Mathematical Methods in Engineering and Science

IIT Kanpur

Assignment Policy of the Course

- Assignments of this course are of the traditional problem-solving type. Not only the result, but also the approach will be considered for evaluation.
- Submissions are invariably to be in the PDF format. PDF version of scanned copy of neat handwritten work is acceptable. Brevity and to the point solutions will be rewarded. Illegible writing, sloppy work and beating around the bush will be penalized.
- For many segments of the course, a programming background will be useful, though it is not essential.
- For some problems in the assignments, programming will be needed to get the complete solution. But, the programme does not need to be submitted.
  Those students who do not have a programming background may conduct the intial few (2 to 5) steps or iterations manually and indicate through what repititive and automated procedure a programme would need to proceed in order to get the final complete solution. Such solutions will be considered for partial credit.
- There may also be some problems, for which programming may not be needed as such, but may be useful in solving the problem systematically. Students having a programming background are encouraged to take advantage of this.

MMES (2017) Assignment 5

(Full marks = 100)

1. Show that the gravitational field due to a particle at $\mathbf{r}_0$, given by Newton's law of gravitation,

$$\mathbf{g} = -c\frac{\mathbf{r} - \mathbf{r}_0}{\|\mathbf{r} - \mathbf{r}_0\|^3}$$

can be obtained as the gradient of a scalar potential function, i.e. *there exists a gravitational potential*. Show further that the potential function satisfies Laplace's equation.

2. The fundamental laws of electromagnetism relate the electric and the magnetic fields as

$$\int_C \mathbf{E} \cdot d\mathbf{R} = -\frac{\partial \phi}{\partial t} \quad \text{(Faraday)}, \qquad \int_C \frac{\mathbf{B}}{\mu} \cdot d\mathbf{R} = i \quad \text{(Ampere)},$$

$$\int_S \int \epsilon \mathbf{E} \cdot \mathbf{N} dS = q \qquad \text{and} \qquad \int_S \int \mathbf{B} \cdot \mathbf{N} dS = 0 \quad \text{(Gauss)}.$$

   (a) Using definitions of magnetic flux (ϕ), charge density (Q) and current density ($\mathbf{J}$) from

$$\phi = \int_S \int \mathbf{B} \cdot \mathbf{N} dS, \quad i = \int_S \int \mathbf{J} \cdot \mathbf{N} dS \quad \text{and} \quad q = \int \int_V \int Q dV,$$

   and noting that $\mathbf{J} = \epsilon \frac{\partial \mathbf{E}}{\partial t}$ for a perfect dielectric, apply integral theorems to develop expressions of divergence and curl of the electric and magnetic fields for a perfect dielectric medium of *constant* properties. [**Maxwell's equations**, in differential form.]

   (b) Further, from these equations, using second order differential relationships, develop individual (uncoupled) wave equations for the electric field ($\mathbf{E}$) and for the magnetic field ($\mathbf{B}$), in the absence of free charge in the medium.

3. For $p(x) = (x - \alpha)^3 q(x)$, show that $p'(\alpha) = 0$, and $p''(\alpha) = 0$.

4. Find the companion matrix of the polynomial $x^4 - 2x^3 - 5x^2 + 10x - 3$ and determine its eigenvalues by the QR-decomposition algorithm. Verify that these are the roots of the given polynomial.

5. Let $P(x)$ be a polynomial on which successive synthetic division by a chosen quadratic polynomial $x^2 + px + q$ produces the successive quotients $P_1(x)$, $P_2(x)$ and remainders $rx + s$, $ux + v$, such that

$$P(x) = (x^2 + px + q)P_1(x) + rx + s, \qquad P_1(x) = (x^2 + px + q)P_2(x) + ux + v.$$

   (a) Observing that the expressions $P_1(x)$, $P_2(x)$ and the numbers r, s, u, v all *depend* upon p and q in the chosen expression $x^2 + px + q$, differentiate the expression for $P(x)$ above partially with respect to p and q, and simplify to obtain expressions for $\frac{\partial r}{\partial p}$, $\frac{\partial r}{\partial q}$, $\frac{\partial s}{\partial p}$, $\frac{\partial s}{\partial q}$. [*Hint:* At its roots, a polynomial evaluates to zero.]

(b) Frame the Jacobian $\mathbf{J}$ of $[r \;\; s]^T$ with respect to $[p \;\; q]^T$ and work out an iterative algorithm based on the first order approximation to iterate over the parameters p and q for obtaining $r = s = 0$. In brief, work out an iterative procedure to isolate a quadratic factor from a polynomial.
[This is the **Bairstow's method**, often found to be an effective way to solve polynomial equations.]

(c) Implement the procedure to find all roots of the polynomial

$$P(x) = x^6 - 19x^5 + 125x^4 - 329x^3 + 66x^2 + 948x - 216$$

up to two places of decimal, starting with $p = 0$ and $q = 0$, i.e. x^2 as the initial divisor expression.

6. Find a solution of the equation $e^{-x} = x$ up to two places of decimal. Is the solution unique?

7. Consider the problem of finding a root of the function $h(x) = 30\sin x + x^3 - 5$ with two points 0 and 1.44 identified with opposite signs of the function. (Up to two places of decimal is enough.)

   (a) Use the method of false position with these two points as starting values to find the root.

   (b) Use the Newton-Raphson method, starting with 0, to find the root and note the comparative performance.

   (c) Attempt the Newton-Raphson method, starting with 1.43, 1.44 and 1.45, and explain your experience of the first four iterations in each case.

8. Starting from $\mathbf{x} = [1 \;\; 1 \;\; 1]^T$, solve the system of equations

$$16x_1^4 + 16x_2^4 + x_3^4 = 16, \quad x_1^2 + x_2^2 + x_3^2 = 3, \quad x_1^3 - x_2 = 0$$

by Newton's method.

9. Find all the maxima and minima of the function $\phi(x) = 25(x - \frac{1}{2})^4 - 2(x - \frac{1}{2})^2$, identify its other salient features and sketch its graph.

10. Consider the function $f : R^2 \to R$ defined by

$$f(\mathbf{x}) = 2x_1^2 - x_1^4 + x_1^6/6 + x_1x_2 + x_2^2/2 \ .$$

Find out all the stationary points and classify them as local minimum, local maximum and saddle points.