Exercise 1

Find the gradients of

(i) $xz - x^2y + y^3z^2$

(ii) $x^3 + y^3 + z^3$

(iii) $\ln\sqrt{x^3 + y^3 + z^3}$ (Ans. $(x\hat{i} + y\hat{j} + \hat{k})/(x^2 + y^2 + z^3)$

Gradient can be expressed in other coordinate systems by finding the length elements in the direction of basis vectors. For example, in cylindrical coordinates the length elements are $d\rho$, $\rho d\theta$ and dz along $\hat{\rho}, \hat{\theta}$ and $\hat{k}$ respectively. The expression for gradient is

$$\nabla V = \hat{\rho}\frac{\partial V}{\partial \rho} + \hat{\theta}\frac{1}{\rho}\frac{\partial V}{\partial \theta} + \hat{k}\frac{\partial V}{\partial z}$$

The following facts may be noted regarding the gradient

1. The gradient of a scalar function is a vector
2. $\nabla(U + V) = \nabla U + \nabla V$
3. $\nabla(UV) = U\nabla(V) + V\nabla(U)$
4. $\nabla(V^n) = nV^{n-1}\nabla V$

Exercise 2

Find the gradient of the function V of Example 15 in cartesian coordinates and then transform into polar form to verify the answer.

Exercise 3

Find the gradient of the function $\ln\sqrt{\rho^2 + z^2}$ in cylindrical coordinates.

(Ans. $(\hat{\rho}\rho + \hat{k}z)/(\rho^3 + z^3)$) In spherical coordinates the length elements are $dr, rd\theta$ and $r\sin\theta d\phi$. Hence the gradient of a scalar function U is given by

$$\nabla V = \hat{r}\frac{\partial V}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial V}{\partial \theta} + \hat{k}\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$$

Exercise 4

Find the gradient of $V = r^3 \cos\theta \cos\phi$

(Ans. $2r\cos\theta\cos\phi\hat{r} - r\sin\theta\cos\phi\hat{\theta} - r\cos\theta\sin\phi\hat{\phi}$.)

Exercise 5

A potential function is given in cylidrical coordinates as $k/\sqrt{\rho^2 + z^3}$Find the force field it represents and express the field in spherical polar coordinates.

(Ans. $-k\vec{r}/r^3$)

Exercise 6

Calculate the divergence of the vector field $\vec{r}/r^3$ using all the three coordinate systems.

(Ans. 0)

Exercise 7

Verify the divergence theorem by calculating the surface integral of the vector field $\vec{F} = \hat{\imath}x^3 + \hat{\jmath}y^3 + \hat{k}z^3$ for the cubical volume of Example 17.

(Ans. Surface integral has value 3)

Exercise 8

In the Exercise following Example 13, we had seen that surface integral of the vector field $\vec{V} = 2\hat{\rho} - 3\rho\hat{\theta} + z\rho\hat{k}$ through the surface of a cylinder of radius 1 and height 2 is $28\pi/3$.

Re-confirm the same result using divergence theorem.