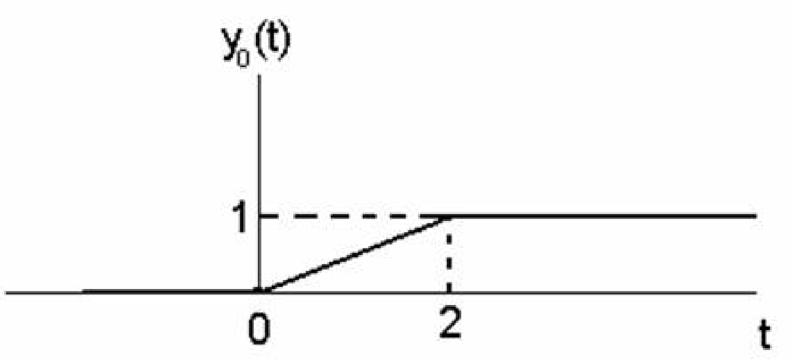
Module 1 : Signals in Natural Domain

Problem 1

We are given a certain linear time - invariant system with impulse response $h_0(t)$. We are told that when the input is $x_0(t)$ the output is $y_0(t)$, which is sketched in figure below. We are then given the following set of inputs to linear time - invariant systems with the indicated impulse responses:

	Input x(t)	**Impulse response h(t)**
(a)	$x(t) = 2x_0(t)$	$h(t) = h_0(t)$
(b)	$x(t) = x_0(t) - x_0(t-2)$	$h(t) = h_0(t)$
(c)	$x(t) = x_0(t-2)$	$h(t) = h_0(t+1)$
(d)	$x(t) = x_0(-t)$	$h(t) = h_0(t)$
(e)	$x(t) = x_0(-t)$	$h(t) = h_0(-t)$
(f)	$x(t) = x_0'(t)$	$h(t) = h_0'(t)$

[Here $x_0'(t)$ and $h_0'(t)$ denote the first derivative of $x_0(t)$ and $h_0(t)$, respectively.]



In each of these cases, determine whether or not we have enough information to determine $y(t)$ when the input is $x(t)$ and the system has impulse response $h(t)$. If it is possible to determine $y(t)$, provide an accurate sketch of it with numerical values clearly indicated on the graph.

Solution 1

Given,

$y_0(t) = x(t) \ * \ h(t)$

$y_0(t) = \int_{-\infty}^{\infty} h_0(\lambda) x_0(t-\lambda) d\lambda = \int_{-\infty}^{\infty} x_0(\lambda) h_0(t-\lambda) d\lambda$ --- (1)

(a)

$y(t) = x(t) \ * \ h(t)$

$= \int_{-\infty}^{+\infty} h(\lambda) x(t-\lambda) d\lambda$

$= \int_{-\infty}^{+\infty} h_0(\lambda) . 2x_0(t-\lambda) d\lambda$

$= 2y_0(t)$ --- From (1)

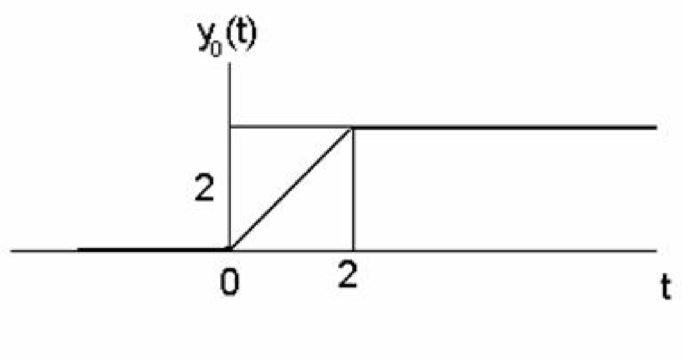


Figure (a)

(b)

$y(t) = x(t) * h(t)$

$= \int_{-\infty}^{+\infty} x(\lambda)h(t-\lambda)d\lambda$

$= \int_{-\infty}^{+\infty} \{x_0(\lambda) - x_0(\lambda-2)\}h(t-\lambda)d\lambda$

$= \int_{-\infty}^{+\infty} x_0(\lambda)h(t-\lambda)d\lambda - \int_{-\infty}^{+\infty} x_0(\lambda-2)h(t-\lambda)d\lambda$

Put $\lambda - 2 = k$

$= \int_{-\infty}^{+\infty} x_0(\lambda)h(t-\lambda)d\lambda - \int_{-\infty}^{+\infty} x_0(k)h(t-k-2)dk$

$= y_0(t) - y_0(t-2)$

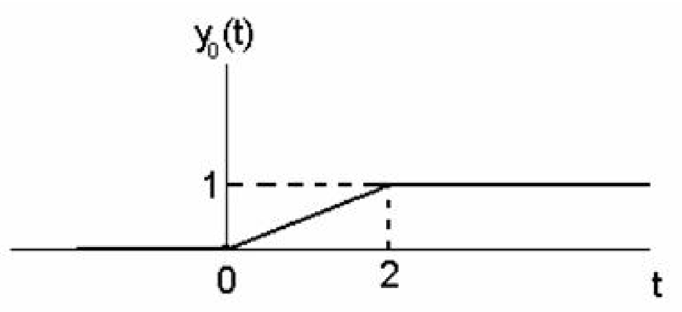


Figure (b)

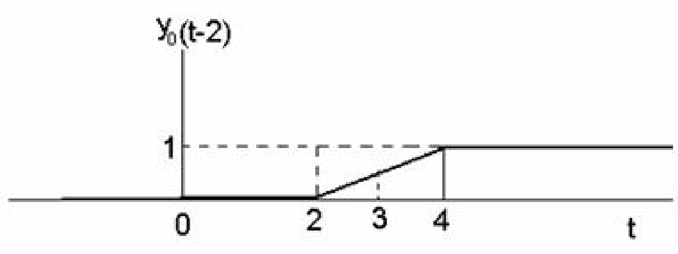


Figure (c)

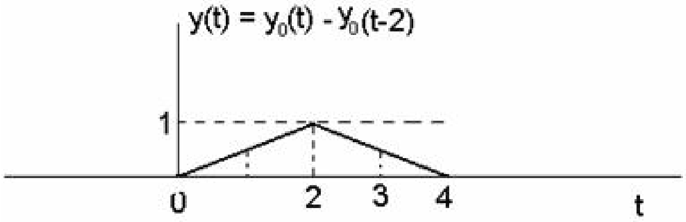


Figure (d)

(c)

$y(t) = x(t) \ * \ h(t)$

$= \int_{-\infty}^{+\infty} h(\lambda)x(t-\lambda)d\lambda$

$= \int_{-\infty}^{+\infty} h_0(\lambda+1)x_0(t-\lambda-2)d\lambda$

Put $\lambda+1=k$

$= \int_{-\infty}^{+\infty} h_0(k)x_0(t-k-1)dk$

$= y_0(t-1)$

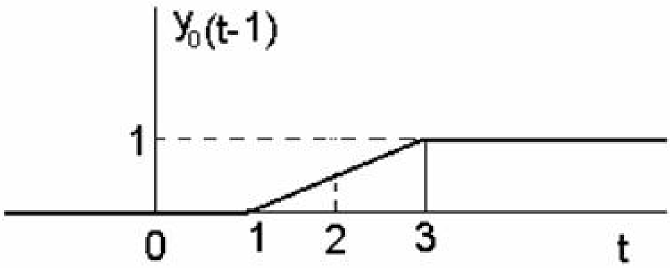


Figure (e)

(d)

$y(t) = \int_{-\infty}^{+\infty} h(\lambda)x(t-\lambda)d\lambda$

$= \int_{-\infty}^{+\infty} h_0(\lambda)x_0(\lambda-t)d\lambda$

In this case, we do not have enough information to predict $y(t)$.

(e)

$y(t) = \int_{-\infty}^{+\infty} h(\lambda)x(t-\lambda)d\lambda$

$= \int_{-\infty}^{+\infty} h_0(-\lambda)x_0(\lambda-t)d\lambda$

Put $\lambda=-k$

$$= \int_{-\infty}^{+\infty} h_0(k) x_0(-k-t) dk$$

$$= y(-t)$$

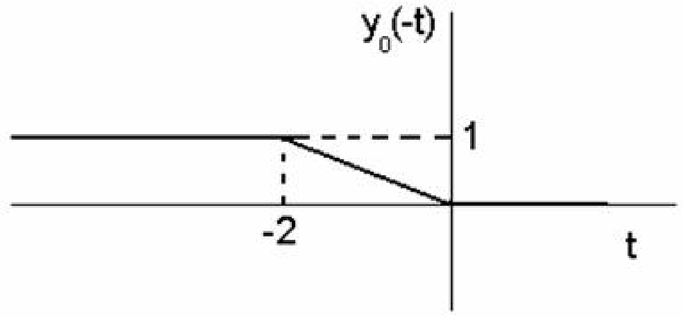


Figure (f)

(f) In this case, we do not have enough information to predict $y(t)$.

Problem 2

Determine whether each of the following statements concerning LTI systems is true or false. Justify your answer.

(a) If $h(t)$ is the impulse response of an LTI system and $h(t)$ is periodic and nonzero, the system is unstable.

(b) The inverse of a causal LTI system is always causal.

(c) If $|h[n]| \leq K$ for each n , where K is a given number, then the LTI system with $h[n]$ as its impulse response is stable.

(d) If a discrete - time LTI system has an impulse response $h[n]$ of finite duration, the system is stable.

(e) If an LTI system is causal, it is stable.

(f) The cascade of a non causal LTI system with a causal one is necessarily non - causal.

(g) A continuous - time LTI system is stable if and only if its step response $s(t)$ is absolutely integrable, that is, if and only if

$$\int_{-\infty}^{+\infty} |s(t)| dt \leq \infty$$

(h) A discrete - time LTI system is causal if and only if its step response $s[n]$ is zero for $n < 0$.

Solution 2

(a) True

Proof: For a system to be stable

$$\int_{-\infty}^{+\infty} |h(t)| dt = M; \quad M \geq 0$$

If h is periodic with period P (say), then

$$\int_{-\infty}^{+\infty} |h(t)| dt = N \int_{-P/2}^{+p/2} |h(t)| dt \qquad \text{(where } N \to \infty)$$

$h(t)$ is non-zero

$$\int_{-\infty}^{+\infty} |h(t)| dt \to \infty$$

Hence unstable.

(b) False

Counter example :

Let an LTI system

$y(t) = x(t - t_0)$

Or

$x(t) \to x(t - t_0)$

Which is causal

For its inverse system, $x(t)$ will be output and $y(t)$ will be input.

i.e.

$$\boldsymbol{x(t - t_0) \to x(t)}$$

$x(t - t_0 + t_0) \to x(t + t_0)$ [Using shift invariance]

Which is non-causal.

(c) False

Counter example :

Let

$h[n] = k/2 \;\; \forall k > 0$

Here

$h[n] \le k \;\; \forall n \in I$

But

$$\sum_{-\infty}^{+\infty} |h[n]| = \sum_{-\infty}^{+\infty} k/2$$

Which is not bounded.

Hence , the impulse response must be absolutely summable , for the system to be stable.

(d) False

Counter example :

Let

$h[n] = \tan(\frac{\pi n}{4})$ for $-4 \le n \le 4$

Which is of finite duration

But

$$\sum_{-\infty}^{+\infty} h[n] = \sum_{n=-4}^{+4} \tan\left(\frac{n\pi}{4}\right)$$

diverges (i.e. it is not absolutely summable) .

Hence the system is unstable.

(e) False

Counter example:

Let

$$h[t] = \begin{cases} 0 & \forall, t < 0 \\ k & \forall, t > 0 \end{cases}$$

which is causal. but unstable.

(f) False.

Counter example:

Let

$$x(t) \xrightarrow{h_1(t)} x(t + t_0)$$

which is non-causal

$$x(t) \xrightarrow{h_2(t)} x(t - t_0)$$

which is causal.

Cascade of the two system

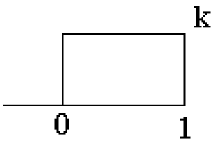
$$x(t) \xrightarrow{h_1(t)} \xrightarrow{h_2(t)} x(t)$$

which is clearly causal.

(g) False

Counter example:

Let

$h(t) =$ 

Here

$$s(t) = u(t) * h(t)$$

$$= \int_{-\infty}^{+\infty} h(\lambda) u(t - \lambda) d\lambda$$

$$= \int_{-\infty}^{t} h(\lambda) d\lambda$$

$$= k$$

but

$$\int_{-\infty}^{+\infty} |h(t)| dt \quad = \quad \int_{-\infty}^{+\infty} k \; dt$$

which is divergent.

(h) True

Sufficiency:

$h[n] = s[n] - s[n-1]$

if

$s[n] = 0 \ \forall \ n < 0$

Let n < 0

$s[n] = s[n-1] = 0$

$h[n] = 0 \ \forall \ n < 0$

=>causality.

Necessity :

$h[n] = 0 \ \forall \ n < 0$

$s[n] = h[n] * u[n]$

$= \sum_{\lambda=-\infty}^{+\infty} u[n] h[n-\lambda]$

$= \sum_{\lambda=0}^{+\infty} h[n-\lambda]$

if

$n < 0$,

then

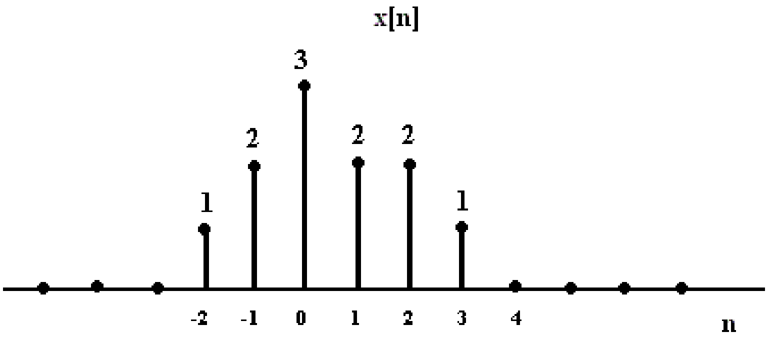
$n - \lambda < 0 \ \forall \ \lambda < 0$

Thus s[n] = 0 for n < 0 .

Problem 3

Consider the LTI system initially at rest and described by the difference equation $y[n]+2y[n-1]=x[n]+2x[n-2]$.

Find the response of this system to the input depicted in the figure by solving the difference equation recursively.



Solution 3

The input can be written as:

$$x[n]: \quad \underset{\substack{\uparrow \\ -2}}{1} \quad 2 \quad 3 \quad 2 \quad 2 \quad 1$$

As given, the LTI system is initially at rest. Hence, since $x[n]=0$ for $n<-3$, thus, $y[n]=0$ for $n<-3$ (By the definition of causality).

Now we recursively insert values of input $x[n]$ in the difference equation.

Putting:

$n=-2$,	$y[-2]+2y[-3]=x[-2]+2x[-4]$	Thus, $y[-2]=1$.
$n=-1$,	y [-1] + 2y [-2] = x [-1] + 2x [-3]	Thus, $y[-1]=0$.
$n=0$,	$y[0]+2y[-1]=x[0]+2x[-2]$.	Thus, $y[0]=5$.
$n=1$,	$y[1]+2y[0]=x[1]+2x[-1]$.	Thus, $y[1]=-4$.
$n=2$,	$y[2]+2y[1]=x[2]+2x[0]$.	Thus, $y[2]=16$.
$n=3$,	$y[3]+2y[2]=x[3]+2x[1]$.	Thus, $y[3]=-27$.
$n=4$,	$y[4]+2y[3]=x[4]+2x[2]$.	Thus, $y[4]=58$.
$n=5$,	$y[5]+2y[4]=x[5]+2x[3]$.	Thus, $y[5]=-114$.
$n=6$,	$y[6]+2y[5]=x[6]+2x[4]$.	Thus, $y[6]=228$.

and so on.

Thus the output y[n] becomes as shown

$$y[n] \; : \; \underset{\substack{\uparrow \\ -2}}{1} \quad 0 \quad 5 \quad -4 \quad 16 \quad -27 \quad 58 \quad -114 \quad 228 \quad \ldots$$

Thus we see that the output is unbounded.

Problem 4

Consider the cascade interconnection of three causal LTI systems, illustrated in Figure (a). The impulse response is $h_2[n]=u[n]-u[n-2]$ and the overall impulse response is as shown in Figure (b).

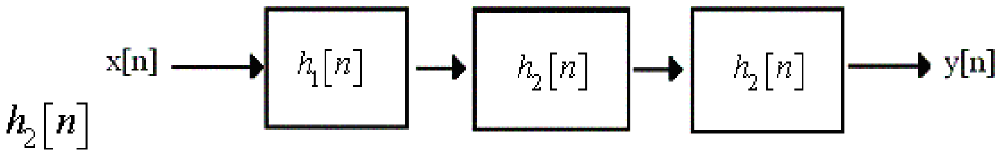


Fig (a)

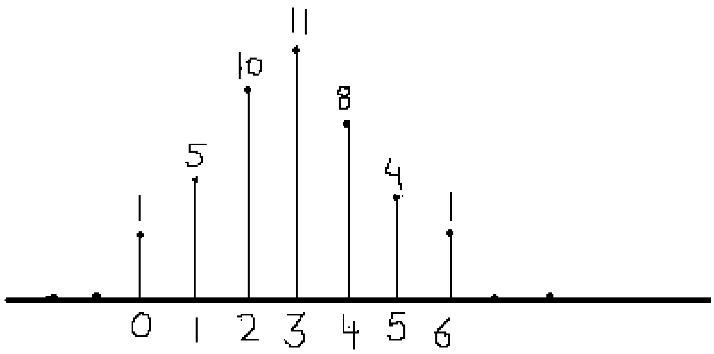


Fig (b)

(a) Find the impulse response $h_1[n]$.

(b) Find the response of the overall system to the input $x[n]=\delta[n]-\delta[n-1]$.

Solution 4

(a) We know that the overall impulse response of cascaded systems is the convolution of the impulse responses of the individual systems.

Let the overall impulse response of the cascaded system be $h[n]$.

Therefore using the above mentioned property, we have

$$h[n] = h_1[n] * h_2[n] * h_2[n]$$

Since the convolution is associative in nature therefore we can say that

$$h[n] = h_1[n] * (h_2[n] * h_2[n])$$

So first convolving $h_2[n]$ with itself:

Let $h_3[n] = h_2[n] * h_2[n]$.

Since $h_2[k]$ is non-zero for k = 0 and 1 only, therefore we can write that

$$h_3[n] = \sum_{k=0}^{1} h_2[k] h_2[n-k]$$

Therefore:

$$h_3[0] = \sum_{k=0}^{1} h_2[k] h_2[-k]$$

$(1)(1)+(1)(0) = 1$

$$h_3[1] = \sum_{k=0}^{1} h_2[k] h_2[1-k]$$

$$= (1)(1) + (1)(1) = 2$$

$$h_3[2] = \sum_{k=0}^{1} h_2[k] h_2[2-k]$$

$$= (1)(0) + (1)(1) = 2$$

Therefore, we have

$$h_3[n]: \quad \underset{\underset{0}{\uparrow}}{1} \qquad 2 \qquad 1$$

Now h[n] is nonzero from 0 to 6 and $h_3[n]$ is nonzero from 0 to 2.

Also, $h[n] = h_1[n] * h_3[n]$.

Therefore $h_1[n]$ will be nonzero from 0 to 4.

(If a signal is non-zero in the interval (a , b)and it is convoluted with another signal which is non zero in the interval (c,d) then the convoluted signal is non zero in the interval (a+c,b+d).)

Let

$$h_1[n] = \underset{\substack{\uparrow \\ 0}}{a} \quad b \quad c \quad d \quad e$$

Now,

$$h[n] = h_1[n] * h_3[n] = \sum_{k=0}^{4} h_1[k]\, h_3[n-k]$$

Therefore, we have

$$h[0] = \sum_{k=0}^{4} h_1[k]\, h_3[-k]$$

$= (a)(1) = 1$ (from the Figure (b) we have h[0] = 1)

$\Rightarrow a = 1$.

$$h[1] = \sum_{k=0}^{4} h_1[k]\, h_3[1-k]$$

$= (a)(1) + (b)(1) = 5$

$\Rightarrow b = 3$

$$h[2] = \sum_{k=0}^{4} h_1[k]\, h_3[2-k]$$

$= (a)(1) + (b)(2) + (c)(1) = 10$

$\Rightarrow c = \mathbf{3}$.

$$h[3] = \sum_{k=0}^{4} h_1[k]\, h_3[3-k]$$

$= (b)(1) + (c)(2) + (d)(1) = 11$

$\Rightarrow d = \mathbf{2}$

$$h[4] = \sum_{k=0}^{4} h_1[k]\, h_3[4-k]$$

$= (c)(1) + (d)(2) + (e)(1) = 8$

$\Rightarrow e = \mathbf{1}$.

Therefore,

$$
\begin{array}{llllll}
h_1[n]: & 1 & 3 & 3 & 2 & 1 \\
 & \uparrow & & & & \\
 & 0 & & & &
\end{array}
$$

(b) We have

$x[n] = \delta[n] - \delta[n-1]$

To get the response, we have to convolve it with h[n] which we obtained in the part **(a)**

$$y[n] = \sum_{-\infty}^{+\infty} h[k]x[n-k]$$

Proceeding as in part **(a)** above, i.e. putting different values of n and correspondingly calculating y[n],finally we get the answer

y[0] =1 , y[1] = 4 , y[2] = 5, y[3] = 1 , y[4] = -3 , y[5] = -4 , y[6] = -3 , y[7] = -1.

Therefore,

$$
\begin{array}{lllllllll}
y[n]: & 1 & 4 & 5 & 1 & -3 & -4 & -3 & -1 \\
 & \uparrow & & & & & & & \\
 & 0 & & & & & & &
\end{array}
$$

Problem 5

Let x(t) be a continuous-time signal, and let

$$y_1(t) = x(2t) \text{ and } y_2(t) = x(t/2).$$

The signal y_1 represents a speeded up version of x(t) in the sense that the duration of the signal is cut in half. Similarly, $y_2(t)$ represents a slowed down version of x(t) in the sense that the duration of the signal is doubled.

Consider the following statements :

(1) If x(t) is periodic, then $y_1(t)$ is peiodic.
(2) If $y_1(t)$ is periodic, then x(t) is peiodic.
(3) If x(t) is periodic, then $y_2(t)$ is peiodic.
(4) If $y_2(t)$ is periodic, then x(t) is peiodic.

For each of these statements, determine whether it is true, and if so, determine the relationship between the fundamental periods of the two signals considered in the statement.

If the statement is not true, produce a counterexample to it.

Solution 5 :

We observe that , all the three functions are compressed / expanded versions of each other .
Thus, if we can prove any one of the given statements to be true/false , the other ones become explicitly implied (in case of continuous signals) .

(1)

From the given information , we have

$$x(t) \xrightarrow[\text{System}]{\text{Linear}} y_1(t) = x(2t)$$

Let T be the period of x(t) . Then , we have

$$x(t) = x(t+T)$$

Inputting x(t + T) in the LSI system, will give the output as follows :

$$\begin{aligned} x(t+T) \xrightarrow[\text{System}]{\text{Linear}} & x(2t+T) \\ &= x\left(2\left\{t+\frac{T}{2}\right\}\right) \\ &= y_1\left(t+\frac{T}{2}\right) \end{aligned}$$

Thus we have

$$y(t) = y\left(t+\frac{T}{2}\right)$$

which proves the first statement to be true. And the period of the resultant signal is also compressed by half.

NOTE : The given system is linear and shift **variant .**

Thus the remaining three statements are also true .
And the period of the expanded signal will be twice that of original signal.

Problem 6

Let x(t) be a continuous-time complex exponential signal,

$$x(t) = e^{j\omega 0t}$$

with fundamental frequency w_0 and fundamental period

$$T_0 = \frac{2\pi}{w_0}$$

Consider the discrete-time signal obtained by taking equally spaced samples of x(t) - that is

$$x[n] = x(nT) = e^{j\omega_0 t}$$

(a) Show that x[n] is periodic if and only if T / T_0 is a rational number - that is, if and only if some multiple of the sampling interval exactly equals a multiple of the period of x(t) .

(b) Suppose that x [n] is periodic - that is, that

$$\frac{T}{T_0} = \frac{p}{q}$$

where p and q are integers. What are the fundamental period and fundamental frequency as a fraction of $w_0 T$.

(c) Again assuming that T / T_0 satisfies the above equation, determine precisely how many periods of x(t) are needed to obtain the samples that form a single period of x[n] .

Solution 6:

(a) For x [n] to be periodic with fundamental period N,

$$\begin{aligned} x[n] = x[n+N] &= e^{j\omega_0 t(n+N)} \\ &= e^{j\omega_0 tn} e^{j\omega_0 tN} \\ &= e^{j\omega_0 tn} \end{aligned}$$

For this to be true , following condition should be satisfied ...

$$\begin{aligned} & e^{j\omega_0 tN} = 1 \\ \therefore\quad & \omega_0 tN = 2k\pi \\ \therefore\quad & \frac{2\pi}{T_0} Nt = 2k\pi \\ \therefore\quad & \frac{T}{T_0} = \frac{k}{N} \end{aligned}$$

as required to prove.

Converse is also easy to observe.

(b) Now if $T / T_0 = p / q$,

fundamental period = q

fundamental frequency = 1 / q = 1/p (k / N)

And thus we have :

$$\begin{aligned} & \because \frac{k}{N} = \frac{\omega_0 t}{2\pi} \\ & \therefore \frac{1}{q} = \frac{\omega_0 t}{2\pi p} \end{aligned}$$

(c) Now assuming $T / T_0 = p / q = k / N$

$$\text{For } k = p,$$
$$N = \frac{pT_0}{T} = \frac{2\pi p}{\omega_0 T}$$

Thus , p / T periods of x(t) are needed to obtain samples that form a single period of x [n].

Problem 7

(a) Show that if a system is either additive or homogenous , it has the property that if the input is identically zero, then the output is also identically zero.

(b) Determine a system (either in continuous or discrete time) that is neither additive nor homogeneous but which has zero output if the input is identically zero .

(c) From part (a) , can you conclude that if the input to a linear system is zero between times t1 and t2 in continuous time or between times n1 and n2 in discrete time, then its output must also be zero between these same times ? Explain your answer.

Solution 7 :

(a)

(i) If the system is additive , following relation is true .

$$x_1(t) + x_2(t) \xrightarrow[\text{System}]{\text{Additive}} y_1(t) + y_2(t)$$

Now if the input is identically zero , that is

$$x_1(t) = x_2(t) = x(t) = 0$$

When this is given as input to the given system , we get

$$x(t) + x(t) \xrightarrow[\text{System}]{\text{Additive}} y(t) + y(t) = 2y(t)$$
$$2x(t) = x(t)$$
$$\text{Thus, } \ 2y(t) = y(t)$$
$$\textit{Thus}, \ y(t) = 0$$

(ii) Now , if the system is homogenous , we proceed as follows :

$$2x(t) \xrightarrow[\text{System}]{\text{Homogenous}} 2y(t)$$
$$\text{Now if x(t)=0,}$$
$$2x(t)=x(t)$$
$$\text{Thus, } 2y(t)=y(t)$$
$$\textit{Thus} \ \ y(t) = 0$$

(b) Consider a system, which is described by the following relation

$$y(t) = \sin x(t)$$

Observe that it has following properties : Neither additive nor homogenous ...

$$\sin a\{x(t)\} \not\to a \sin x(t)$$
$$\sin\{x_1(t) + x_2(t)\} \not\to \sin x_1(t) + \sin x_2(t)$$

But if the input is identically zero , sine of the input would also be identically zero .

(c) NO. It cannot be concluded that if the input is zero for a time interval , the output must also be zero for the same interval.

For example, consider a system which gives an output which is a delayed version of the input :-

$$y(t) = x(t-1)$$

Now if $\quad x(t) = 0 \quad 3 < t < 4$
$\quad\quad = 1 \quad$ elsewhere

Then the output will not be zero in the interval (3,4) ...

Problem 8 :

In the concept of correlation functions , it is often important in practice to compute the correlation function $\phi_{hx}(t)$, where h (t) is a fixed given signal, but where x (t) may be any of a wide variety of signals. Let S be the system which takes x (t) as the input and gives $\phi_{hx}(t)$ as the output .

(a) Is S linear ? Is S time invariant ? Is S causal ? Explain your answers .

(b) Do any of your answer to above part change if we take as the output $\phi_{xh}(t)$ instead of $\phi_{hx}(t)$?

Solution 8 :

By definition ;

$$\phi_{hx}(t)=\int_{-\infty}^{\infty} h(t+\tau)x(\tau)d\tau$$

Replacing x (t) by cx(t) where c is a real number ,

$$\tilde{\phi}_{hx}(t)=\int_{-\infty}^{\infty} h(t+\tau)cx(\tau)d\tau=c\phi_{hx}(t)$$

Similarly putting ,

$\tilde{x}(t)=x_1(t)+x_2(t)$, we get

$$\tilde{\phi}_{hx}(t)=\int_{-\infty}^{\infty} h(t+\tau)\tilde{x}(\tau)d\tau=\int_{-\infty}^{\infty} h(t+\tau)(x_1(\tau)+x_2(\tau))d\tau$$

$$=\int_{-\infty}^{\infty} h(t+\tau)x_1(\tau)d\tau+\int_{-\infty}^{\infty} h(t+\tau)x_2(\tau)d\tau$$

$$=\phi_{hx_1}(t)+\phi_{hx_2}(t)$$

Thus , $\phi_{hx}(t)$ obeys homogeneity and additivity .

Implies $\phi_{hx}(t)$ is LINEAR .

NEXT , substituting x (t) by x $(t-t_0)$

$$\tilde{\phi}_{hx}(t)=\int_{-\infty}^{\infty} h(t+\lambda+t_0)x(\lambda)d\lambda$$

Now , Substituting , $\lambda=\tau-\tau_0$,we get $\tilde{\phi}_{hx}(t)=\int_{-\infty}^{\infty} h(t+\lambda+t_0)x(\lambda)d\lambda$

$$\phi_{hx}(t-t_0)=\int_{-\infty}^{\infty} h(t-t_0+\tau)x(\tau)d\tau$$

Hence $\phi_{hx}(t-t_0)$ need not be equal to $\tilde{\phi}_{hx}(t)$ for all x (t) and t_0.

Hence $\phi_{hx}(t)$ is shift-variant .

Let $h(t)=e^{-t}$ and let x (t) = u (t)

Now , $$\phi_{hx}(t)=\int_{-\infty}^{\infty} h(t+\tau)x(\tau)d\tau=\int_{0}^{\infty} e^{-t-\tau}d\tau=e^{-t}$$

Even though x (t)=0 for t < 0 , $\phi_{hx}(t)$ is non-zero for t < 0 .

This counterexample shows that the auto correlating system is non-causal.

(b) The system remains linear (homogeneous and additive) and non-causal. The system becomes shift invariant.

$$\phi_{hx}(t)=\int_{-\infty}^{\infty} h(t+\tau)x(\tau)d\tau$$

Replacing x (t) by x ($t-t_0$) , we get

$$\tilde{\phi}_{xh}(t)=\int_{-\infty}^{\infty} x(t-t_0+\tau)h(\tau)d\tau$$ and $$\phi_{xh}(t-t_0)=\int_{-\infty}^{+\infty}(t-t_0+\tau)h(\tau)d\tau$$

Hence $\tilde{\phi}_{xh}(t)=\phi_{xh}(t-t_0)$ for all signals x (t) and all values of t_0 .

Thus $\phi_{xh}(t)$ is a shift-invariant system .

Problem 9 :

(a) Show that if the response of an LTI system to $x(t)$ is the output $y(t)$, then the response of the system to

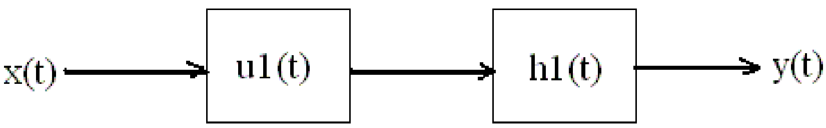
$$x'(t)=\frac{dx(t)}{dt}$$

is $y'(t)$. Do this problem in three different ways:

(i) Directly from the properties of linearity and time invariance and the fact that:

$$x'(t)=\lim_{h\to 0}\frac{x(t)-x(t-h)}{h}$$

(ii) By differentiating the convolution integral.
(iii) By examining the system in Figure 1.



(b) Demonstrate the validity of the following relationships :-

$$(i)\ y'(t) = x(t) * h'(t)$$

$$x(t)=\sum_{k=0}^{\infty} A_k \cos(w_k t+\phi_k)$$

[Hint: These are easily done using block diagrams as in (iii) of part (a) and the fact that
$u_1(t) * u_{-1}(t)=\delta(t)$]

(c) An LTI system has the response $y(t)$ = sinwot to input $x(t)$ = exp [-5t] . $u(t)$. Use the result of part (a) to aid in determining the impulse response of this system.

(d) Let $s(t)$ be the unit step response of a continuous-time LTI system. Use part (b) to deduce that the response $y(t)$ to the input $x(t)$ is

$$y(t) = \int_{-\infty}^{+\infty} x'(\tau) s(t-\tau) d\tau$$

(I)

Show also that

$$x(t) = \int_{-\infty}^{+\infty} x'(\tau) \mathbf{u}(t-\tau) d\tau. \quad (II)$$

(e) Use equation (I) to determine the response of an LTI system with step response

$$s(t) = (e^{-3t} - 2e^{-2t} + 1)u(t)$$

to the input *x (t) = exp [t] . u (t).*

(f) Let *s*[*n*] be the unit step response of a discrete - time LTI system. What are the discrete - time counterparts of equations (I) and (II)?

Solution 9 :

(a)

x(*t*) --------LTI system---------> y(t)

=> x(t+h) -----------------> y(t+h) ....(1) [from shift invariance property]

& -*x*(*t*) -------------------> -y(t) ....(2) [from homogeneity]

now applying additivity in above 2 equations

x(t+h) - *x*(*t*) --------------------> y(t+h) - y(t) ....(3)

now applying homogeneity in equation (3)

(1/h) [x(t+h) - x(t)] -----------------> (1/h) [y(t+h) - y(t)]

$$\lim_{h\to 0} (1/h) [x(t+h) - x(t)] \text{ -----------------> } \lim_{h\to 0} (1/h) [y(t+h) - y(t)]$$

x'(t) -------------------> y'(t)

$$(ii) \quad y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$$

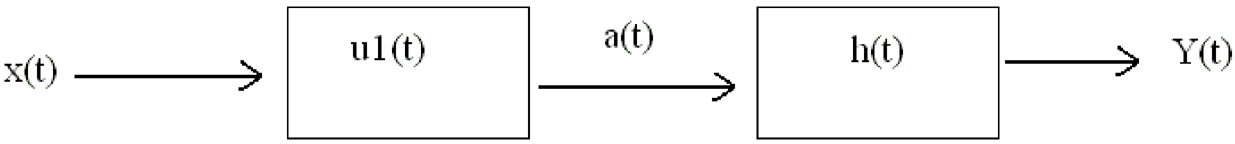
Differentiating both sides w. r. t. t

$$y'(t) = \int_{-\infty}^{\infty} h(\lambda)\, x'(t-\lambda)\, d\lambda$$

$$= h * x'(t)$$

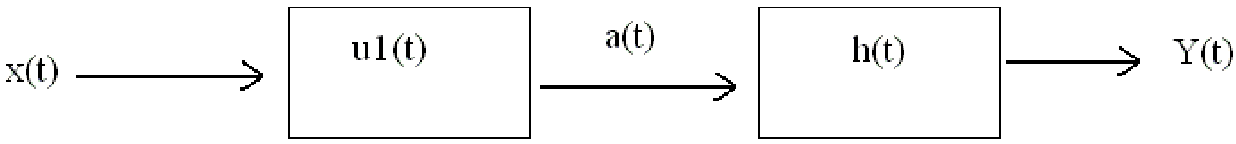
which is the output to the input *x*(*t*)

(iii)

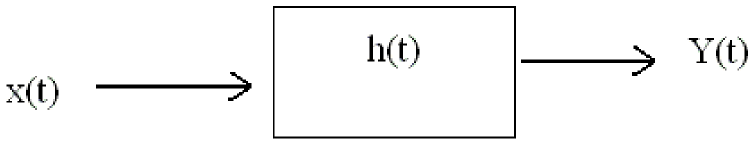


........(1)

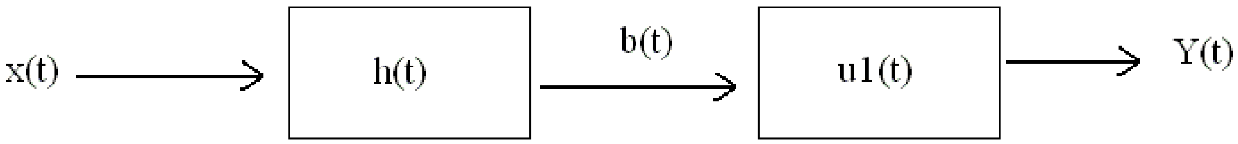
Let



Hence a(t) = x'(t)



We can also write (1) as



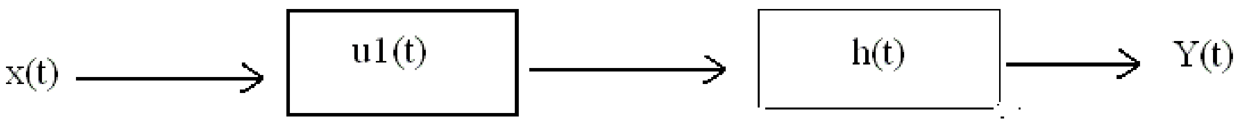
[As Convolution is Commutative]

=> y(t) = b(t)

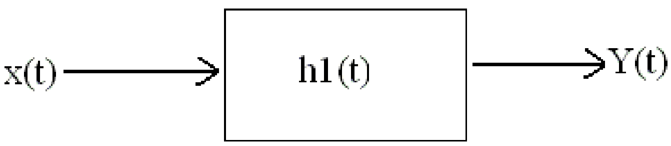
=> Y(t) = y'(t)

hence output to the input x'(t) is y'(t)

(b)



or equivalently



where

$$
\begin{aligned}
h1(t) &= u_1(t) * h(t) \\
&= \int_{-\infty}^{\infty} h(\lambda)\, u_1(t-\lambda)\, d\lambda \\
&= \int_{-\infty}^{\infty} h(\lambda)\, \frac{d}{dt} u(t-\lambda)\, d\lambda \\
&= \frac{d}{dt} \int_{-\infty}^{\infty} h(\lambda)\, u(t-\lambda)\, d\lambda \\
&= h'(t)
\end{aligned}
$$

$$\therefore \quad y'(t) = x(t) * h'(t)$$

$$(ii)\ \ \mathrm{y(t)} = \left(\int_{-\infty}^{t} x(\tau)\, d\tau \right) * h'(t) = x'(t) * \left(\int_{-\infty}^{t} h(\tau)\, d\tau \right)$$

$$x(t) \longrightarrow \boxed{\ h(t)\ } \longrightarrow y(t)$$

$$\Rightarrow \ x(t) \xrightarrow{\delta(t)} \xrightarrow{h(t)} y(t)$$

$$\Rightarrow \ x(t) \xrightarrow{u_{-1}(t)} \xrightarrow{u_1(t)} \xrightarrow{h(t)} y(t)$$

$$\left[\because u_{-1}(t) * u_1(t) = \delta(t) \right]$$

$$\Rightarrow \ \int_{-\infty}^{t} x(\tau)\, d\tau \xrightarrow{u_1(t)} \xrightarrow{h(t)} y(t)$$

$$\Rightarrow \ \int_{-\infty}^{t} x(\tau)\, d\tau \xrightarrow{h'(t)} y(t) \qquad \left[\because h(t) * u_1(t) = h'(t) \right]$$

$$\therefore \ y(t) = \left(\int_{-\infty}^{t} x(\tau) d\tau \right) * h'(t)$$

Again

$$\Rightarrow \ x(t) \xrightarrow{\delta(t)} \xrightarrow{h(t)} y(t)$$

$$\Rightarrow \ x(t) \xrightarrow{u_1(t)} \xrightarrow{u_{-1}(t)} \xrightarrow{h(t)} y(t)$$

$$\left[\because u_{-1}(t) * u_1(t) = \delta(t) \right]$$

$$\Rightarrow \ x'(t) \xrightarrow{u_{-1}(t)} \xrightarrow{h(t)} y(t)$$

$$\Rightarrow \ x'(t) \xrightarrow{h_{eq}(t)} y(t)$$

$$\text{where} \quad \left\{ h_{eq}(t) = u_{-1}(t) * h(t) = \int_{-\infty}^{t} h(\tau) d\tau \right\}$$

$$\therefore \ y(t) = x'(t) * \left(\int_{-\infty}^{t} h(\tau)\, d\tau \right)$$

(c)

$$x(t) = e^{-5t}u(t) \xrightarrow{h(t)} y(t) = \sin \omega_0 t$$

$$\therefore\ x'(t) \xrightarrow{h(t)} y'(t)$$

$$\Rightarrow\ -5e^{-5t}u(t) + e^{-5t}\delta(t) \xrightarrow{h(t)} \omega_0 \sin\omega_0 t$$

$$\Rightarrow\ -5e^{-5t}u(t) + \delta(t) \xrightarrow{h(t)} \omega_0 \sin\omega_0 t \quad \ldots(1)$$

$$\left[\because\ e^{-5t}\delta(t) = \delta(t)\right]$$

now $\quad 5e^{-5t}u(t) \xrightarrow{h(t)} 5\sin\omega_0 t \quad \ldots(2)$

[from Homogeinity]

appling linearity property in (1) and (2)

$$\delta(t) \longrightarrow \omega_0 \sin\omega_0 t + 5\sin\omega_0 t$$

(d)

$$s(t) = h(t) * u(t)$$

$$= \int_{-\infty}^{\infty} h(\lambda)\, u(t-\lambda)\, d\lambda$$

$$= \int_{-\infty}^{t} h(\lambda)\, d\lambda \quad \ldots(1)$$

now from part (b)

$$y(t) = x'(t) * \left(\int_{-\infty}^{t} h(\lambda)\, d\lambda\right) \quad \ldots(2)$$

$$\Rightarrow\ y(t) = x'(t) * s(t)$$

now put $h(t) = \delta(t)$ *in equation* (2)

$$x(t) = x'(t) * \left(\int_{-\infty}^{t} \delta(\lambda)\, d\lambda\right)$$

$$\Rightarrow\ x(t) = x'(t) * u(t) \qquad \left[\because \int_{-\infty}^{t} \delta(\lambda)\, d\lambda = u(t)\right]$$

$(e) \quad x(t) = e^t u(t)$

$\Rightarrow x'(t) = e^t u(t) + \delta(t) e^t$

$\Rightarrow x'(t) = e^t u(t) + \delta(t)$

$s(t) = \left(e^{-3t} - 2e^{-2t} + 1\right) u(t)$

$\therefore$

$$y(t) = x'(t) * s(t) \qquad \left[\text{from part}(d)\right]$$

$$= \int_{-\infty}^{+\infty} \left\{e^{\lambda} u(\lambda) + \delta(\lambda)\right\} \left\{e^{-3(t-\lambda)} - 2e^{-2(t-\lambda)} + 1\right\} u(t-\lambda)\, d\lambda$$

$$= \int_0^t e^{\lambda} \left\{e^{-3(t-\lambda)} - 2e^{-2(t-\lambda)} + 1\right\} d\lambda + \left(e^{-3t} - 2e^{-2t} + 1\right) u(t)$$

$$= \int_0^t \left(e^{-3t+4\lambda} - 2e^{-2t+3\lambda} + e^{\lambda}\right) d\lambda \;\; + s(t)$$

$$= \frac{1}{4} e^{-3t}\left(e^{4t} - 1\right) - \frac{2}{3} e^{-2t}\left(e^{3t} - 1\right) + \left(e^t - 1\right) s(t)$$

$$= -\frac{1}{4} e^{-3t} + \frac{2}{3} e^{-2t} \frac{7}{12} e^t - 1 + s(t)$$

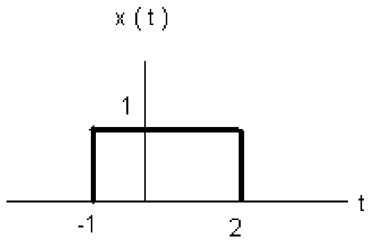
Problem 10 :

(a) Consider an LTI system with input and output related through the equation :-

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau - 2)\, d\tau$$

What is the impulse response h (t) for this system ?

(b) Determine the response of the system when the input x (t) is as shown below :-



Solution 10 :

(a) Comparing the given input output relation with the standard form :

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau - 2)\, d\tau$$

and

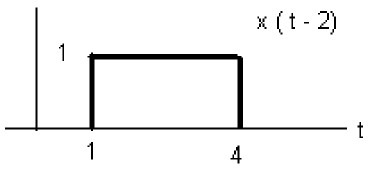
$$y(t) = \int_{-\infty}^{\infty} h(t-\tau) x(\tau)\, d\tau$$

we observe that ,

$$h(t) = e^{-t+2} u(t-2)$$

Also, the input gets shifted by 2 units to the right and then convolved with the impulse response h (t) ...

(b) Now for the given input, first of all shifting the input to the right , we get



Now using the given relation , we get the output as follows :

$$y(t) = \int_{1}^{t} 1.e^{-(t-\tau)} d\tau \qquad \text{for } -1 < t < 2$$

And

$$y(t) = \int_{1}^{4} 1.e^{-(t-\tau)} d\tau \qquad \text{for } t > 2$$

That is :

$$y(t) = 1 - e^{-t+1} \qquad \text{for } -1 < t < 2$$

$$y(t) = e^{-t+4} - e^{-t+1} \qquad \text{for } t > 2$$

Problem 11 :

Let h(t) be the impulse response of a causal and stable LTI system with a rational system function.

Is the system with the impulse response dh(t)/dt guaranteed to be causal and stable ?

(b) Is the system with impulse response $\int_{-\infty}^{t} h(\tilde{t})\, d\tilde{t}$ guaranteed to be causal and unstable ?

Solution 11 :

(a) Causal system implies that the impulse response is zero for $t \leqslant 0$

When we take the derivative, strictly speaking we are not doing a point-wise operation, but are involving h(t-Δt) also. However, the derivative is still zero for any given time $t_0<0$. so, the system remains causal.

Stable system requires the impulse response to be absolutely integrable, i.e. $\int_{-\infty}^{t} \left|\frac{dh}{dt}\right| d\tilde{t} < \infty$

We cannot generalise anything about this without sufficient information about h(t). For example, if h (t) = $e^{-t}u(t)$ then the impulse response of the new function shows that it is stable. But if h (t) = sin (t^2) then dh/dt is 2t $\cos(t^2)$ which is not absolutely integrable, and the system is thus unstable.

(b)

Since h (t) = 0 for t<0, the given integral can be written as $\int_{0}^{t} h(\tilde{t})\, d\tilde{t}$ for t>0, and the integral is easily seen to be zero for t<0. Hence the system is necessarily causal.

For stability, we can consider a system with impulse response h(t) = $e^{-t}u(t)$ so that the integral gives the response of the new system as $\tilde{h}(t) - \int_{0}^{t} (e^{\ t})\, dt - (e^{\ t} - 1)u(t)$ which is bounded above by e^{-t} u (t) and hence is absolutely integrable thus giving a stable system.

On the other hand, we see that systems with impulse responses like h (t) = $1/t^2$ will not lead to a stable derived system.

Problem 12 :

(a) Show that the three LTI systems with impulse response

$$h_1(t) = u(t)$$

$$h_2(t) = -2\,\delta(t) + 5\,e^{-2t}\,u(t)$$

And

$$h_3(t) = 2te^{-t}u(t)$$

All have same responses to

$$x(t) = \cos(t).$$

(b) Find the impulse responses of another LTI system with the same response to cos (t). This problem illustrate the fact that the response to cos (t) cannot be used to specify an LTI system uniquely.

Solution 12 :

(a) $x(t) = \mathrm{Cos}\ t$

Taking Fourier transform

$$X(f) = \frac{1}{2}\{\ \delta(f - \frac{1}{2\pi}) + \delta(f + \frac{1}{2\pi})\ \}$$

Impulse response of the system 1 is

$$h_1(t) = u(t)$$

Frequency response of system 1 is

$$H_1(f) = \frac{1}{2j\pi f} + \pi\delta(f)$$

Similarly for other systems:-

$$h_2(t) = -2\delta(t) + 5e^{-2t}u(t)$$

$$\Rightarrow H_2(f) = \int_{-\infty}^{\infty} [\ -2\delta(t) + 5e^{-2t}u(t)\ ]\ e^{-j2\pi ft}\ dt$$

$$= -2 + \int_0^{\infty} 5\,e^{-t(2+j2\pi f)}\ dt$$

$$H_2(f) = -2 + \frac{5}{2+j2\pi f}$$

$$h_3(t) = 2te^{-t}u(t)$$

Let

$$x_1(t) = e^{-t}u(t)$$

$$X_1(f) = \int_{-\infty}^{\infty} e^{-t}\ e^{-j2\pi ft}\ u(t)\ dt$$

$$= \int_0^{\infty} e^{-t(1+j2\pi f t)}\ dt$$

$$X_1(f) = \frac{1}{1+j2\pi f}$$

$$h_3(t) = 2t\ x_1(t)$$

Therefore,

$$H_3(f) = \frac{2j}{2\pi}\frac{d\,X_1(f)}{df}$$ as frequency response of $t\,x(t) = \frac{j}{2\pi}\frac{d\,X(f)}{df}$ if x(t)'s Fourier transform is X(f)

$$\Rightarrow H_3(f) = \frac{2}{(1+j2\pi f)^2}$$

(b) Let there be another system with the frequency response $H_4(f)$ having the same response to input $x(t) = \text{Cos } t$

$$\Rightarrow \; Y_4(f) = X(f)\, H_4(f)$$

By part (a) we know that

$$Y_4(f) = \frac{1}{2j}\left\{\delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right)\right\}$$

We can rewrite $Y_4(f)$ as:-

$$Y_4(f) = \left[\frac{1}{j}\left\{\delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right)\right\}\right]\left[\frac{1}{2}\left\{\delta\left(f - \frac{1}{2\pi}\right) + \delta\left(f + \frac{1}{2\pi}\right)\right\}\right]$$

$$\Rightarrow Y_4(f) = \left[\frac{1}{j}\left\{\delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right)\right\}\right] X(f)$$

$$\Rightarrow H_4(f) = \left[\frac{1}{j}\left\{\delta\left(f - \frac{1}{2\pi}\right) - \delta\left(f + \frac{1}{2\pi}\right)\right\}\right]$$

By taking Inverse Fourier transform, we get :-

$$h_4(t) = 2 \text{ Sin } t$$