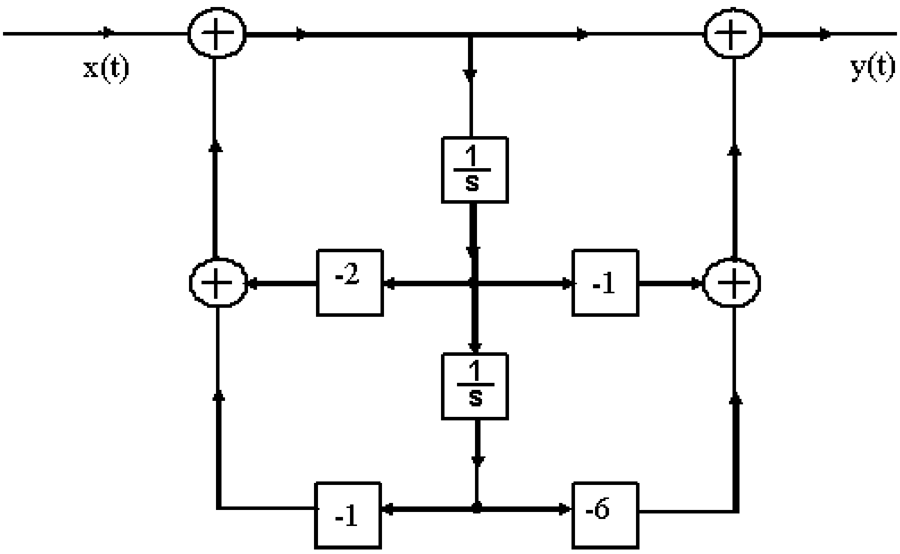
Module 4 : Laplace and Z Transform
Problem Set 4

Problem 1

The input x(t) and output y(t) of a causal LTI system are related to the block diagram representation shown in the figure.

(a) Determine a differential equation relating y(t) and x(t).

(b) Is the system stable?



Solution 1

(a) Let the signal at the bottom node of the block diagram be denoted by *e* (*t*). Then we have the following relations

$$x(t) - 2\frac{de(t)}{dt} - e(t) = \frac{d^2e(t)}{dt^2}$$

$$\frac{d^2e(t)}{dt^2} - \frac{de(t)}{dt} - 6e(t) = y(t)$$

Writing the above equations in Laplace domain, we have

$$X(s) - 2sE(s) - E(s) = s^2E(s)$$

$$s^2E(s) - sE(s) - 6E(s) = Y(s)$$

Eliminating the auxiliary signal *e* (*t*), we get

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2 - s - 6}{s^2 + 2s + 1}$$

$$\Rightarrow s^2Y(s) + 2sY(s) + Y(s) = s^2X(s) - sX(s) - 6X(s)$$

Hence we get the following differential equation,

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = \frac{d^2x(t)}{dt^2} - \frac{dx(t)}{dt} - 6x(t)$$

(b) From the system function *H* (*s*) found in the previous part we see that the poles of the system are at *s* = -1. Since the system is given to be causal, and the right most pole of the system is left of the imaginary axis, the system is stable.

Problem 2

Draw a direct – form representation for the causal LTI systems with the following system functions :

(a) $H_1(s) = \dfrac{s+1}{s^2+5s+6}$

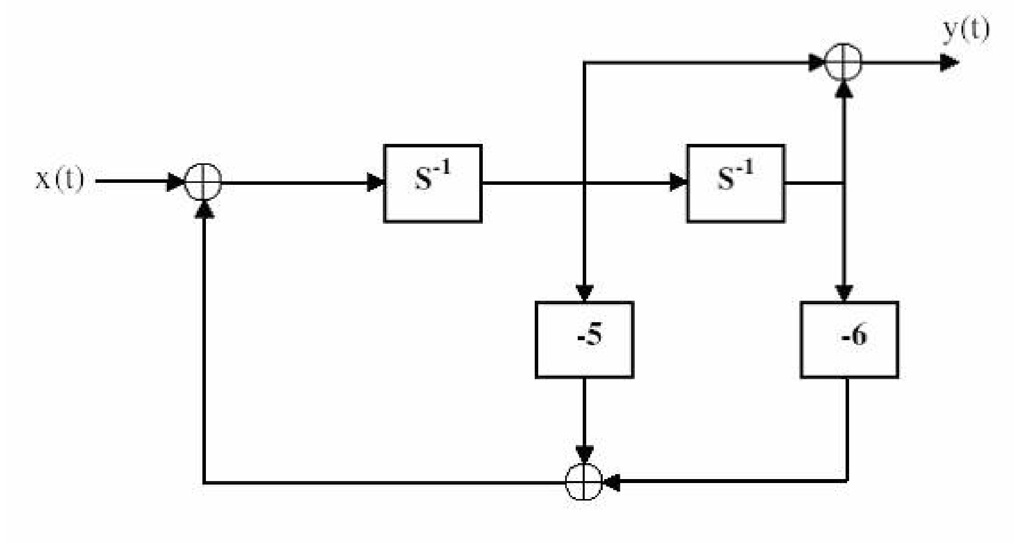
(b) $H_2(s) = \dfrac{s^2-5s+6}{s^2+7s+10}$

(c) $H_3(s) = \dfrac{s}{(s+2)^2}$

Solution 2

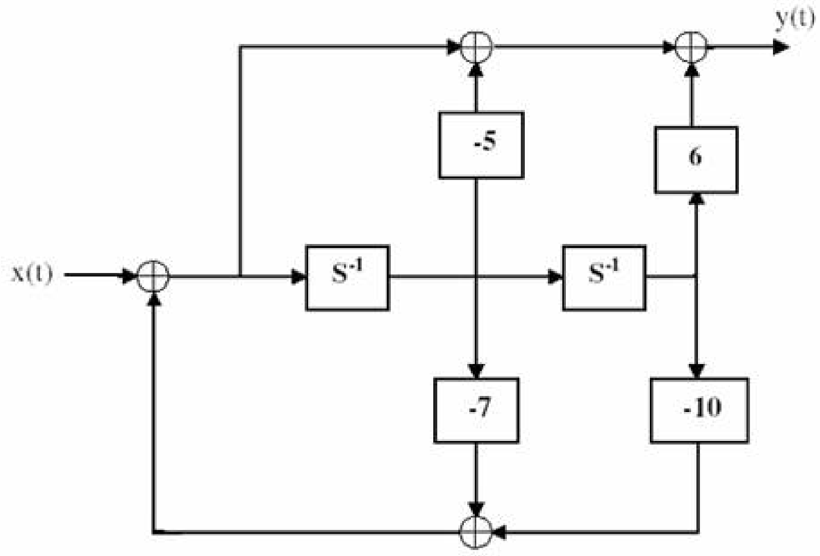
(a) $H_1(s) = \dfrac{s+1}{s^2+5s+6}$

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + x(t)$$



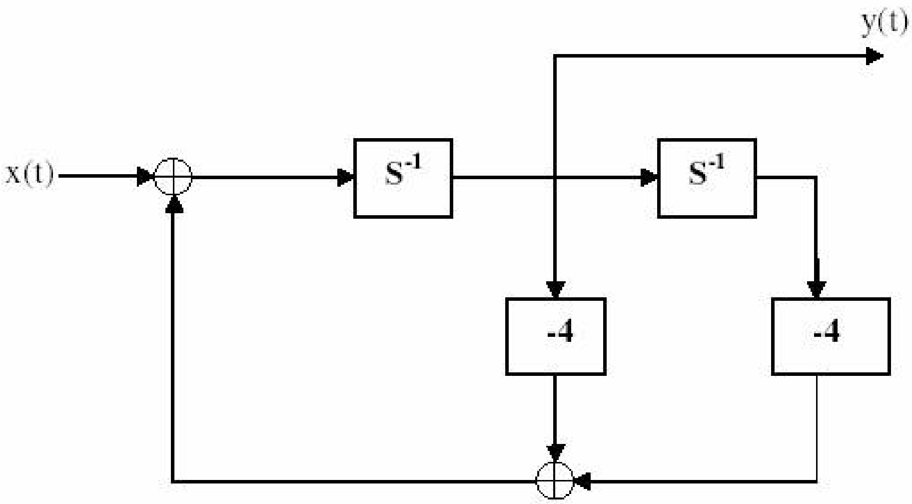
(b) $H_2(s) = \dfrac{s^2-5s+6}{s^2+7s+10}$

$$\frac{d^2y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 10y(t) = \frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 6x(t)$$



(c) $H_3(s) = \dfrac{s}{(s+2)^2}$

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy}{dt} + 4y(t) = \frac{dx(t)}{dt}$$



Problem 3

The following is known about a discrete – time LTI system with input x[n] and output y[n] :

1) If $x[n]=(-2)^n$ for all n , then y[n] = 0 for all n.

2) If $x[n]=\left(\frac{1}{2}\right)^n U(n)$ for all n , then y[n] for all n is of the form $y[n]=\delta[n]+a\left(\frac{1}{4}\right)^n u[n]$,

where a is a constant.

(a) Determine the value of the constant a .

(b) Determine the response y[n] if the input x[n] is x[n] =1 for all n.

Solution 3

(a) It is given that for the input $x[n]=\left(\frac{1}{2}\right)^n u[n]$, the output of the LTI system is of the form $y[n]=\delta[n]+a\left(\frac{1}{4}\right)^n u[n]$. From this fact we can calculate the transfer function *H* (*z*) to be

$$H(z)=\frac{Y(z)}{X(z)}$$

$$\frac{1+\left(\frac{a}{1-\frac{1}{4}z^{-1}}\right)}{\frac{1}{1-\left(\frac{1}{2}z^{-1}\right)}}$$

$$\frac{(1+a)-\frac{1}{4}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$$

Let the above equation be equation (1).

Since the ROC of Y(z) is $|z|>\frac{1}{4}$ and that of *X* (*z*) is $|z|>\frac{1}{2}$ and since the ROC of *H* (*z*) must be such that its intersection with the ROC of *X* (*z*) is contained in the ROC of *Y* (*z*) we must have that the ROC of *H* (*z*) is $|z|>\frac{1}{2}$.

It is also given that the output of this system to the input $x[n]=(-2)^n$ is *y* [*n*] = 0 for all *n* . Since the function z_0^n is an eigen function for a discrete time LTI system, the output to this input is $H(z_0)z_0^n$. From this we can infer that, $H(-2)=0$.

Using this in (1) we can calculate the value of *a* from to be $-\frac{9}{8}$.

(b) The response *y* [*n*] to the input *x* [*n*] = 1, for all *n* is given by *y* [*n*] = *H* (1) for all *n* . Using the value of *a* obtained in the previous part we can find the value of *H* (1) to be -1. Hence, *y* [*n*] = -1.

Problem 4

A causal LTI system is described by the difference equation

y[n] =y[n-1] +y[n-2] +x[n-1] .

(a) Find the system function H(z) = Y(z) / X(z) for this system . Plot the poles and zeroes of H(z) and indicate the region of convergence.

(b) Find the unit sample response of the system.

(c) You should have found the system to be unstable . Find a stable (but non- causal) unit sample response which satisfies the difference equation.

Solution 4

(a) The difference equation describing a causal LTI system is given by

$$y[n] = y[n-1] + y[n-2] + x[n-1]$$

$$\Rightarrow Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + z^{-1}X(z)$$

$$\Rightarrow H(z) = \frac{z^{-1}}{1-z^{-1}-z^{-2}}$$

$$= \frac{z}{z^2 - z - 1}$$

From the system function, we can see that the poles of the system are at

$p_1 = \frac{1-\sqrt{5}}{2}$ and $p_2 = \frac{1+\sqrt{5}}{2}$. Since the system is given to be causal, the ROC

is $|z| > |p_2|$. Figure gives the pole-zero plot for this system.

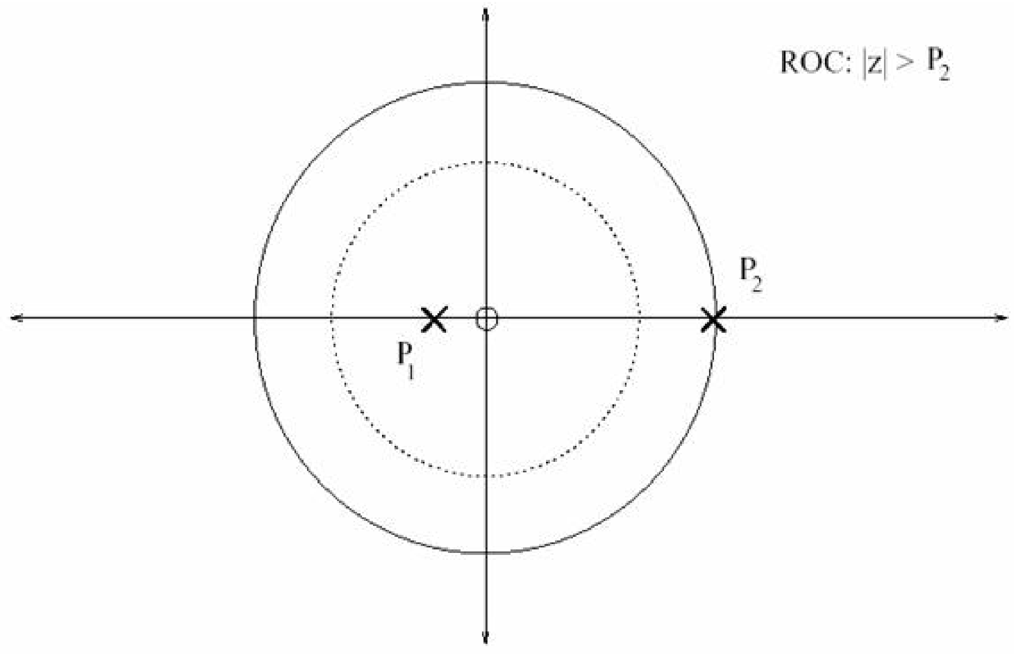


Figure: Pole-zero plot of *H* (*z*). The dotted line represents the unit circle.The ROC is outside the bigger circle.

NOTE : Since the ROC doesn't contain the unit circle, the system is NOT stable.

(b) We can write *H* (*z*) using a partial fraction expansion as

$$H(z) = \frac{A_1}{1-p_1 z^{-1}} + \frac{A_2}{1-p_2 z^{-1}}, \quad |z| > |p_2|$$

for constants given by $A_1 = \frac{1}{p_1 - p_2}$, $A_2 = \frac{1}{p_2 - p_1}$. Taking the Inverse Z-transform of *H(z)* , we get

$$h[n] = [A_1(p_1)^n + A_2(p_2)^n]u[n]$$

(c) Since $|p_2| > 1$, it is clear that the impulse response found in part (b) is unstable. For the system to be stable, the ROC must include the unit circle. Hence the ROC we need to consider is $|p_1| < z < |p_2|$. For this ROC, the impulse response will be given by

$$h_1[n] = A_1(p_1)^n u[n] - A_2(p_2)^n u[-n-1]$$

Problem 5

We are given the following 5 facts about a discrete time signal x[n] with Z – transform X (z) :

(a) x[n] is real and right sided.

(b) X(z) has exactly 2 poles.

(c) X(z) has 2 zeroes at the origin.

(d) X(z) has a pole at z = $\frac{1}{2}e^{j\frac{\pi}{3}}$.

(e) X(1) = $\frac{8}{3}$.

Determine X(z) and specify its region of convergence.

Solution 5

It is given that $x[n]$ is real, and that $X(z)$ has exactly two poles with one of the pole at $z = \frac{1}{2}e^{\frac{j\pi}{3}}$. Since, $x[n]$ is real, the two poles must be complex conjugates of each other. Thus, the other pole of the system is at $z = \frac{1}{2}e^{\frac{-j\pi}{3}}$. Also, it is given that $X(z)$ has two zeros at the origin. Therefore, $X(z)$ will have the following form:

$$X(z) = \frac{Kz^2}{\left(z - \frac{1}{2}e^{\frac{j\pi}{3}}\right)\left(z - \frac{1}{2}e^{\frac{-j\pi}{3}}\right)}$$

$$= \frac{Kz^2}{z^2 - \frac{1}{2}z + \frac{1}{4}}$$

for some constant K to be determined. Finally, it is given that $X(1) = \frac{8}{3}$. Substituting this in the previous equation, we get $K = 2$. Thus,

$$X(z) = \frac{2z^2}{z^2 - \frac{1}{2}z + \frac{1}{4}}$$

Since $x[n]$ is right sided its ROC is $|z| > \frac{1}{2}$ (note that both poles have magnitude $\frac{1}{2}$).

Problem 6

Determine the z-transform for each of the following sequences. Sketch the pole zero plot, and indicate the region of convergence. Indicate whether or not the Fourier transform of the sequence exist

(a) $\delta[n + 5]$

(b) $\delta[n - 5]$

(c) $(-1)^n u[n]$

(d) $\left(\frac{1}{2}\right)^{n+1} u[n+3]$

(e) $\left(\frac{-1}{3}\right)^{n} u[-n-2]$

(f) $\left(\frac{1}{4}\right)^{n} u[3-n]$

(g) $2^n u[-n] + \left(\frac{1}{4}\right)^{n} u[n-1]$

(h) $\left(\frac{1}{3}\right)^{n-2} u[n-2]$

Solution 6

(a) $\delta[n + 5]$

The Z-transform of $\delta[n]$ is 1, now the Z-transform of normal;'>$\delta[n + 5]$ will be Z^5, by the property that if Z-transform of x[n] is X(Z) then theZ-transform ofx[n-m] will be $Z^{-m}X(Z)$. . The region of convergence in this case is the entire z plane except $|Z| = \infty$

(b) $\delta[n - 5]$

The Z-transform of $\delta[n]$ is 1, now the Z-transform of $\delta[n - 5]$ will be Z^{-5} , by the property that if Z-transform of x[n] is X(Z) then the Z-transform of x[n-m] will be $Z^{-m}X(Z)$. The region of convergence in this case is the entire z plane except $|Z| = 0$

(c) $(-1)^n u[n]$

$\alpha^n u[n]$ has the Z-transform $\dfrac{1}{1-\alpha Z^{-1}}$ hence the sequence $(-1)^n u[n]$ will have the Z-transform $\dfrac{1}{1+Z^{-1}}$ with the region of convergence $|z| > 1$

(d) $\left(\frac{1}{2}\right)^{n+1} u[n+3]$

$\left(\frac{1}{2}\right)^{n+1} u[n+3] = 4\left(\frac{1}{2}\right)^{n+3} u[n+3]$ hence from combining the shifting property and the above used property we can get the Z-transform to be as follows $4.\frac{1}{1-\frac{1}{2}Z^{-1}}.Z^3 = \frac{8Z^3}{2-Z^{-1}}$. The region of convergence will be $\frac{1}{2} < |Z| < \infty$

note **that infinity is not in the ROC**

(e) $\left(\frac{-1}{3}\right)^{n} u[-n-2]$

If the X(Z) is the Z-transform of x[n] then we shall use the following properties to solve the above problem.

1. Z-transform of x[n-m] will be $Z^{-m}X(Z)$ **.** The ROC is all Z except 0(if m > 0) or infinity(if m < 0)

2. $-\alpha^n u[-n-1]$ has the Z-transform $\frac{1}{1-\alpha Z^{-1}}$ The ROC is $|Z| < |\alpha|$

$$\left(\frac{-1}{3}\right)^{n} u[-n-2] = 3\left(\frac{-1}{3}\right)^{n+1} u[-n-2] = 3\left(\frac{-1}{3}\right)^{k} u[-k-1]$$

Hence the Z-Transform will be, $\frac{-3}{1-\frac{1}{3}Z^{-1}}$ (in k, we now shift it by m = -1)

Hence the final transform will be $Z\frac{-3}{1-\frac{1}{3}Z^{-1}}$ with region of convergence $|Z| < \left|\frac{1}{3}\right|$.

(f) $\left(\frac{1}{4}\right)^{n} u[3-n]$

$\left(\frac{1}{4}\right)^{n} u[3-n] = \left(\frac{1}{4}\right)^{4}\left(\frac{1}{4}\right)^{n-4} u[-(n-4)-1] = \left(\frac{1}{4}\right)^{4}\left(\frac{1}{4}\right)^{K} u[-(K)-1]$, hence now the Z-transform using a procedure similar to the one above will be $\left(\frac{1}{4}\right)^{4} Z^{-4} \frac{-1}{1-\frac{1}{4}Z^{-1}}$

with a region of convergence $0 < |Z| < \left|\frac{1}{4}\right|$ **note here 0 is not in the ROC**

(g) $2^n u[-n] + \left(\frac{1}{4}\right)^n u[n-1]$ $2^n u[-n] + \left(\frac{1}{4}\right)^n u[n-1] = 2.2^{n-1} u[-(n-1)-1] + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^{n-1} u[n-1]$, hence the Z-transform will be $2Z^{-1}\frac{-1}{1-2Z^{-1}} + \left(\frac{1}{4}\right)Z^{-1}\frac{1}{1-\frac{1}{4}Z^{-1}}$, with region of convergence $\frac{1}{4} < |Z| < 2$

(h) $\left(\frac{1}{3}\right)^{n-2} u[n-2]$

By the shifting property and the property $\alpha^n u[n]$ has the Z-transform $\frac{1}{1-\alpha Z^{-1}}$,

we get the Z-transform to be $Z^{-2}\frac{1}{1-\frac{1}{3}Z^{-1}}$ with the region of convergence $0 < |Z| < \left|\frac{1}{4}\right|$

Problem 7

A pressure gauge that can be modeled as an LTI system has a time response to a unit step input given by $(1-e^{-t}-te^{-t})u(t)$. For the certain input $x(t)$, the output is observed to be $(2-3e^{-t}+e^{-3t})u(t)$

For this observed measurement, determine the true pressure input to the gauge as a function of time.

Solution 7 _

The given model is a LTI system.

When the given system is fed with the input $u(t)$, the output produced is . $(1-e^{-t}-te^{-t})u(t)$ Hence, the **system function H(s)**, is the ration of the Laplace transform of the output divided by the Laplace transform of the input.

$H(s) = L(h(t)) = L((1-e^{-t}-te^{-t})u(t)) / L(u(t))$

$H(s) = (1/s) - (1/s+1) + d/ds(1/s+1) \div (1/s)$

$(1.\therefore H(s) = 1/(s+1)^2/s)$

Now, for the given input $x(t)$, the observed output $y(t) = (2-3e^{-t}+e^{-3t})u(t)$

Let the Laplace transforms of the input and the output be $X(s)$ and $Y(s)$.

We know that,

$$H(s) = Y(s)/X(s)$$

$$\because Y(s) = L(y(t)) = (2/s) - (3/(s+1)) + (1/(s+3))$$

$$X(s) = Y(s)/H(s)$$

$$X(s) = \frac{(2/s) - (3/(s+1)) + (1/(s+3))}{(1/(s+1))^2}$$

$X(s) = 6(s+1)/(s^2+3s)$
$X(s) = 4(1/s+3) + 2/s$ {Method of partial fractions}

$\therefore x(t) = 4e^{-3t}u(t) + 2u(t)$

Therefore, the true pressure input to the gauge is $x(t) = 4e^{-3t}u(t) + 2u(t)$

Problem 8 :

Determine the function of time, **x(t)**, for each of the following Laplace Transforms and their associated regions of convergence:

$$(a)\ \frac{1}{s^2+9}, \qquad \mathrm{Re}\{s\} > 0$$

$$(d)\ \frac{s+2}{s^2+7s+12}, \qquad -4 < \mathrm{Re}\{s\} < -3$$

$$(f)\ \frac{(s+1)^2}{s^2-s+1}, \qquad \mathrm{Re}\{s\} > \frac{1}{2}$$

Solution 8 :

$$(a)\ \frac{1}{s^2+9} \qquad \mathcal{R}e\{s\} > 0$$

$$\frac{1}{s^2+9} = \frac{1}{3}\times\frac{3}{s^2+3^2} \xrightarrow{\mathcal{L}^{-1}} \frac{1}{3}[\sin 3t]u(t) = x(t)$$

$$(d)\ \frac{s+2}{s^2+7s+12} \qquad -4 < \mathcal{R}e\{s\} < -3$$

$$Dr^n = s^2+7s+12 = 0$$

$$\text{roots} = -3, -4 \Rightarrow \text{ real roots}$$

$\therefore$ using partial fraction we get,

$$\frac{s+2}{s^2+7s+12} = \frac{2}{s+4} - \frac{1}{s+3}$$

$$\frac{2}{s+4} \xrightarrow{\mathcal{L}^{-1}} 2e^{-4t}u(t) \qquad \mathcal{R}e\{s\} > -4$$

$$\frac{1}{s+3} \xrightarrow{\mathcal{L}^{-1}} -e^{-3t}u(-t) \qquad \mathcal{R}e\{s\} < -3$$

$$\therefore \frac{s+2}{s^2+7s+12} \xrightarrow{\mathcal{L}^{-1}} 2e^{-4t}u(t) + e^{-3t}u(-t) \quad -4 < \mathcal{R}e\{s\} < -3$$

$$(f)\ \frac{(s+1)^2}{s^2-s+1}, \qquad \mathcal{R}e\{s\} > \frac{1}{2}$$

$$Dr^n = s^2 - s + 1 = 0$$

$$s = \frac{1\pm\sqrt{3}i}{2} \Rightarrow \text{imaginary roots}$$

$$\frac{(s+1)^2}{s^2-s+1} = \frac{s^2-s+1+3s}{s^2-s+1}$$

$$= 1 + \frac{3s}{(s-\frac{1}{2})^2 - \frac{3}{4}}$$

$$= 1 + \frac{3s}{\sqrt{\frac{3}{4}}}\ \frac{\sqrt{\frac{3}{4}}}{(s-\frac{1}{2})^2 - \frac{3}{4}}$$

Now consider $\mathcal{L}^{-1}\left(\dfrac{\sqrt{\frac{3}{4}}}{(s-\frac{1}{2})^2-\frac{3}{4}}\right) = e^{t/2}\left[\sin\left(\dfrac{\sqrt{3}}{2}t\right)\right]u(t)$, $\mathcal{Re}\{s\} > \dfrac{1}{2}$

on differenciating we get,

$$\mathcal{L}^{-1}\left(s\times\dfrac{\sqrt{\frac{3}{4}}}{(s-\frac{1}{2})^2-\frac{3}{4}}\right) = \left[e^{t/2}\sin\left(\dfrac{\sqrt{3}}{2}t\right)+\dfrac{\sqrt{3}}{2}e^{t/2}\cos\left(\dfrac{\sqrt{3}}{2}t\right)\right]u(t)$$

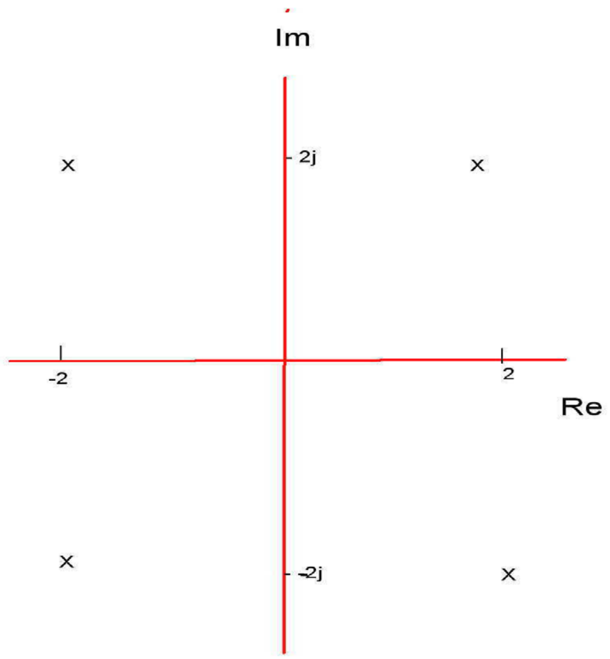
$$\therefore \dfrac{(s+1)^2}{s^2-s+1} = 1 + \dfrac{3s}{\sqrt{\frac{3}{4}}}\;\dfrac{\sqrt{\frac{3}{4}}}{(s-\frac{1}{2})^2-\frac{3}{4}}$$

$$\xrightarrow{\mathcal{L}^{-1}} \delta(t) + e^{t/2}\left[\sin\left(\dfrac{\sqrt{3}}{2}t\right)+\dfrac{\sqrt{3}}{2}\cos\left(\dfrac{\sqrt{3}}{2}t\right)\right]u(t)$$

Problem 9 :

For the signal x(t) below and for the four pole-zero plot in **Fig. 9.01** ,determine the corresponing constraint on **ROC**:

1. $x(t)e^{-3t}$ is absolutely integrable



Solution 9 :

$x(t)e^{-3t}$ is absolutely integrable

$\Rightarrow \int |h(t)| dt < \infty$

$\Rightarrow \int |h(t)e^{-j\omega t}| dt < \infty$

$\Rightarrow$ imaginary axis lies within ROC

$x(t) \xrightarrow{\mathcal{L}} X(s) \quad [s \in R]$

$\therefore x(t)e^{-3t} \xrightarrow{\mathcal{L}} X(s+3) \quad [s+3 \in R]$

by replacing s by s+3

$\rightarrow$ poles shift towards left as in fig 9.02

[multiplying by e^{-3t}, x(t) which may not be absolutely integrable otherwise may become integrable]

$\therefore ROC: \mathcal{Re}\{s\} > -1$

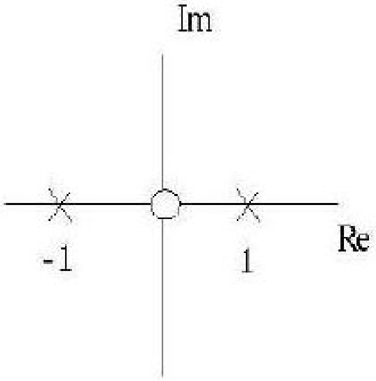
its Pole-Zero plot :

Problem 10 :

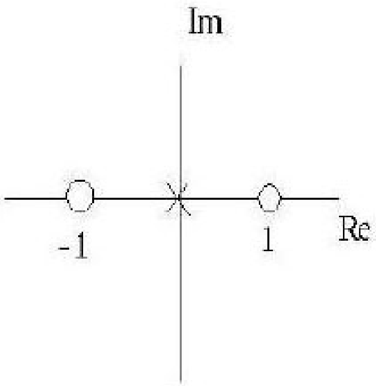
(a) Show that if x(t) is an even function, so that x(t) =x(-t), then X(s) =X(-s).

(b) Show that if x(t) is an odd function, so that x(t) =-x(-t), then X(s) =-X(-s).

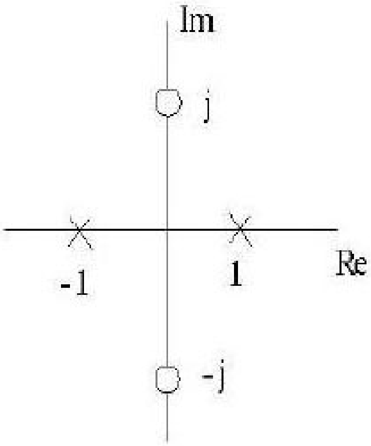
(c) Determine which, if any, of the pole-zero plots in Figure below could correspond to an even function of time. For those that could, indicate the required ROC.



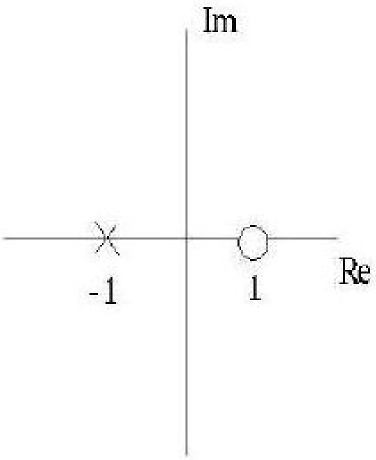
(a)



(b)



(c)



(d)

9.42 Determine whether each of the following statements is true or false. If a statement is true, construct a convincing argument for it. If it is false, give a counterexample.

(a) The Laplace transform os $t^2u(t)$ does not converge anywhere on the s-plane.

(b) The Laplace transform os $e^{t^2}u(t)$ does not converge anywhere on the s-plane.

(c) The Laplace transform os e^{jw_0t} does not converge anywhere on the s-plane.

(d) The Laplace transform os $e^{jw_0t}\, u(t)$ does not converge anywhere on the s-plane.

(e) The Laplace transform os $|t|$ does not converge anywhere on the s-plane.

Solution 10 :

EE 210 Tutorial
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PROBLEMS

9.41

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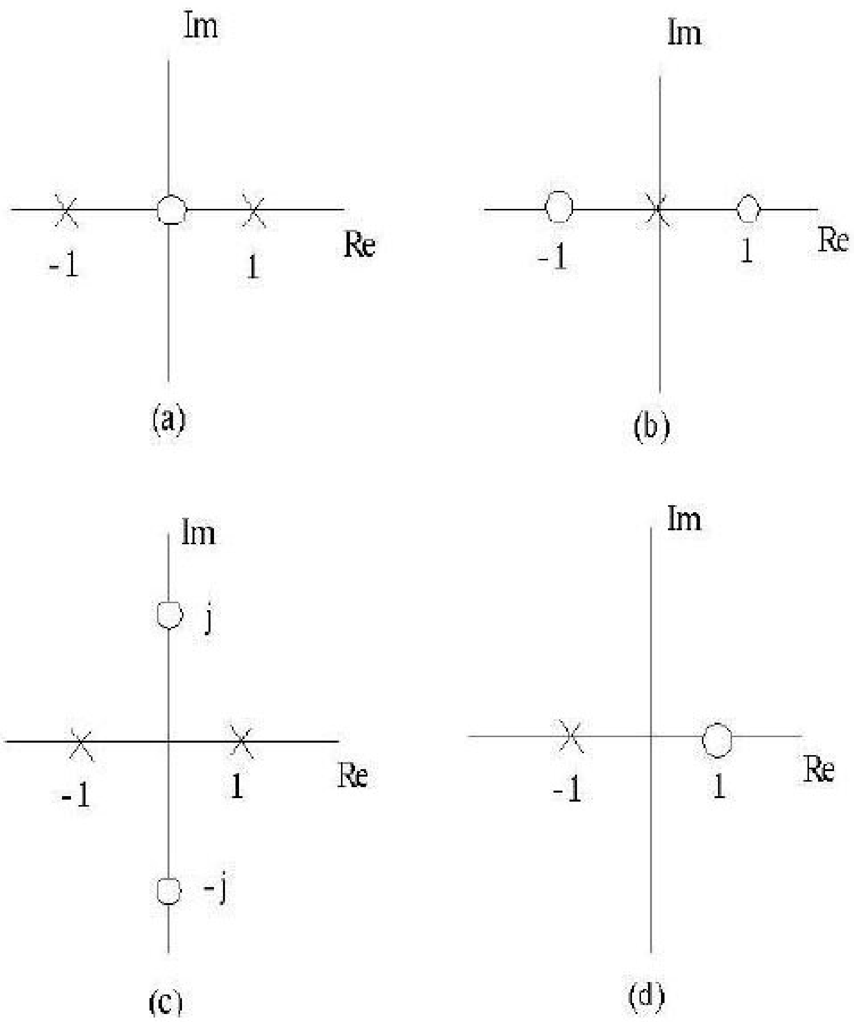

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(b) The Laplace transform os $e^{t^2}u(t)$ does not converge anywhere on the s-plane.

(c) The Laplace transform os e^{jwot} does not converge anywhere on the s-plane.

(d) The Laplace transform os e^{jwot} u(t) does not converge anywhere on the s-plane.

(e) The Laplace transform os |t|does not converge anywhere on the s-plane.

SOLUTIONS

9.41

(a)

$$X(s)=\int_{-\infty}^{+\infty} x(t)\mathrm{e}^{-st}dt$$

$$X'(s)=\int_{-\infty}^{+\infty} x(-t)\mathrm{e}^{-st}dt$$

put v=-t, dv=-dt

limits are now from +∞ to - ∞

thus:

$$X'(s)=\int_{+\infty}^{-\infty} x(v)\mathrm{e}^{-sv}(-dv)=X(-s)$$

Since x(t)=x(-t) then $\mathcal{L}$x(t)= $\mathcal{L}$x(-t)

Thus X(s) = X(-s)

(b)
Here x(t)=- x(-t) then $\mathscr{L}$x(t)= -$\mathscr{L}$x(-t)
Thus X(s) = -X(-s)

(c) The function x(t) can only be even if X(s)=X(-s), i.e. X(s) is even. Thus the ROC of X(s) and X(-s) must be same and so poles should be symmetric wrt to origin, with no poles on the y-axis. Also the zeroes must be symmetric wrt to origin. If the zero is at origin, it must be a double zero.
(a) Can't be even function unless there isa double zero at the origin.
(b) Can't be even function since ROC can be either to the left or right of y-axis and so X(s) and X(-s) cannot have same ROC.
(c) Can be even with ROC: Re(s) ϵ (-1,1)
(d) Can't be even function since ROC can be either to the left or right of x=-1 and so X(s) and X(-s) cannot have same ROC.

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(a) False. X(s) converges to $2/s^3$ with ROC: Re(s)>0. This is clear from the fact that for limit $t \to +\infty$, $t^2e^{-st}=t^2e^{-(\sigma+j\omega)t}$ tends to zero for σ=Re(s)>0 as the exponent term dominates and its integral from 0 to $+\infty$ converges.
(b) True. $e^{(t^2-st)}$ does not tend to zero as $t \to +\infty$, which means the integral from 0 to $+\infty$ is unbounded for all s. Thus the Laplace transform cannot converge for any value of s.
(c) True. For Re(s)>0, e^{jw_0t-st} does not tend to zero for $t \to -\infty$. For Re(s)<0, it does not converge for $t \to +\infty$. Hence for no value of s does the Laplace transform converge.
(d) False. Here the limits of integration for the Laplace transform are from 0 to $+\infty$. Thus for Re(s)>0, e^{jw_0t-st} tends to zero for $t \to +\infty$. And the integral conveges too for Re(s)>0. The Laplace transform isgiven by $1/(s-jw_0)$
(e) True. |t|=t u(t) - t u(-t). For t u(t) the ROC is Re(s)>0, for t u(-t) ROC is Re(s)<0. Hence there is no value of s common to both the ROCs.

Problem 11 :

The inverse of a stable LSI system H(s) is defined as a system that, when cascaded with H(s), results in an overall transfer function of unity or, equivalently, an overall impulse response that is an impulse.

(a) If $H_1(s)$ denotes the transfer function of an inverse system of H(s), determine the general algebraic relationship between H(s) and $H_1(s)$.

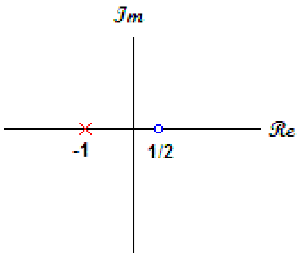
(b) Shown in the figure below (at left) is the pole-zero plot for a stable, causal system H(s). Determine the pole-zero plot for the associated inverse system.

Solution 11 :

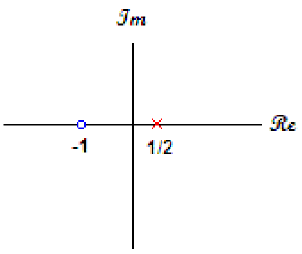
(a) We first note the subtle point that the question talks of H(s) and $H_1(s)$, thus implying the assumptions that the transfer function and the inverse function have Laplace transforms. Then we get the relation :

First, we note that the total transfer function is unity. This can also be noted from the fact that h (t) = u(t) and the laplace transform of u(t) is U(s) =1. Then we use the convolution property of the Laplace transform to write, 1 = U(s) = H(s) $H_1(s)$. Hence, $H_1(s) = 1/H(s)$

(b) Since $H_1(s) = 1/H(s)$ we see that the poles and zeroes will get interchanged. Or, we can deal with the given figure explicitly to note that H(s) is of the form $k\frac{(s-1/2)}{(s+1)}$ where k is some arbitrary nonzero constant. Then we get $H_1(s)$ to be $\frac{1}{k}\frac{(s-1/2)}{(s+1)}$ The pole - zero plot of this function is given at the right.



H(s)



$H_1(s)$

Problem 12 :

A class of systems, known as minimum-delay or minimum-phase systems, is sometimes defined through the statement that these systems are causal and stable and that the inverse systems are also causal and stable

Based on the preceding definition, develop an argument to demonstrate that all poles and zeroes of the transfer function of a minimum-delay system must be in the left half of the s plane.

Solution 12 :

In this solution, we use the idea of the problem 11 of this module

Our argument is based on the regions of convergence of Laplace transforms and the corresponding implications for the rational systems. We recall that the region of convergence (ROC) of the the Laplace transform is always bounded by two vertical lines.

For a system to be causal, the region of convergence of the Laplace transform must include the part of the s-plane as $\lim s \to \infty$

For a system to be stable, the imaginary axis has to be a part of the ROC.

So, a causal and stable system has its ROC as the right side of a pole of the transfer function H(s), with the poles all lying in the left half plane s<0 (else the imaginary axis will not be in the ROC).

Now, from the problem above, we see that the zeroes of the transfer function become the poles of its inverse. So, for the inverse to be stable and causal, all the poles of the inverse (i.e. the zeroes of the original transfer function) must also be in the left half plane s<0.

This completes the argument.