Practise Quiz – 2 (Modules 6 to 9)

Dr. Niket Kaisare, IIT-Madras

Instructions

- This practice quiz is intended to give you an overview of you may expect in the final exam
- This quiz is *lengthier* than the exam, since we intend to cover a lot of material from various modules here
- You are not expected to *memorize* the formulae. Instead, focus on *understanding* the techniques, derivation strategy, and method to solve problems.
- I personally believe in problem solving as best way to learn. In order to promote that, I will repeat approx. 15% worth material from the practise tests and assignments in the final exam.
- You will require a calculator for the exam. You should be comfortable using the calculator. Please practise well.

Prob. 1: Multiple Choice Questions

In the following problems, indicate only one correct answer. Each correct answer earns 2 points, and wrong answer loses -1 point. There is no negative marking for not attempting the question.

1. "Quadrature" refers to

a. A method to obtain roots of a nonlinear equation
b. Quadratic approximation of a function
c. Inner (dot) product with quadratic weighting functions
d. Representing integral as a weighted sum of function values at certain points

2. Which of the following methods can be used to solve Stiff ODEs:

a. Midpoint Method
b. Adaptive step-size Runge Kutta
c. Explicit Euler's method
d. None of the above are suitable for stiff ODEs

3. The optimum step-size h that will give highest accuracy for the numerical differentiation:

$$\frac{d^2y}{dt^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \vartheta(h^2)$$

a. $h \propto \sqrt{\varepsilon}$
b. $h \propto \varepsilon^2$
c. $h \propto \varepsilon^{2/3}$
d. None of the above

4. Which of the following is true about a general ODE Boundary Value Problem (BVP)?

a. Finite difference approximation leads to linear algebraic equations
b. One can always find an analytical solution to any ODE-BVP

c. Neumann and mixed boundary conditions are handled using "ghost point" approach
d. It can be solved using "method of lines"

5. Which of the following is not a closed formula for integration?
a. Trapezoidal Rule b. Simpson's Rule c. Cubic Spline d. None of the above

Prob. 2: Objective Type Questions

1. Write down the Simpson's 3/8th rule to compute

$$\int_0^{15} f(x)dx$$

if the entire domain is split into 15 equal intervals. Don't leave the solution in terms of h. Single implementation of Simpson's 3/8th rule: $\frac{3h}{8}\big(f(a) + 3f(a+h) + 3f(a+2h) + f(a+3h)\big)$.

2. Technique to solve an ODE-BVP, where one assumes an initial condition and solves the ODE repeatedly to match the desired boundary value is known as ____________________ method.

3. In error analysis for differentiation as well as other methods, which type of error increased as the step-size, h, was decreased?

4. The Laplace equation, $\nabla^2 T = 0$, is an example of (parabolic / hyperbolic / elliptic) PDE.

5. Give an example of Neumann boundary condition for #4 above.

Prob. 3: Short Answer Questions

1. Explain how you will use Trapezoidal method to numerically evaluate the integral

$$\int_0^{\infty} f(x)\, dx$$

2. Our friend, Saanvi, says that solving the ODE problem: $y' = e^{-t}$, $y(0) = 0$ using single step of Heun's method with $h = 1$ is numerically equivalent to using single application of the Trapezoidal rule. Do you agree or disagree? Please explain.

3. Each step of RK-4 method requires four computations of the ODE function, whereas Euler's method requires just one. Explain briefly why RK-4 is still preferred over Euler's method.

Prob. 4: Numerical Differentiation to find a Jacobian

Find numerical Jacobian for the following function of two variables at $(1, 1)$ using central difference formula.

$$f(x, y) = \begin{bmatrix} xe^{-y} \\ x^2 + 3xy + y^3 \end{bmatrix}$$

Choose appropriate step-size to minimize errors, assuming the calculator precision is 10^{-12}.

Prob. 5: ODE-Initial Value Problem using RK-2 Midpoint Method

Consider the following ODE-IVP:

$$\frac{dy}{dt} = te^{-y}, \qquad y(0) = 0$$

Solve the ODE-IVP using RK-2 Midpoint Method with $h = 0.5$ to obtain $y(1)$.

Prob. 6: ODE-IVP Using Euler's Method and Compare with RK-2

Re-solve the above problem using explicit Euler's method with $h = 0.25$ to obtain $y(1)$.

Obtain the true solution of the original equation algebraically. Compare the errors obtained using RK-2 and Euler's methods.

[Note to students: Use this result to understand what your answer should be for Prob. 3–3.]

Prob. 7: Error Analysis for Numerical Differentiation

- Show that the following four-point backward difference formula can be used to compute the second derivative:

$$f''(x_i) = \frac{2f(x_i) - 5f(x_{i-1}) + 4f(x_{i-2}) - f(x_{i-3})}{h^2}$$

- Show that the truncation error of the above formula is $\mathcal{O}(h^2)$

Hint: Use Taylor's series expansion for each of the terms on the right hand side of the final equation. Retain up to fourth derivatives in each of these Taylor's Series Expansion.

Prob. 8: Comparing integration formulae

For $h = 1$, compare the results from Trapezoidal rule and Simpson's 1/3rd rule for computing

$$\int_0^6 \frac{dt}{1+t}$$

by comparing the numerical value of the integral with its true value, $I = \ln(7)$.

Prob. 9: Shooting Method

Warning: This is a lengthy problem

We will use the shooting method to solve the following ODE-BVP:

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - e^y = 0$$

$$y(0) = 0 \quad y'(1) = 0$$

- Substituting $z = y'$, write down the above equation as a set of two first-order ODEs with appropriate boundary conditions
- Replace the second boundary condition with an initial condition $y'(0) = 0$, resulting in an ODE-IVP. Solve this ODE-IVP using explicit Euler's method with step size $h = 0.25$.
- Solving the ODE-IVP with Euler's method, we will also obtain $y'(1)$. Note down this value of $y'(1)$, which corresponds to the initial condition of $y'(0) = 0$.
- Repeat the above two steps for another initial condition, $y'(0) = -10$. Again note the value of $y'(1)$. Verify that the sign of $y'(1)$ has changed when we changed the initial condition.
- Use a bracketing method to get the next guess of $y'(0)$. With this new guess of $y'(0)$, solve the ODE-IVP. Note down the new value of $y'(1)$.
- Use the bracketing method again to determine new guess of $y'(0)$. [You do not need to solve the ODE-IVP again for this exercise.]

Prob. 10: Finite Difference Scheme

The above problem is to be solved using a second-order accurate finite difference scheme with a step-size of $h = 0.2$. Write down the resultant equations.

Prob. 11: Method of Lines

We wish to solve the following parabolic PDE to obtain $T(t, x)$ using method of lines:

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial t^2} + g(T)$$

The boundary conditions are $T(t, 0) = 100,\ T(t, 1) = 30$. Using a step-size of $h = 0.2$, use method of lines to convert the above PDE into a set of ODEs. The initial condition is that the value of T is uniformly 100 at time $t = 0$.

Formulae and Hints

Numerical Differentiation

Forward difference formula $$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h}$$

Backward difference formula $$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h}$$

Central difference formula $$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$$

Central difference formula (second derivative) $$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$$

Forward difference formula (second derivative) $$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2}$$

Numerical Integration

Trapezoidal Rule $$\frac{h}{2}[\, f(a) + f(a+h)\,]$$

Simpson's 1/3rd Rule $$\frac{h}{3}[\, f(a) + 4f(a+h) + f(a+2h)\,]$$

Simpson's 3/8th Rule $$\frac{h}{3}[\, f(a) + 3f(a+h) + 3f(a+2h) + f(a+3h)\,]$$

Runge-Kutta (Classic) formulae

In all the formulae below, $k_1 = f(y_i, t_i)$

RK-2: Heun's $$y_{i+1} = y_i + \frac{h}{2}[k_1 + k_2] \qquad k_2 = f(y_i + hk_1,\ t_i + h)$$

RK-2: Midpoint $$y_{i+1} = y_i + h[k_2] \qquad k_2 = f\left(y_i + \frac{h}{2}k_1,\ t_i + \frac{h}{2}\right)$$

RK-4 Classic $$y_{i+1} = y_i + \frac{h}{6}[k_1 + 2k_2 + 2k_3 + k_4] \qquad k_2 = f\left(y_i + \frac{h}{2}k_1,\ t_i + \frac{h}{2}\right)$$

$$k_3 = f\left(y_i + \frac{h}{2}k_2,\ t_i + \frac{h}{2}\right)$$

$$k_4 = f(y_i + hk_3,\ t_i + h)$$