) = y'''(L) = 0$. With the numerical value $L = 2$ and $a = 1$, use relaxation method twice, with 10 and 20 steps, to develop two estimates of the deflection of the beam. By extrapolation, determine the deflections at $x = 1, 1.6, 2$ more accurately.

5. Solve the following differential equations.

   (a) $(2\cos y + 4x^2)dx = x \sin y\, dy$.

   (b) $\frac{dy}{dx} = \frac{2x-3y+2}{3x+2y+5}$.

   (c) $\frac{dy}{dx} = \frac{3y+2x+4}{4x+6y+5}$.

   (d) $2xyy' + (x-1)y^2 = x^2 e^x$.

   (e) $(x^2 + y/x)dx + (\ln x)dy = 0;\ y(e) = -e^3/3$.

   (f) $y'' = 1 + (y')^2$.

   (g) $yy'' = y^2 y' + (y')^2$.

6. Given the initial value problem $y' = f(x, y)$, $y(0) = 0$, test the existence of a solution, its uniqueness, continuous dependence on the initial condition, satisfaction of a Lipschitz condition in a finite domain containing $(0, 0)$ and the existence, continuity and boundedness of $\frac{\partial f}{\partial y}$ at this point, for the following cases of $f(x, y)$: (a) y/x, and (b) $x^2|y|$.

7. Find out the family of curves orthogonal to the family of parabolas $y = kx^2$. Generalize the procedure to determine the family of curves orthogonal to the family of curves $y = kx^n$. Any interesting observations?

8. A young enterpreneur invests an amount Rs D per year at an interest rate r. If the investment is made continuously and the interest is compounded continuously, then find out

   (a) an expression for the rate at which the accumulated amount grows,

   (b) an expression for the accumulated amount at any time t, and

   (c) the required yearly investment D so as to yield an accumulated retirement return of 10 million rupees after 40 years, if the rate of interest is 6%.

9. Solve the following differential equations.

   (a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$.

   (b) $4\ddot{y} - 4\dot{y} - 3y = 0;\ y(-2) = e,\ \dot{y}(-2) = -e/2$.

   (c) $(x^2D^2 - 3xD + 4)y = 0;\ y(1) = 0,\ y'(1) = 3$.

10. Develop the Maclaurin's series (Taylor's series around $x = 0$) for the function $y(x)$, for which it is given that $y(0) = 1$, $y'(0) = 1$ and $y'' = -xy$ (work only up to the term x^9). Plot this series in superimposition with the series obtained by truncation of the highest order non-zero contribution.