Practise Quiz – 2 (Modules 6 to 9)

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Prob. 1: Multiple Choice Questions

1. "Quadrature" refers to

d. Representing integral as a weighted sum of function values at certain points

2. Which of the following methods can be used to solve Stiff ODEs:

d. None of the above are suitable for stiff ODEs

3. The optimum step-size h for a $\vartheta(h^2)$ numerical differentiation is:

a. $h \propto \sqrt{\varepsilon}$

4. For an ODE Boundary Value Problem (BVP)?

c. Neumann and mixed boundary conditions are handled using "ghost point" approach

5. **c.** Cubic Spline: is not a closed formula for integration?

Prob. 2: Objective Type Questions

1. Simpson's 3/8th rule to compute: $\int_0^{15} f(x)dx$ with $h = 1$

$$\frac{3}{8}\begin{bmatrix} (f_0 + 3f_1 + 3f_2 + f_3) + (f_3 + 3f_4 + 3f_5 + f_6) + (f_6 + 3f_7 + 3f_8 + f_9) + \\ (f_9 + 3f_{10} + 3f_{11} + f_{12}) + (f_{12} + 3f_{13} + 3f_{14} + f_{15}) \end{bmatrix}$$

This can alternatively be written as:

$$\frac{3}{8}[f_0 + f_{15} + 2(f_3 + f_6 + f_9 + f_{12}) + 3(f_1 + f_2 + f_4 + f_5 + f_7 + f_8 + f_{10} + f_{11} + f_{13} + f_{14})]$$

2. Shooting Method

3. Round-off Error increases as the step-size is decreased

4. The Laplace equation is an elliptic PDE.

5. Neumann boundary condition: (i) Flux condition, (ii) Insulating boundary, (iii) Symmetry condition, and (iv) $aT_x + b = 0$ are all going to be acceptable solutions to this question

Prob. 3: Short Answer Questions

1. First, I will transform the variable into another appropriate variable so that the limits of integration are finite. An example is $y = \frac{1}{1+x}$ or $y = \tan^{-1}(x)$, etc.

With the first choice, the integral becomes

$$\int_0^1 f\left(\frac{1-y}{y}\right)\frac{1}{y^2}dy$$

We can now use trapezoidal rule with an appropriate choice of h.

2. Yes, Heun's method is numerically equivalent to RK-2 Heun's method if the function $f(t, y)$ is function only of t and not of y. However, this is not all. We need to factor in that the solution of ODE is of the form $y(t) = p(t) + c$, whereas integral is nothing but $I = y(t) - y(t_0)$. Thus, the two methods are equivalent provided we account for this factor ("constant bias") also.

3. Each step of RK-4 method requires four computations of the ODE function. However, RK-4 is a fourth order accurate method (local truncation error is $\mathcal{O}(h^5)$ and global truncation error is $\mathcal{O}(h^4)$). This means that halving the step size will result in a factor of 16 (i.e., $1/2^4$) improvement in accuracy with only four additional computations of the ODE function. Hence, RK-4 is preferred over Euler's method.

Prob. 4: Numerical Differentiation to find a Jacobian

Numerical Jacobian is given by:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{f_1(1+h,1) - f_1(1-h,1)}{2h} & \frac{f_1(1,1+k) - f_1(1,1-k)}{2k} \\ \frac{f_2(1+h,1) - f_2(1-h,1)}{2h} & \frac{f_2(1,1+k) - f_2(1,1-k)}{2k} \end{bmatrix}$$

Since central difference formula is $\mathcal{O}(h^2)$ accurate, the optimum step-size is $h \propto \varepsilon^{\frac{1}{3}}$. Since we are taking derivative about $(1,1)$, we choose $h = k = 10^{-4}$.

$$f(x,y) = \begin{bmatrix} xe^{-y} \\ x^2 + 3xy + y^3 \end{bmatrix}$$

In order to get the desired precision as per the problem statement, we will need to retain all the digits displayed on the calculator.

(x,y)	$f_1 = xe^{-y}$	$\partial f_1/\partial(\cdot)$	$f_2 = x^2 + 3xy + y^3$	$\partial f_1/\partial(\cdot)$
$(1+10^{-4},1)$	0.36791623	0.3679	5.000500010	5.0
$(1-10^{-4},1)$	0.36784265		4.999500010	
$(1,1+10^{-4})$	0.36784266	–0.3679	5.000600030	6.0
$(1,1-10^{-4})$	0.36791623		4.999400030	

Thefore, the Jacobian is given by (please note the elements):

$$J = \begin{bmatrix} 0.3679 & -0.3679 \\ 5.0 & 6.0 \end{bmatrix}$$

PROBLEM 5

$$\frac{dy}{dt} = te^{-y} \qquad y(0) = 0$$

Although most books solve such problems in step-wise manner with each paragraph devoted to one iteration, I prefer TABULAR format we used in the class with EXCEL sheets. It is more organized and easier to follow.

RK-2 MIDPOINT METHOD

$$k_1 = f(t_i, y_i) = t_i * \exp(-y_i)$$

$$k_2 = f\left(t_i + \frac{h}{2}, y_i + \frac{hk_1}{2}\right) = (t_i + 0.25) * \exp\left(-\left(y_i + \frac{k_1}{4}\right)\right)$$

$$y_{i+1} = y_i + hk_2 = y_i + \frac{k_2}{2}$$

t_i	y_i	k_1	$t_i + \frac{h}{2}$	$y_i + \frac{hk_1}{2}$	k_2	y_{i+1}
0	0	0	0.25	0	0.25	0.125
0.5	0.125	0.4412	0.75	0.2353	0.5927	0.4214
1	0.4214					

PROBLEM 6

$$y_{i+1} = y_i + hk_1 = y_i + k_1/4 \qquad (h = 0.25)$$

t_i	y_i	$f(t_i, y_i)$	y_{i+1}
0	0	0	0
0.25	0	0.25	0.0625
0.5	0.0625	0.4697	0.1799
0.75	0.1799	0.6265	0.3365
1	0.3365		

PROBLEM 7

TAYLOR'S SERIES

$x_i - x_{i-1} = h$; $\quad x_i - x_{i-2} = 2h$; $\quad x_i - x_{i-3} = 3h$

$$f(x_{i-1}) = f(x_i) - h f'(x_i) + \frac{h^2}{2} f''(x_i) - \frac{h^3}{6} f'''(x_i) + \frac{h^4}{24} f''''(x_i) + \ldots$$

$$f(x_{i-2}) = f(x_i) - 2h f'(x_i) + 2h^2 f''(x_i) - \frac{4h^3}{3} f'''(x_i) + \frac{2h^4}{3} f''''(x_i) + \ldots$$

$$f(x_{i-3}) = f(x_i) - 3h f'(x_i) + \frac{9h^2}{2} f''(x_i) - \frac{27h^3}{6} f'''(x_i) + \frac{27h^4}{8} f''''(x_i)$$

AT THIS STAGE, WE WILL TAKE A SHORT-CUT. We will multiply first equation by (-5), second by (4) and third by (-1) and add them up.

$$-5 f(x_{i-1}) + 4 f(x_{i-2}) - f(x_{i-3}) =$$

$$(-5+4-1) f(x_i) - (-5+8-3) h f'(x_i) + h^2 f''(x_i)\left[-\frac{5}{2} + 8 - \frac{9}{2}\right] - \frac{h^3}{6}\left[-5 + 32 - 27\right] f'''(x_i) + h^4 f''''(x_i)\left[-\frac{5}{24} + \frac{64}{24} - \frac{81}{24}\right] + \ldots$$

$$= -2 f(x_i) - 0 + h^2 f''(x_i) - 0 - \frac{22}{24} h^4 f''''(x_i)$$

REARRANGING :

$$2 f(x_i) - 5 f(x_{i-1}) + 4 f(x_{i-2}) - f(x_{i-3}) + \underbrace{\frac{11}{12} h^4 f''''(x_i)}_{\text{LEADING ERROR TERM}} = h^2 f''(x_i)$$

$$\therefore\ f''(x_i) = \frac{2 f(x_i) - 5 f(x_{i-1}) + 4 f(x_{i-2}) - f(x_{i-3})}{h^2} + \underbrace{\frac{11}{12} f''''(x_i)\, h^2}_{\mathcal{O}(h^2)}$$

PROBLEM 8

With $h = 1$

COMPUTE $\Rightarrow \displaystyle\int_0^6 \frac{dt}{1+t}$

TRAPEZOID $\frac{1}{2}[f_i + f_{i+1}]$	t	$f(t)$	SIMPSON'S $\frac{1}{3}$RD $\frac{1}{3}[f_i + 4f_{i+1} + f_{i+2}]$
	0	1	
$\frac{1}{2}\left(1+\frac{1}{2}\right) = 0.75$	1	$\frac{1}{2}$	1.1111
0.4167	2	$\frac{1}{3}$	
0.2917	3	$\frac{1}{4}$	0.5111
0.225	4	$\frac{1}{5}$	
0.1833	5	$\frac{1}{6}$	0.3365
0.1548	6	$\frac{1}{7}$	

SUM $\Downarrow$ $\boxed{I = 2.0215}$

$\Downarrow$ SUM $\boxed{I = 1.9587}$

$\log(7) = 1.9459$

ERROR $= 0.0756$

ERROR $= 0.0128$

Thus, error using SIMPSON'S $\frac{1}{3}$RD RULE is LOWER than TRAPEZOIDAL RULE.

PROBLEM 9

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - e^y = 0 \qquad y(0)=0, \quad y'(1)=0$$

let $z = \frac{dy}{dt} \Rightarrow \frac{dz}{dt} + 3z - e^y = 0$

$$\underbrace{\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} z \\ e^y - 3z \end{bmatrix}}_{F(Y)} \qquad \begin{bmatrix} y(0) \\ z(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

GUESS-1 : $y'(0) = 0$ $\quad (h = 0.25)$

t	Y	F	$Y^{(i+1)} = Y + hF$
0	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0.25 \end{bmatrix}$
0.25	$\begin{bmatrix} 0 \\ 0.25 \end{bmatrix}$	$\begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix}$	$\begin{bmatrix} 0.0625 \\ 0.3125 \end{bmatrix}$
0.5	$\begin{bmatrix} 0.0625 \\ 0.3125 \end{bmatrix}$	$\begin{bmatrix} 0.3125 \\ 0.1270 \end{bmatrix}$	$\begin{bmatrix} 0.1406 \\ 0.3442 \end{bmatrix}$
0.75	$\begin{bmatrix} 0.1406 \\ 0.3442 \end{bmatrix}$	$\begin{bmatrix} 0.3442 \\ 0.1182 \end{bmatrix}$	$\begin{bmatrix} 0.2687 \\ 0.3738 \end{bmatrix}$

$\Rightarrow y'(1) = 0.3738$

GUESS-2 : $y'(0) = -10$

0	$\begin{bmatrix} 0 \\ -10 \end{bmatrix}$	$\begin{bmatrix} -10 \\ 31 \end{bmatrix}$	$\begin{bmatrix} -2.5 \\ -2.25 \end{bmatrix}$
	$\begin{bmatrix} -2.5 \\ -2.25 \end{bmatrix}$	$\begin{bmatrix} -2.25 \\ 6.8321 \end{bmatrix}$	$\begin{bmatrix} 3.0625 \\ -0.5420 \end{bmatrix}$
	$\begin{bmatrix} 3.0625 \\ -0.5420 \end{bmatrix}$	$\begin{bmatrix} -0.5420 \\ 1.6727 \end{bmatrix}$	$\begin{bmatrix} -3.1980 \\ -0.1238 \end{bmatrix}$
	$\begin{bmatrix} -3.1980 \\ -0.1238 \end{bmatrix}$	$\begin{bmatrix} -0.1238 \\ 0.4123 \end{bmatrix}$	$\begin{bmatrix} -3.2289 \\ -0.0207 \end{bmatrix}$

$\Rightarrow y'(1) = -0.0207$

LETS USE REGULA FALSI

$x^{(u)}$ = FIRST GUESS = 0 $\qquad f^u = 0.3738$

$x^{(l)}$ = SECOND GUESS = -10 $\qquad f^l = -0.0207$

$$x^{(i+1)} = -10 + 0.0207\ (0.3738 + 0.0207)/10 = -9.9992$$

t	Y	F	$Y^{(i+1)}$
0	$\begin{bmatrix} 0 \\ -9.9992 \end{bmatrix}$	$\begin{bmatrix} -9.9992 \\ 30.9976 \end{bmatrix}$	$\begin{bmatrix} -2.4998 \\ -2.2498 \end{bmatrix}$
0.25	$\begin{bmatrix} -2.4998 \\ -2.2498 \end{bmatrix}$	$\begin{bmatrix} -2.2498 \\ 6.8315 \end{bmatrix}$	$\begin{bmatrix} -3.0622 \\ -0.5419 \end{bmatrix}$
0.50	$\begin{bmatrix} -3.0622 \\ -0.5419 \end{bmatrix}$	$\begin{bmatrix} -0.5419 \\ 1.6926 \end{bmatrix}$	$\begin{bmatrix} -3.1977 \\ -0.1238 \end{bmatrix}$
0.75	$\begin{bmatrix} -3.1977 \\ -0.1238 \end{bmatrix}$	$\begin{bmatrix} -0.1238 \\ 0.4122 \end{bmatrix}$	$\begin{bmatrix} -3.2287 \\ -0.0206 \end{bmatrix}$

$$x^{(i+1)} = -9.9992 + 0.0206 \frac{[0.3738 + 0.0206]}{9.9992}$$

$$= -9.9984$$

We can again use EULER'S METHOD once again with $y'(0) = -9.9984$ as starting guess.

WE WILL SOLVE GUESS-1 OF PROBLEM-9 USING RK-2 HEUN'S METHOD

$$\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} z \\ e^{y} - 3z \end{bmatrix} \qquad \begin{aligned} y(0) &= 0 \\ z(0) &= 0 \end{aligned}$$

HEUN'S : $k_1 = F(Y)$ is a 2×1 VECTOR

TO CALCULATE k_2, we need: $(Y + hk_1)$

$$k_2 = F(Y + hk_1)$$

$$Y^{(i+1)} = Y^{i} + \frac{h}{2}(k_1 + k_2)$$

$$\boxed{h = 1/4 \qquad \frac{h}{2} = \frac{1}{8}}$$

t	Y	k_1	$Y + \frac{k_1}{4}$	$F(Y + hk_1) = k_2$	$Y^{(i+1)} = Y + \frac{k_1 + k_2}{8}$
0	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0.25 \end{bmatrix}$	$\begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix}$	$\begin{bmatrix} 0.0312 \\ 0.1562 \end{bmatrix}$
0.25	$\begin{bmatrix} 0.0312 \\ 0.1562 \end{bmatrix}$	$\begin{bmatrix} .1562 \\ .5630 \end{bmatrix}$	$\begin{bmatrix} .0703 \\ .2970 \end{bmatrix}$	$\begin{bmatrix} .2970 \\ .1818 \end{bmatrix}$	$\begin{bmatrix} 0.0879 \\ 0.2494 \end{bmatrix}$
0.5	$\begin{bmatrix} .0879 \\ .2494 \end{bmatrix}$	$\begin{bmatrix} .2494 \\ .3438 \end{bmatrix}$	$\begin{bmatrix} .1502 \\ .3353 \end{bmatrix}$	$\begin{bmatrix} .3353 \\ .1562 \end{bmatrix}$	$\begin{bmatrix} 0.1610 \\ 0.3119 \end{bmatrix}$
0.75	$\begin{bmatrix} 0.1610 \\ 0.3119 \end{bmatrix}$	$\begin{bmatrix} .3119 \\ .2391 \end{bmatrix}$	$\begin{bmatrix} 0.2390 \\ 0.3716 \end{bmatrix}$	$\begin{bmatrix} 0.3716 \\ 0.1550 \end{bmatrix}$	$\begin{bmatrix} 0.2464 \\ 0.3611 \end{bmatrix} \Rightarrow Y'(1) = 0.3611$

Lets RECAP STEP-1

$$Y = \underset{\text{COLUMN-2}}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} \Rightarrow F(Y) = k_1 = \underset{\text{COLUMN-3}}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \Rightarrow Y + \frac{k_1}{4} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/4 \end{bmatrix} = \underset{\text{COLUMN-4}}{\begin{bmatrix} 0 \\ 0.25 \end{bmatrix}}$$

Using $(Y + hk_1)$ CALCULATE k_2 $\Big\}$ $k_2 = F\left(\begin{bmatrix} 0 \\ 0.25 \end{bmatrix}\right) = \begin{bmatrix} 0.25 \\ 1 - 0.75 \end{bmatrix} = \underset{\text{COLUMN-5}}{\begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix}}$

$$\therefore Y^{(i+1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{8}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix}\right\} = \begin{bmatrix} 0.0312 \\ 0.1562 \end{bmatrix} \leftarrow \text{LAST COLUMN}$$

PROBLEM 10

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - e^y = 0$$

$$y(0) = 0 \qquad y'(1) = 0$$

$$\text{SOLUTION VARIABLES} \left\} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} \right. \begin{matrix} \Rightarrow y(0) \\ \Rightarrow y(0.2) \\ \Rightarrow y(0.4) \\ \Rightarrow y(0.6) \\ \Rightarrow y(0.8) \\ \Rightarrow y(1) \end{matrix}$$

FOR y_1 to y_4, we have

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{(0.2)^2} + 3\,\frac{y_{i+1} - y_{i-1}}{2(0.2)} - e^{y_i} = 0$$

$$25\,y_{i+1} - 50\,y_i + 25\,y_{i-1} + 7.5\,y_{i+1} - 7.5\,y_{i-1} - e^{+y_i} = 0$$

$$32.5\,y_{i+1} - 50\,y_i + 17.5\,y_{i-1} = e^{y_i}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 17.5 & -50 & 32.5 & 0 & 0 & 0 \\ 0 & 17.5 & -50 & 32.5 & 0 & 0 \\ 0 & 0 & 17.5 & -50 & 32.5 & 0 \\ 0 & 0 & 0 & 17.5 & -50 & 32.5 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{y_1} \\ e^{y_2} \\ e^{y_3} \\ e^{y_4} \\ 0 \end{bmatrix}$$

PROBLEM 11

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial x^2} + g(T)$$

We split the domain into 5 intervals

$x =$	0	0.2	0.4	0.6	0.8	1.0
	T_0	T_1	T_2	T_3	T_4	T_5

Since we have DIRICHLET BOUNDARY CONDITIONS, we need to write equations only for T_1 to T_4

$$\left.\frac{\partial T}{\partial x}\right|_i = \frac{T_{i+1} - T_{i-1}}{2h} = 2.5\left(T_{i+1} - T_{i-1}\right)$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_i = \frac{T_{i+1} - 2T_i + T_{i-1}}{h^2} = 25\left[T_{i+1} - 2T_i + T_{i-1}\right]$$

THEREFORE:

$$\frac{dT_i}{dt} + 2.5v\left[T_{i+1} - T_{i-1}\right] = 25\alpha\left[T_{i+1} - 2T_i + T_{i-1}\right] + g(T_i)$$

$$\frac{dT_i}{dt} = T_{i+1}(25\alpha - 2.5v) + T_{i-1}(25\alpha + 2.5v) - (50\alpha T_i) + g(T_i)$$

$$\frac{d}{dt}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 100(25\alpha + 2.5v) + (25\alpha - 2.5v)T_2 - 50\alpha T_1 + g(T_1) \\ T_1(25\alpha + 2.5v) + (25\alpha - 2.5v)T_3 - 50\alpha T_2 + g(T_2) \\ T_2(25\alpha + 2.5v) + (25\alpha - 2.5v)T_4 - 50\alpha T_3 + g(T_3) \\ T_3(25\alpha + 2.5v) + 30(25\alpha - 2.5v) - 50\alpha T_4 + g(T_4) \end{bmatrix}$$