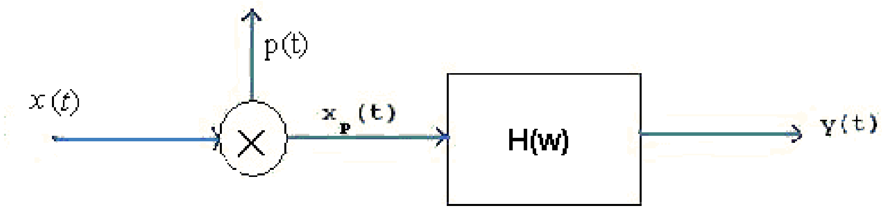
Module 3 : Sampling and Reconstruction
Problem Set 3

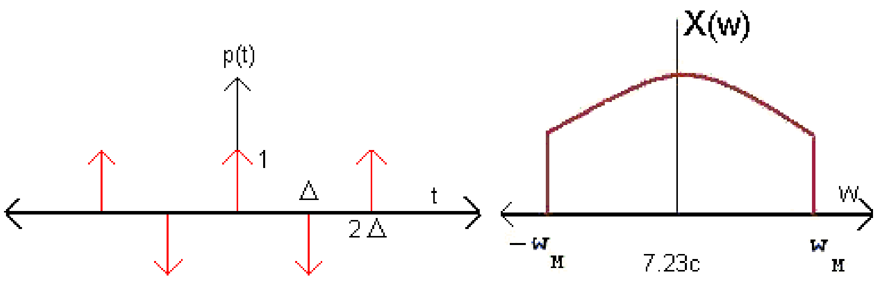
Problem 1

Shown in figure below is a system in which the sampling signal is an impulse train with alternating sign.



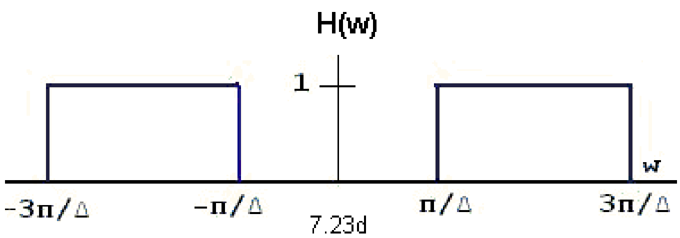
7.23a

The sampling signal p(t), the Fourier Transform of the input signal x(t) and the frequency response of the filter are shown below:



7.23b

7.23c



7.23d

(a) For $\Delta < \dfrac{\pi}{2\omega_M}$, sketch the Fourier transform of $x_p(t)$ and y(t) .

(b) For $\Delta < \dfrac{\pi}{2\omega_M}$, determine a system that will recover x(t) from x_p (t) and another that will recover x(t) from y(t).

(c) What is the *maximum* value of Δ in relation to w_M for which x(t) can be recovered from either x_p(t) or y(t)?

Solution 1

(a) As $\mathbf{x_p(t) = x(t)\ p(t)}$

By dual of convolution theorem we have $X_p(\omega) = X(\omega)\ P(\omega)$.

So we first find the Fourier Transform of **p(t)** as follows :-

The Fourier Transform of a periodic function is an impulse train at intervals of $\omega = \dfrac{2\pi}{2\Delta} = \dfrac{\pi}{\Delta}$.

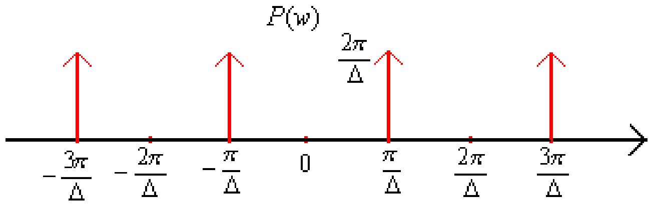
Strength of impulse at $\dfrac{k\pi}{\Delta}$ being:

$$c_k = \frac{\pi}{\Delta}\int_{(2\Delta)} p(t)\, e^{j\frac{2\pi k_t}{2\Delta}}\, dt$$

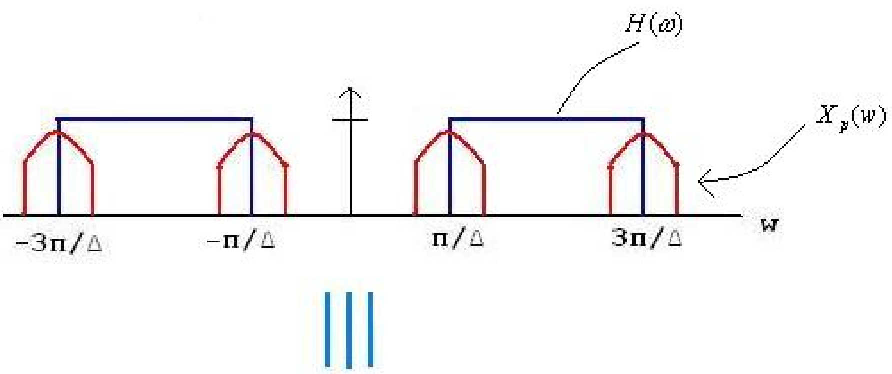
$$= \frac{\pi}{\Delta}(1 - e^{j\frac{2\pi k}{2\Delta}\Delta}) = \frac{\pi}{\Delta}(1 - e^{jk\pi})$$

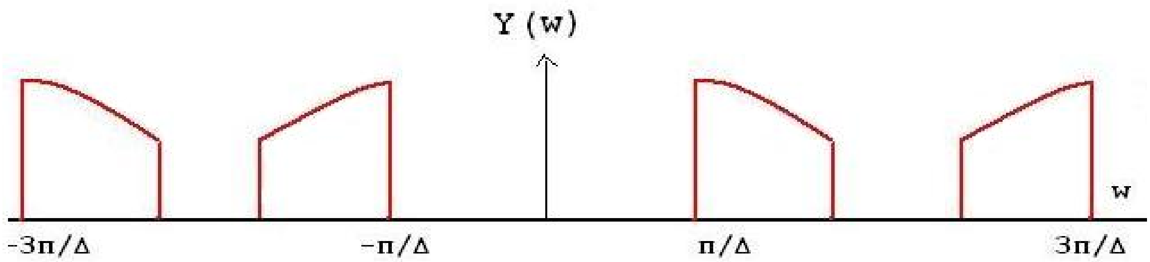
$$= \frac{\pi}{\Delta}\left(1 - (-1)^k\right)$$

Thus, we have can sketch $P(w)$:



Thus we can also sketch $X_p(w)$ and hence $Y(w)$:





(b) To recover x(t) from $x_p(t)$:

Modulate $x_p(t)$ with $\cos(\frac{\pi}{\Delta}t)$.

$\cos(\frac{\pi}{\Delta}t)$ has a spectrum with impluses of equal strength at $\frac{\pi}{\Delta}$ & $-\frac{\pi}{\Delta}$. Thus the new signal will have copies of the original spectrum (modulated by a constant of-course) at all even multiples of $\frac{\pi}{\Delta}$. Now an appropriate Low-pass filter can extract the original spectrum!

To recover x(t) from y(t):

Here too, notice from the figures that modulation with $\cos(\frac{\pi}{\Delta}t)$ will do the job. Here too, the modulated signal will have copies of the original spectrum at all even multiples of $\frac{\pi}{\Delta}$.

(c) So long as adjacent copies of the original spectrum do not overlap in $X_p(w)$, theoretically one can reconstruct the original signal. Therefore the condition is:

$$2\omega_M < \frac{2\pi}{\Delta} \Rightarrow \Delta < \frac{\pi}{\omega_M}$$

Problem 2

The signal y(t) is obtained by convolving signals $x_1(t)$ ans $x_2(t)$ where:

$|X_1(\omega)| = 0 \quad for \quad |\omega| > 1000\pi \qquad \&$

$|X_2(\omega)| = 0 \quad for \quad |\omega| > 2000\pi$

Impulse train sampling is performed on y(t) to get $y_p(t) = \sum_{-\infty}^{\infty} y(nT)\delta(t - nT)$.

Specify the range of values of T so that y(t) may be recovered from $y_p(t)$.

Solution 2

By the Convolution Theorem,

$Y(\omega) = X_1(\omega)X_2(\omega)$

$Y(\omega) = 0 \quad for \quad |\omega| > 1000\pi$

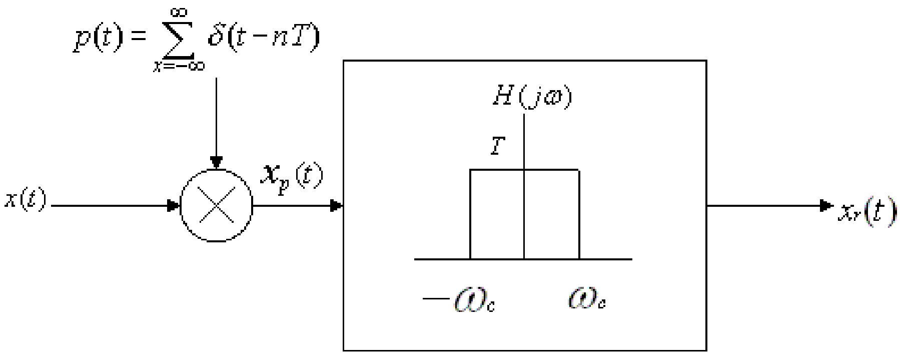
Thus from the Sampling Theorem, the sampling rate must exceed $2 * \frac{1000\pi}{2\pi} = 1000$.

Thus T must be less than 10^{-3}, i.e: 1millisecond.

Problem 3

In the figure below, we have a sampler, followed by an ideal low pass filter, for reconstruction of $x(t)$ from its samples $x_p(t)$. From the sampling theorem, we know that if $\omega_s = \frac{2\pi}{T}$ is greater than twice the highest frequency present in $x(t)$ and $\omega_c = \frac{\omega_s}{2}$, then the reconstructed signal will exactly equal $x(t)$. If this condition on the bandwidth of $x(t)$ is violated, then $x_r(t)$ will not equal $x(t)$. We seek to show in this problem that if $\omega_c = \frac{\omega_s}{2}$, then for any choice of T, $x_r(t)$ and $x(t)$ will always be equal at the sampling instants;

that is, $x_r(kT) = x(kT),\ k = 0, \pm 1, \pm 2, \ldots$



To obtain this result, consider the following equation which expresses $x_r(t)$ in terms of the samples of $x(t)$:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT) T \frac{\omega_c}{\pi} \frac{Sin\left[\omega_c (t - nT)\right]}{\omega_c (t - nT)}$$

With $\omega_c = \frac{\omega_s}{2}$, this becomes

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT) \frac{Sin\left[\frac{\pi}{T}(t - nT)\right]}{\frac{\pi}{T}(t - nT)}$$

By considering the value of μ for which $\frac{\left[Sin(\mu)\right]}{\mu} = 0$, show that, without any restrictions on $x(t)$, $x_r(kT) = x(kT)$ for any integer value of k.

Solution 3

In order to show that $x_r(t)$ and $x(t)$ are equal at the sampling instants, consider

$$\lim_{t \to kT} x_r(t) = \lim_{t \to kT} \sum_{n=-\infty}^{\infty} x(nT) \frac{Sin\left[\frac{\pi}{T}(t - nT)\right]}{\frac{\pi}{T}(t - nT)} \qquad (k \in Z)$$

$$= \sum_{n=-\infty}^{\infty} \left(\lim_{t \to kT} x(nT) \frac{Sin\left[\frac{\pi}{T}(t-nT)\right]}{\frac{\pi}{T}(t-nT)} \right) \quad \text{(assuming limit and summation are interchangeable)}$$

$$= \sum_{n=-\infty, n \neq k}^{\infty} \left(x(nT) \frac{Sin\left[\frac{\pi}{T}(kT-nT)\right]}{\frac{\pi}{T}(kT-nT)} \right) + \lim_{t \to kT} \left(x(kT) \frac{Sin\left[\frac{\pi}{T}(t-kT)\right]}{\frac{\pi}{T}(t-kT)} \right)$$

$$= \sum_{n=-\infty, n \neq k}^{\infty} \left(x(nT) \frac{Sin\left[\pi(k-n)\right]}{\pi(k-n)} \right) + x(kT) \lim_{t \to kT} \left(\frac{Sin\left[\frac{\pi}{T}(t-kT)\right]}{\frac{\pi}{T}(t-kT)} \right)$$

$$= 0 + x(kT) \times 1 \quad \left(\begin{array}{l} \because \ (k-n) \in Z \ \Leftrightarrow \ Sin\left[\pi(k-n)\right] = 0 \\ and \quad \lim_{x \to 0} \frac{Sin\ x}{x} = 1 \end{array} \right)$$

Thus, $\lim_{x \to kT} x_r(t) = x(kT)$

Assuming the continuity of $x_r(t)$ at $t = kT$,

$x_r(kT) = x(kT), \quad \forall k \in Z$

Problem 4

This problem deals with one procedure of bandpass sampling and reconstruction. This procedure, used when $x(t)$ is real, consists of multiplying $x(t)$ by a complex exponential and then sampling the product. The sampling system is shown in **fig. (a)** below. With $x(t)$ real and with $X(j\omega)$ nonzero only for $\omega_1 < |\omega| < \omega_2$, the frequency is chosen to be $\omega_0 = \left(\frac{1}{2}\right)(\omega_2 + \omega_1)$, and the lowpass filter $H_1(j\omega)$ has cutoff frequency $\left(\frac{1}{2}\right)(\omega_2 + \omega_1)$.

(a) For $X(j\omega)$ shown in **fig. (b)** , sketch $X_p(j\omega)$.

(b) Determine the maximum sampling period T such that $x(t)$ is recoverable from $x_p(t)$.

(c) Determine a system to recover $x(t)$ from $x_p(t)$.

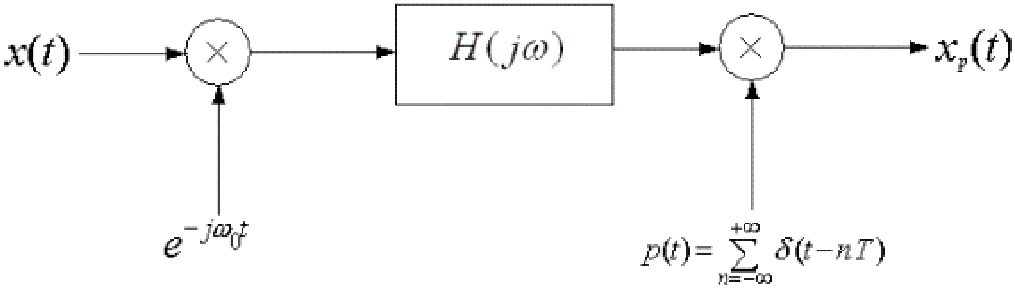


Fig. (a)

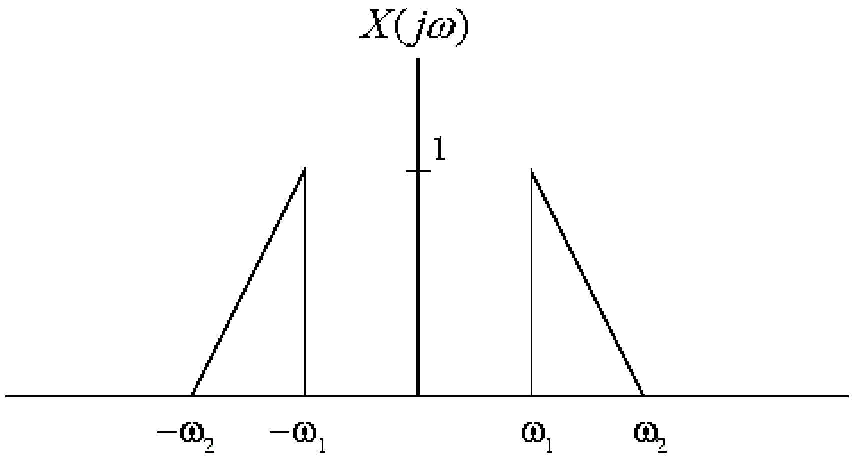
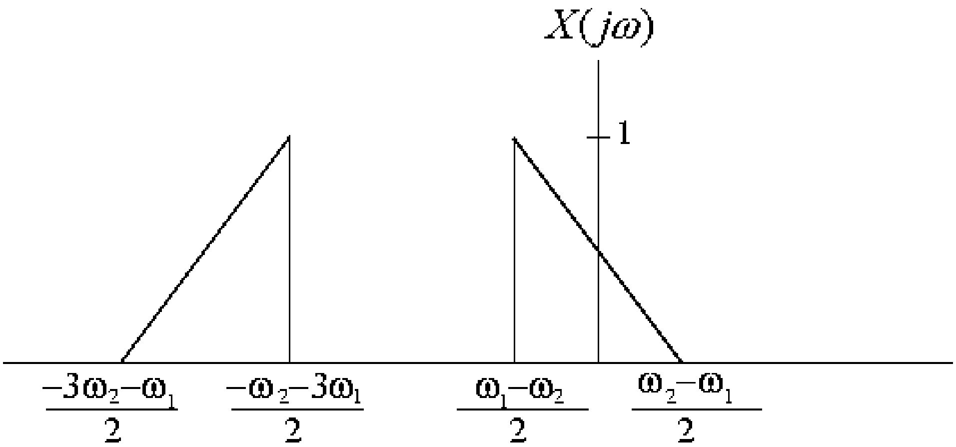


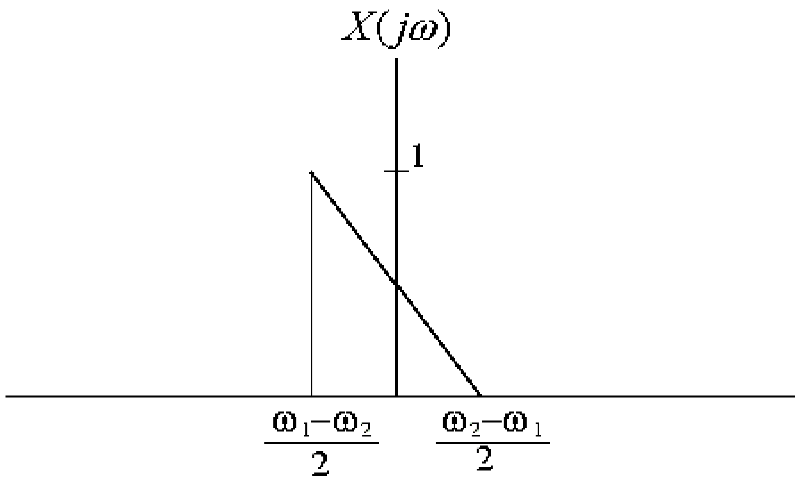
Fig. (b)

Solution 4

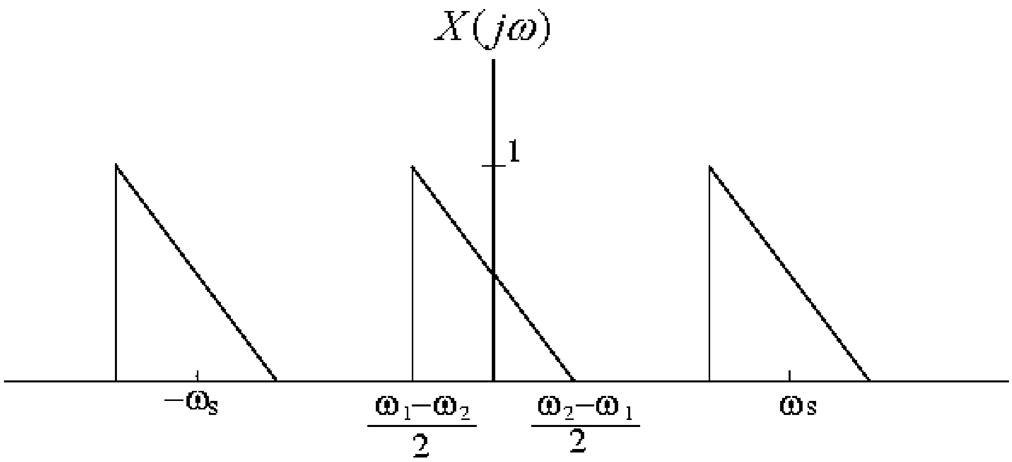
(a) Multiplication by the complex exponential $e^{-j\omega t}$ in the time domain is equivalent to shifting left the Fourier transform by an amount ' ω ' in the frequency domain. Therefore, the resultant transform looks as shown below:



After passing through the filter, the Fourier transform becomes:



Now sampling the signal amounts to making copies of the Fourier Transform, the center of each separated from the other by the sampling frequency in the frequency domain. Thus $X_p(j\omega)$ has the following form (assuming that the sampling frequency is large enough to avoid overlapping between the copies):



(b) $x(t)$ is recoverable from $x_p(t)$ only if the copies of the Fourier Transform obtained by sampling do not overlap with each other. For this to happen, the condition set down by the Shannon-Nyquist Sampling Theorem for a band-limited signal has to be satisfied, i.e. the sampling frequency should be greater than twice the bandwidth of the original signal. Mathematically,

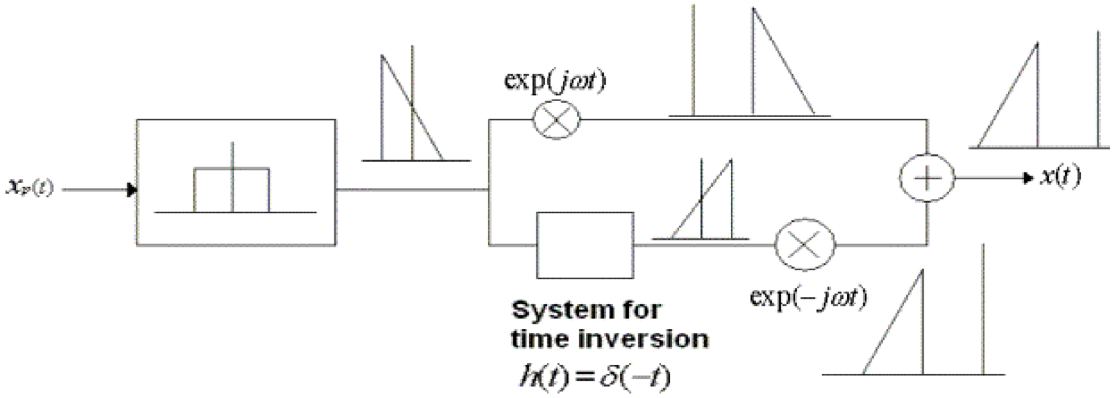
$$\omega_s > 2\omega_m$$

$$\Rightarrow \frac{2\pi}{T} > 2\left(\frac{\omega_2 - \omega_1}{2}\right)$$

$$\Rightarrow T < \left(\frac{2\pi}{\omega_2 - \omega_1}\right)$$

Hence, the maximum sampling period for $x(t)$ to be recoverable from $x_p(t)$ is $\left(\frac{2\pi}{\omega_2 - \omega_1}\right)$.

(c) The system to recover $x(t)$ from $x_p(t)$ is outlined below :



Problem 5

The procedure for interpolation or upsampling by an integer factor N can be thought of as the cascade of two operations. The first operation, involving system A, corresponds to inserting N-1 zero-sequence values between each sequence value of $x[n]$, so that

$$x_p[n] = \begin{cases} x_d\left[\frac{n}{N}\right], & n = 0, \pm N, \pm 2N, \ldots \\ 0, & otherwise \end{cases}$$

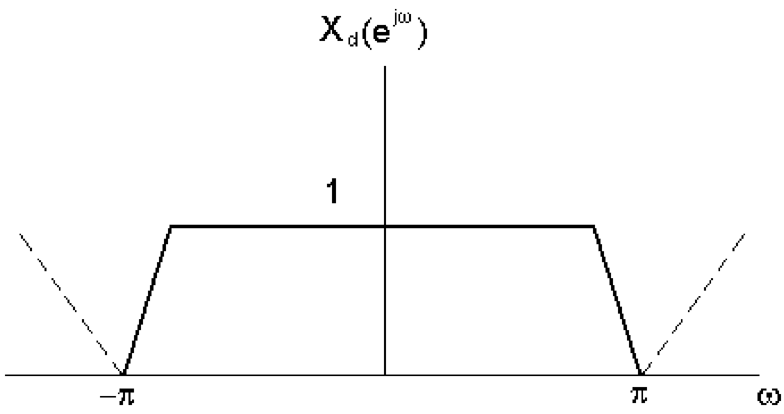
For exact band-limited interpolation, $H(e^{j\omega})$ is an ideal lowpass filter.

(a) Determine whether or not system A is linear.

(b) Determine whether or not system A is time variant.

(c) For $X_d(e^{j\omega})$ as sketched in the figure and with N = 3, sketch $X_p(e^{j\omega})$.

(d) For N = 3, $X_d(e^{j\omega})$ as in the figure, and $H(e^{j\omega})$ appropriately chosen for exact band-limited interpolation, sketch $X(e^{j\omega})$.



Solution 5

(a) Let $x_1[n]$ and $x_2[n]$ be two inputs with corresponding outputs $y_1[n]$ and $y_2[n]$ respectively. Now, suppose the new input to this system is $\alpha x_1[n]+\beta x_2[n]$. Then, the corresponding output is given by, $y[n]=\alpha x_1\left[\frac{n}{N}\right]+\beta x_2\left[\frac{n}{N}\right]$ if $n=0,\pm N,\pm 2N,\ldots.$ and 0 otherwise. From this it is clear that $y[n]=\alpha y_1[n]+\beta y_2[n]$ and hence the system is linear.

(b) The system *A* is not time invariant. We can see this from the following example. Let $x[n]=1$, for n = 0, 1 and $x[n]=0$ otherwise. Also assume that N=2. Thus, the output corresponding to $x[n]$ will be $y[n]=1$ if n = 0, 2 and $y[n]=0$ otherwise. Now, let $x_1[n]=x[n-1]=1$ if n = 1, 2 and 0 otherwise. The corresponding output is $y_1[n]=1$ if n = 2, 4 and $y_1[n]$ = 0 otherwise. Thus, $y_1[n]=y[n-2]\neq y[n-1]$ and hence this shows that the system is not time invariant.

(c) Refer to Fig. 1 and Fig. 2

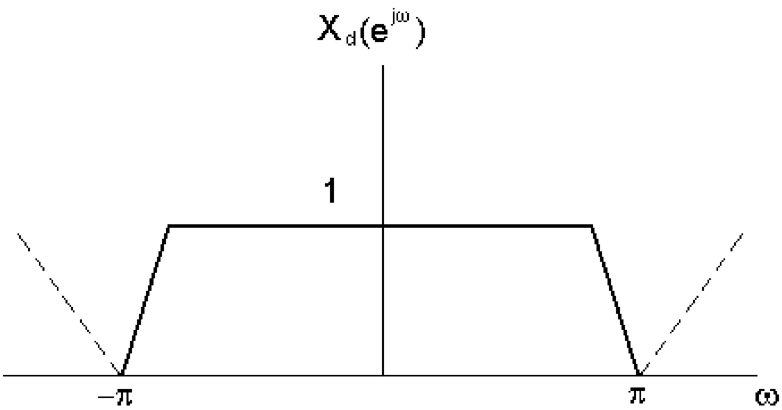


Figure 1: Spectrum given in the problem.

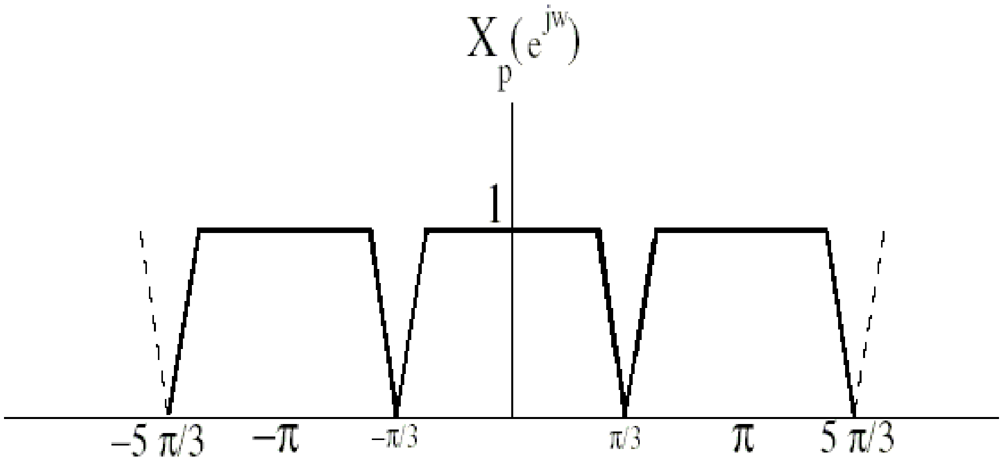


Figure 2: Spectrum of the interpolated signal.

(d) Refer to Fig. 3.

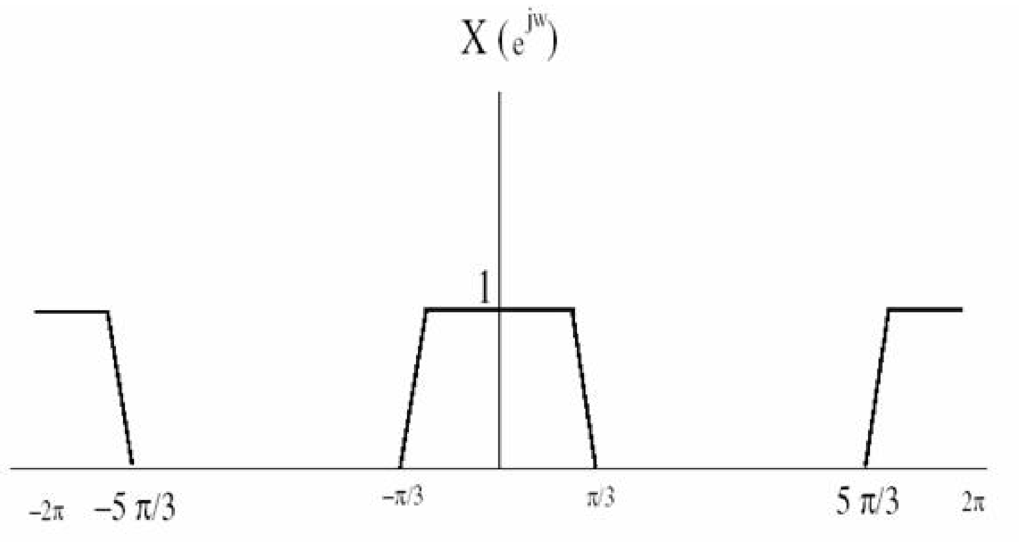


Figure 3: Spectrum of the upsampled signal.

Problem 6

Shown in the figures is a system in which the sampling signal is an impulse train with alternating sign. The Fourier Transform of the input signal is as indicated in the figures:

(i) For $\Delta < \pi/(2\omega_M)$, sketch the Fourier transform of $x_p(t)$ and $y(t)$.

(ii) For $\Delta < \pi/(2\omega_M)$, determine a system that will recover $x(t)$ from $x_p(t)$.

(iii) For $\Delta < \pi/(2\omega_M)$, determine a system that will recover $x(t)$ from $y(t)$.

(iv) What is the *maximum* value of Δ in relation to ω_M for which $x(t)$ can be recovered from either $x_p(t)$ or $y(t)$?

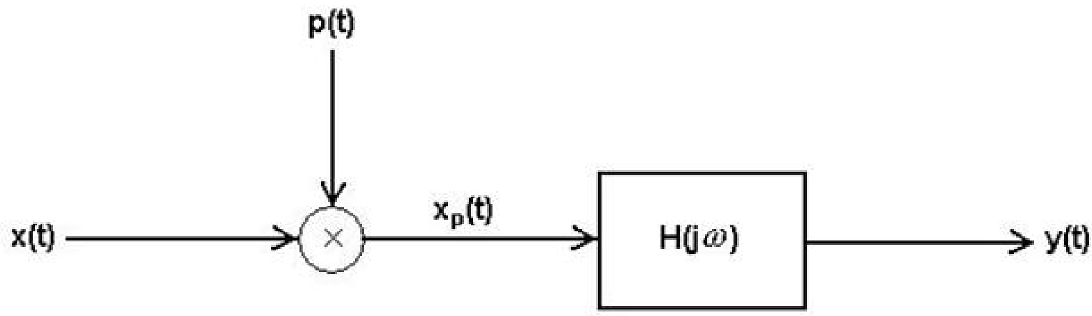


Fig (a)

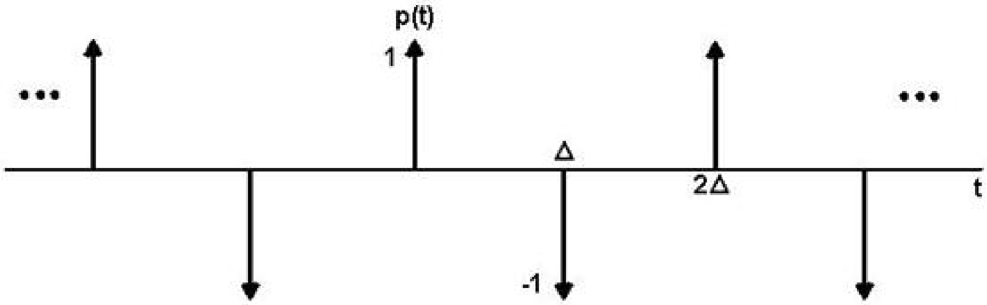


Fig (b)

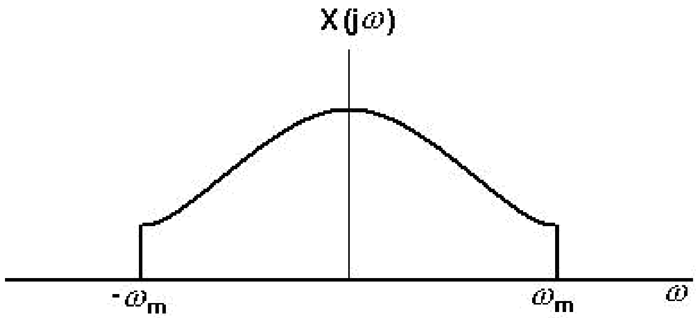


Fig (c)

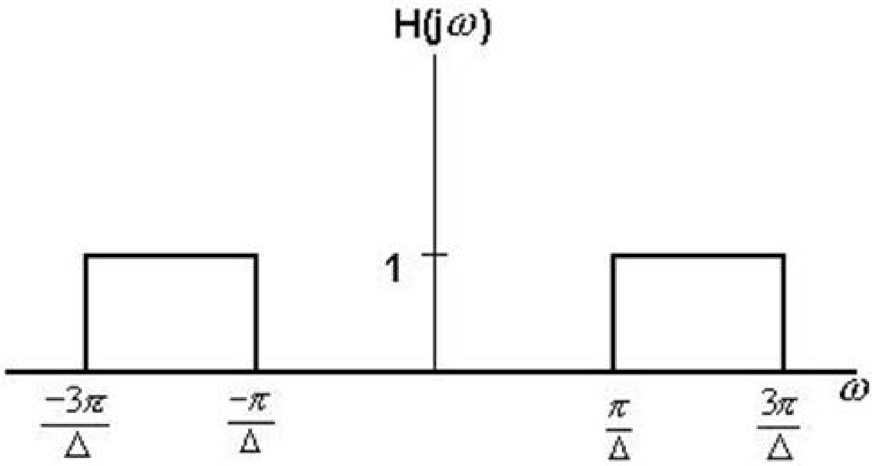


Fig (d)

Solution 6

(a) As $x_p(t) = x(t)p(t)$, by dual of convolution theorem we have $X_p(j\omega) = X(j\omega)P(j\omega)$. So, we first find the Fourier Transform of $p(t)$ as follows:

The Fourier Transform of a periodic function is an impulse train at intervals of $\omega = 2\pi/2\Delta = \pi/\Delta$, each impulse being of magnitude:

$$P(j\omega)_k = 2\pi / 2\Delta \int_{(2\Delta)} p(t)\exp(-jk\omega_0 t)dt$$
$$= \pi / \Delta \left(1 - Cos(\pi k)\right)$$

Here w e see that the impulses on the ω axis vanish at even values of k .

Hence, the Fourier Transform of $X_p(j\omega)$ is as shown in **figure (a).** I n the frequency domain, the output signal Y can be found by multiplying the input with the frequency response . Hence $Y(j\omega)$ is as shown below in the **figure (b).**

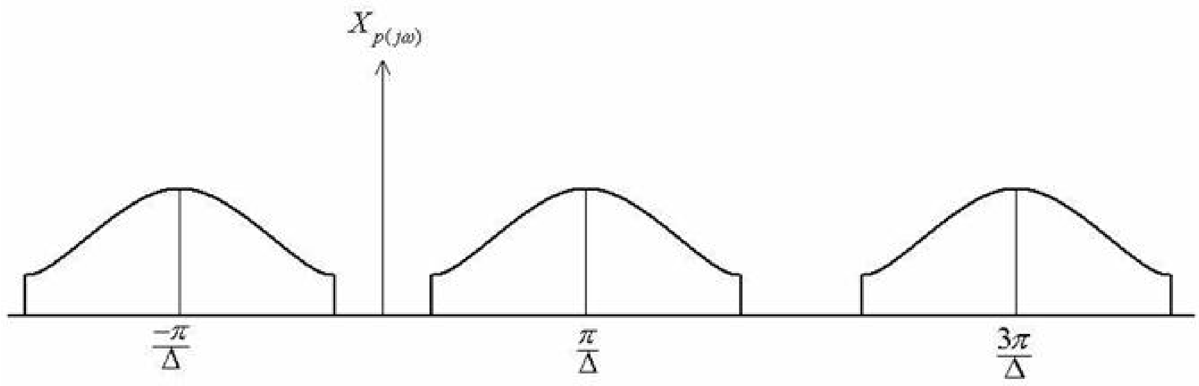


Figure (a)

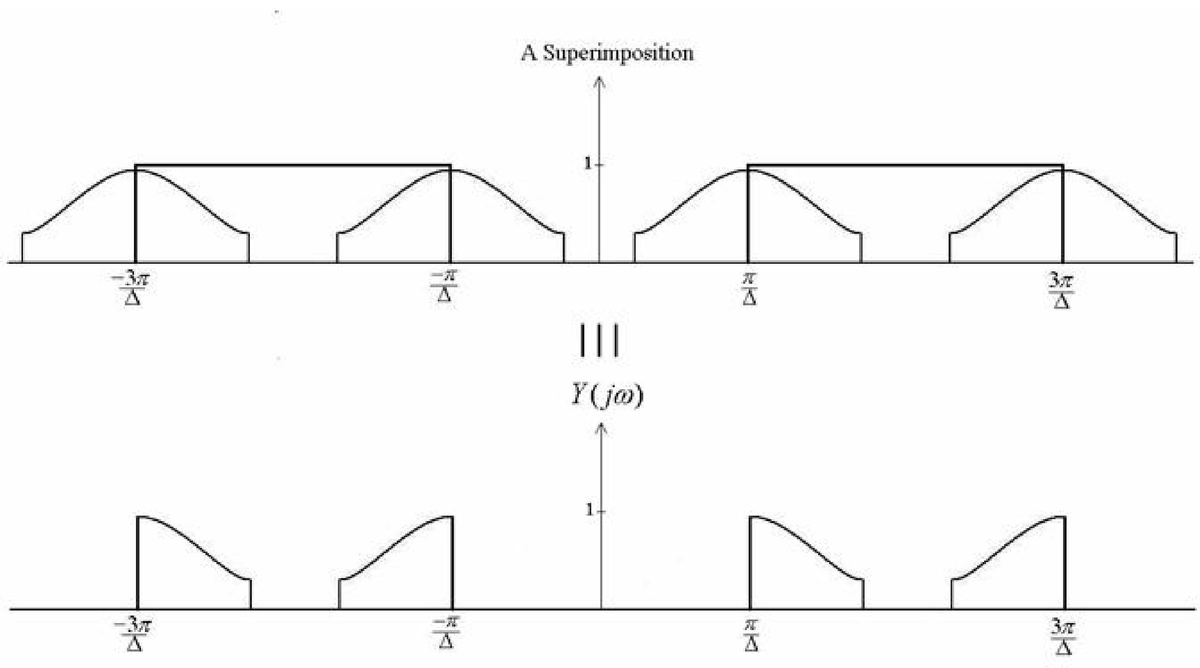


Figure (b)

(b) To recover $x(t)$ from $x_p(t)$, we do the following two things:

1) Modulate the signal by .

$$Cos\left(\left(2\pi/\Delta\right)t\right)$$

2) Apply a low pass filter of bandwidth $\pi/2\Delta$.

(c) To recover $x\left(t\right)$ from $y(t)$, we do the following two things:

1) Modulate the signal by $2Cos\left(\left(2\pi/\Delta\right)t\right)$.
2) Apply a low pass filter of bandwidth $\pi/2\Delta$.

(d) Maximum value for recoverability is π/ω_M as can be seen from the graphs.

Problem 7

A signal $x(t)$ with Fourier transform $X(j\omega)$ undergoes impulse –train sampling to generate

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT),$$

where $T = 10^{-4}$.

For each of the following set of constraints on $x(t)$ and/or $X(j\omega)$, does the sampling theorem guarantee that $x(t)$ can be recovered exactly from $x_p(t)$?

(a) $X(j\omega)=0 \quad for \quad |\omega|>5000\pi$

(b) $X(j\omega)=0 \quad for \quad |\omega|>15000\pi$

(c) $\Re e\{X(j\omega)\}=0 \quad for \quad |\omega|>5000\pi$

(d) $x(t)$ real and $X(j\omega)=0 \quad for \quad \omega>5000\pi$

(e) $x(t)$ real and $X(j\omega)=0 \quad for \quad \omega<-15000\pi$

(f) $X(j\omega) * X(j\omega)=0 \quad for \quad |\omega|>15000\pi$

(g) $|X(j\omega)|=0 \quad for \quad \omega>5000\pi$

Solution 7

From Sampling Theorem we know that if $x(t)$ be a band-limited signal with

$X(j\omega)=0 \quad for \quad |\omega|>\omega_M$, then $x(t)$ is uniquely determined by its samples

$x(nT)$, $n=0,\pm1,\pm2,\pm3\ldots$, if

$\omega_s > 2\omega_M$,

where

$$\omega_s = \frac{2\pi}{T}$$

Now,

$T = 10^{-4}$

$\omega_s = 20000\pi$

(a) $X(j\omega) = 0 \quad for \quad |\omega| > 5000\pi$

$2\omega_M = 10000\pi$

Here, obviously, $\omega_s < 2\omega_M$.

Hence $x(t)$ can be recovered exactly from $x_p(t)$.

(b) $X(j\omega) = 0 \quad for \quad |\omega| > 15000\pi$

$2\omega_M = 30000\pi$

Here, obviously $\omega_s < 2\omega_M$,

Hence $x(t)$ can be recovered exactly from $x_p(t)$.

(c) $\Re e\{X(j\omega)\} = 0 \quad for \quad |\omega| > 5000\pi$

Real part of $X(j\omega) = 0$, but we can't say anything particular about imaginary part of the $X(j\omega)$, thus not necessary that $X(j\omega) = 0$ for this particular range.

Hence $x(t)$ cannot be recovered exactly from $x_p(t)$.

(d) $x(t)$ real and $X(j\omega) = 0 \quad for \quad \omega > 5000\pi$

As $x(t)$ is real we have $X(j\omega) = \overline{X(-j\omega)}$

Taking mod on both sides

$X(j\omega) = \overline{X(-j\omega)} = 0 \quad for \; \omega > 5000\pi$

$\Rightarrow \left|\overline{X(-j\omega)}\right| = |X(-j\omega)| = 0 \; for \; \omega > 5000\pi$

$\Rightarrow X(-j\omega) = 0 \; for \; \omega > 5000\pi$

$\Rightarrow X(j\omega) = 0 \quad for \; \omega < -5000\pi$

So, we get

$X(j\omega) = 0 \; for \; |\omega| > 5000\pi$

Here , obviously $\omega_s > 2\omega_M$,

Hence x(t) can be recovered exactly from $x_p(t)$.

(e) $x(t)$ real and $X(j\omega) = 0 \quad for \quad \omega < -15000\pi$

Proceeding as above we get

$X(j\omega) = 0 \; for \quad |\omega| > 15000\pi$

Here, obviously $\omega_s < 2\omega_M$,

Hence x(t) cannot be recovered exactly from $x_p(t)$.

(f) $X(j\omega) * X(j\omega) = 0 \quad for \quad |\omega| > 15000\pi$

When we convolve two functions with domain ω_1 to ω_2 and ω_3 to ω_4 then the domain of their convolution function varies from $\omega_1 + \omega_3$ to $\omega_2 + \omega_4$.

Here, $\omega_1 = \omega_3$ and $\omega_2 = \omega_4$

$2\omega_1 = 15000$

$\Rightarrow \omega_1 = 7500$

Therefore,

$X(j\omega) = 0 \quad for \quad |\omega| > 7500\pi$

Here, obviously $\omega_s > 2\omega_M$,

Hence x(t) can be recovered exactly from $x_p(t)$.

(g) $|X(j\omega)| = 0 \quad for \quad \omega > 5000\pi$

We cannot say anything about $X(j\omega) \quad for \quad \omega < -5000\pi$,

Hence x(t) cannot be recovered exactly from $x_p(t)$.

Problem 8

Shown in the figures below is a system in which the input signal is multiplied by a periodic square wave. The period of $s(t)$ is T. The input signal is band limited with

$|X(j\omega)| = 0 \, for \, |\omega| \geq \omega_M$

(a) For $\Delta = T/3$, determine, in terms of ω_M, the maximum value of T for which there is no aliasing among the replicas of $X(j\omega)$ in $W(j\omega)$.

(b) For $\Delta = T/4$, determine, in terms of ω_M, the maximum value of T for which there is no aliasing among the replicas of $X(j\omega)$ in $W(j\omega)$.

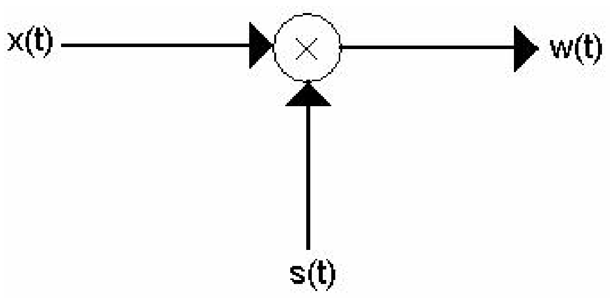


Figure (a)

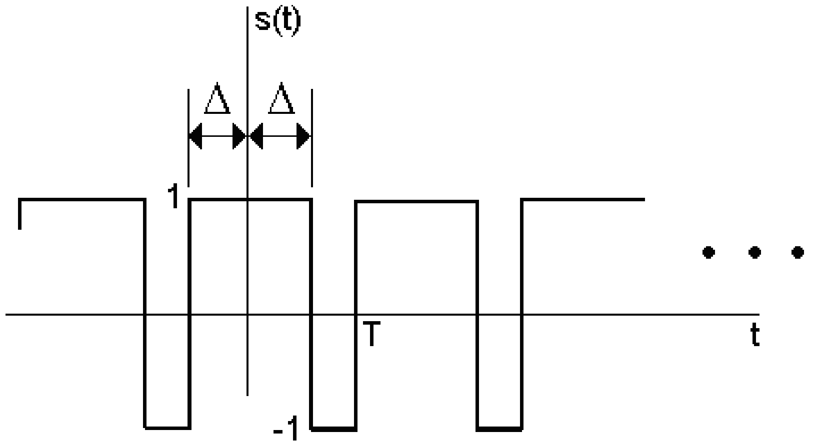


Figure (b)

Solution 8

(a) We know that

$x(t) \,.\, s(t) \;\rightarrow\; (FT) \;\rightarrow\; X(j\omega) \;*\; S(j\omega)$

$s(t)$ is a periodic square wave of period T.

With $\Delta = T/3$ as shown in the figure.

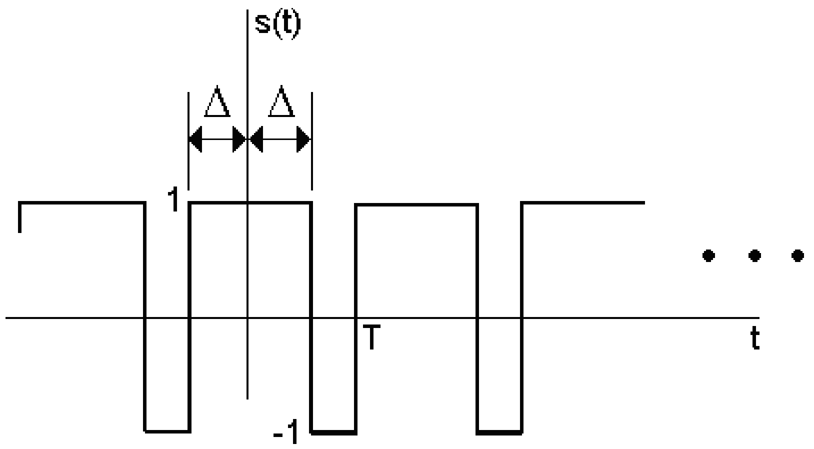


Figure (a)

We calculate $S(j\omega)$ as follows: (FT of a periodic signal)

$$S(j\omega) \quad = \quad \sum a_k \delta(\omega - \omega_0)$$

where,

$$a_k \quad = \quad \frac{1}{T}\int_{(T)} s(t)\exp(-jk\omega_0 t)dt$$

Considering any one period T, (say from 0 to T) and

Substituting $\omega_0 = 2\pi / T$

$$a_k = \frac{1}{(j2\pi k)}\{1 - 2\exp(-j4\pi k/3) + \exp(-j2\pi k)\}$$

$$= \frac{-1}{(j\pi k)}\{\exp(-j4\pi k/3)\}$$

which can never be 0.

Thus, $S(j\omega)$ is an impulse train situated at intervals of ω_0.

And ω_M has a maximum value of $1/2(2\pi/T)$.

$T \leq \pi/\omega_M$ (Maximum value of T without aliasing).

(b) We know that
$x(t)\,.\,s(t) \;\rightarrow\; (FT) \;\rightarrow\; X(j\omega) \;*\; S(j\omega)$
$s(t)$ is a periodic square wave of period T.

With $\Delta = T/4$ as shown in the figure.

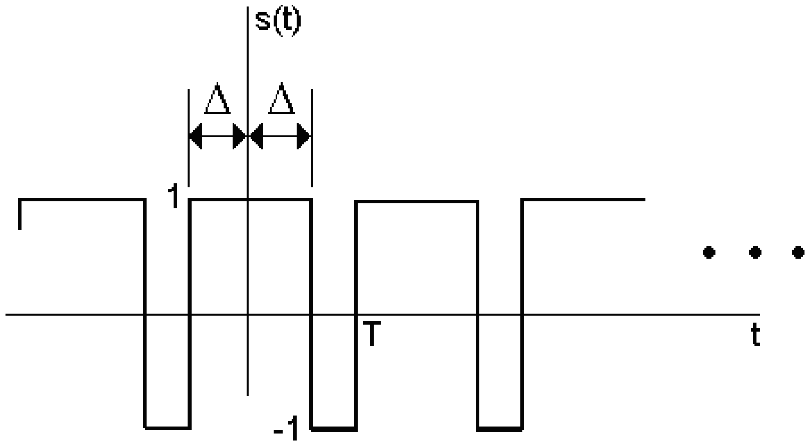


Figure (b)

We calculate $S(j\omega)$ as follows: (FT of a periodic signal)

$$S(j\omega) \quad = \quad \sum a_k \delta(\omega - \omega_0)$$

where

$$a_k \quad = \quad \frac{1}{T}\int_{(T)} s(t)\exp(-jk\omega_0 t)dt$$

Considering any one period T, (say from 0 to T) and

Substituting $\omega_0 = 2\pi/T$

$$a_k = \frac{1}{(j2\pi k)}\{1 - 2\exp(-j\pi k) + \exp(-j2\pi k)\}$$

$= 0$ for $k = 2m$ (i.e. k is even)

Thus, $S(j\omega)$ is an impulse train situated at intervals of $2\omega_0$.

And ω_M has a maximum value of $1/2(2*2\pi/T)$.

$T \leq 2\pi/\omega_M$ (Maximum value of T without aliasing).

Problem 9 :

Figure I shows the overall system for filtering a continuous-time signal using a discrete time filter. If $X_c(jw)$ and H(exp(jw)) are as shown in **Figure II**, with 1/T=20kHz, sketch $X_p(jw)$, X(exp(jw)), Y(exp(jw)), $Y_p(jw)$ and $Y_c(jw)$.

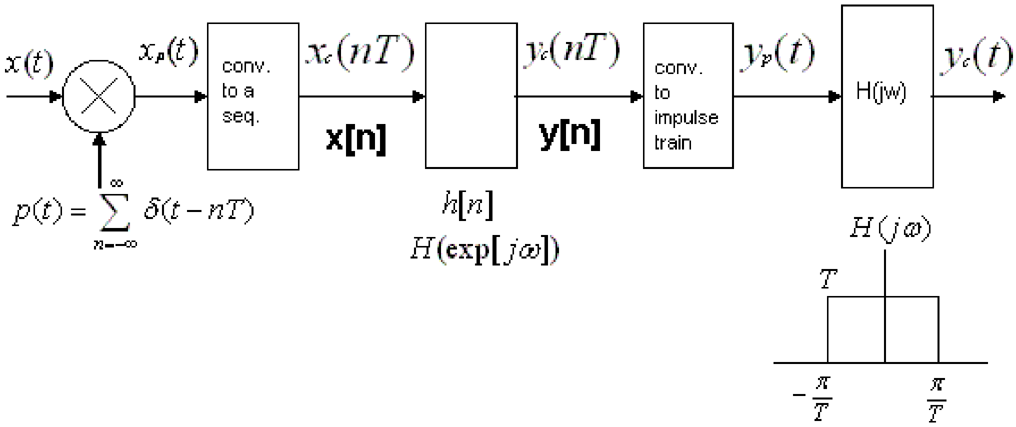


Figure (I)

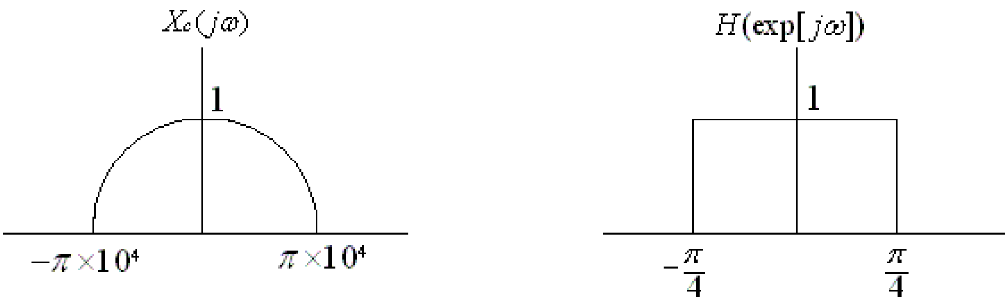
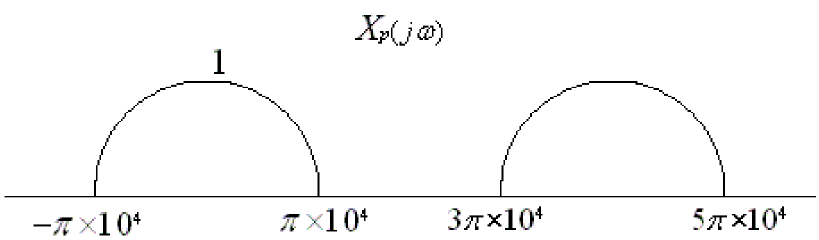
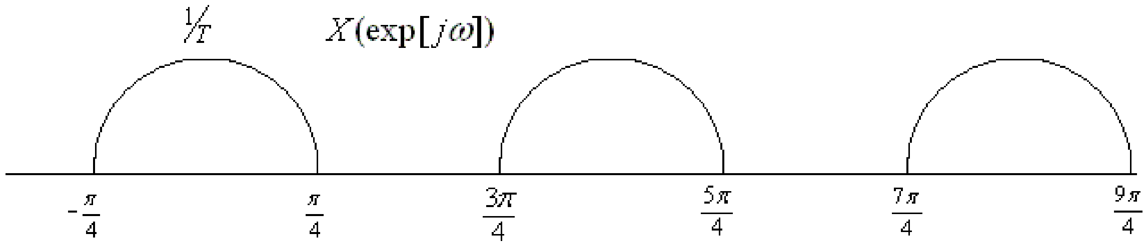
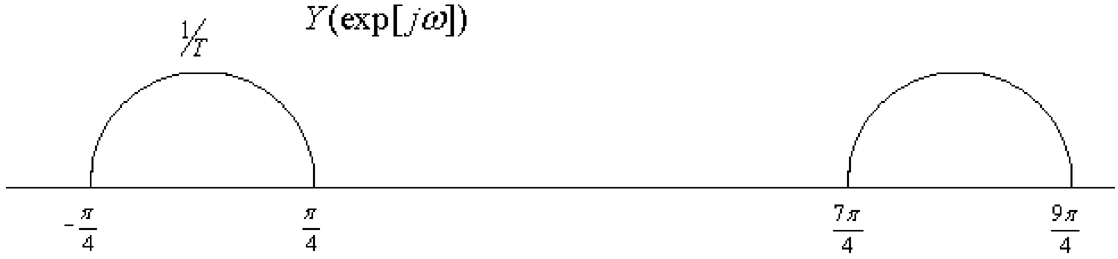


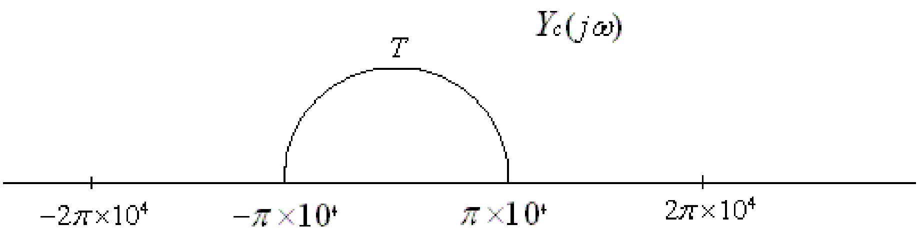
Figure (II)

Solution 9 :







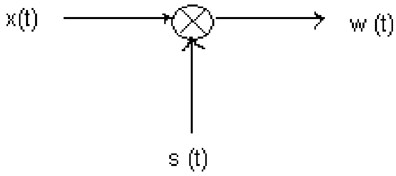


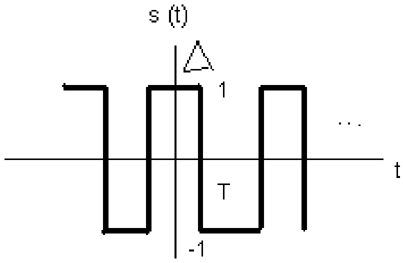
Problem 10 :

Shown in figure below is a system in which the input signal is multiplied by a periodic square wave. The period of s (t) is T. The input signal is band limited with | X(jw) |=0 for |w|>w_m .

(a) For $\Delta = T/3$ determine, in terms of w_m , the maximum value of T for which there is no aliasing among the replicas of X (jw) in W(jw).

(b) For $\Delta = T/4$ determine, in terms of w_m , the maximum value of T for which there is no aliasing among the replicas of X (jw) in W(jw).


x(t)
w (t)
s (t)


s (t)
Δ
1
. . .
t
T
-1