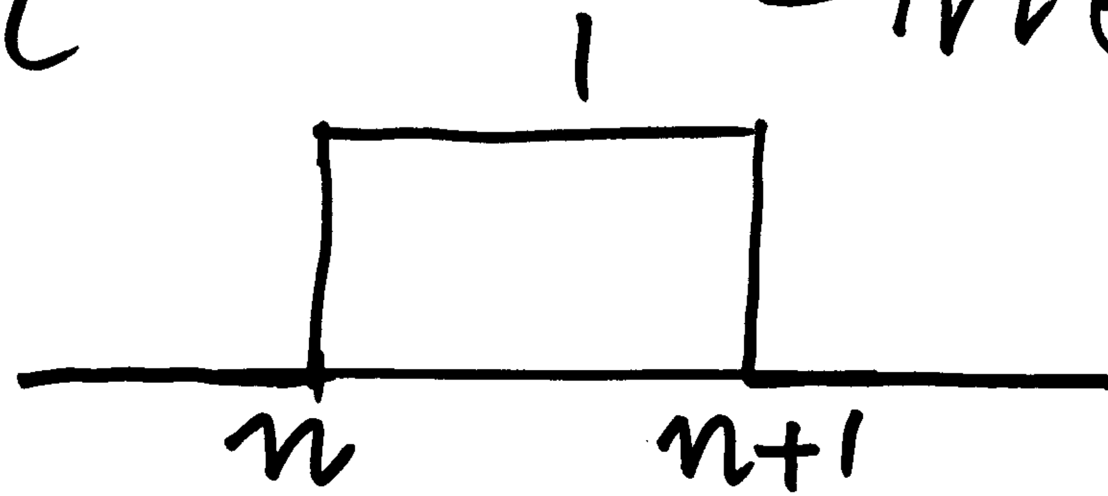
Prof. V. M. Gadr
Date-14-1-10

LECTURE 5

EQUIVALENCE : FUNCTIONS AND SEQUENCES

Basis for V_0

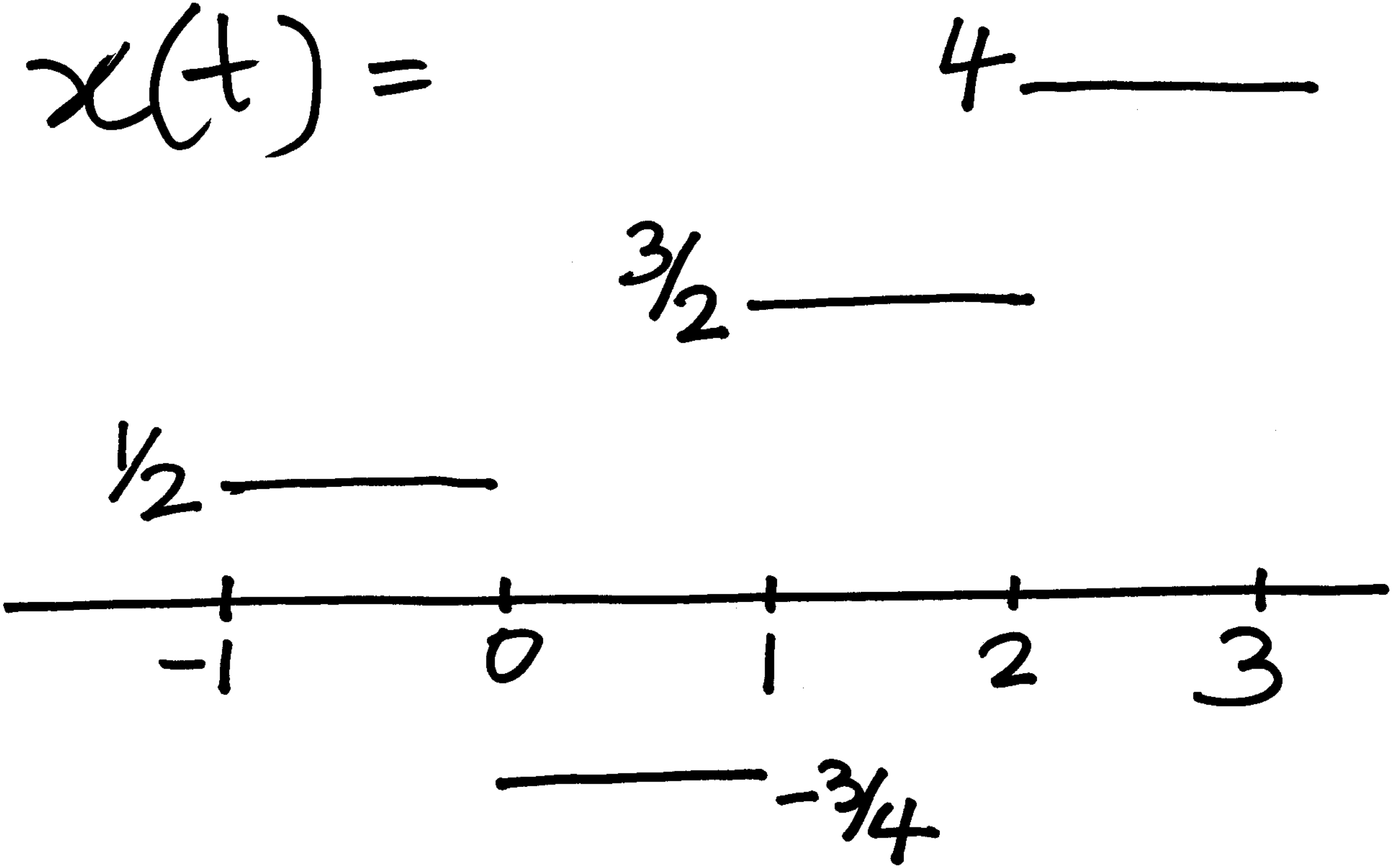
$$\left\{\phi(t-n)\right\}_{m\in\mathbb{Z}}$$



This is an <u>orthonormal</u> basis for V_0!

$$\langle \phi(t-n), \phi(t-m) \rangle \quad n, m \in \mathbb{Z} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

$x(t) =$



$$x(t) = \cdots + \frac{1}{2}\phi(t+1) + \left(-\frac{3}{4}\right)\phi(t) + \frac{3}{2}\phi(t-1) + 4\phi(t-2) + \cdots$$

There is an equivalence between

x(t) and the sequence

$$\cdots \quad \tfrac{1}{2} \quad -3/4 \quad 3/2 \quad 4 \quad \cdots$$

$$\uparrow$$

$$0$$

We could also write the same sequence

as

$$\cdots \quad \tfrac{1}{2} \quad -3/4 \quad 3/2 \quad 4 \quad \cdots$$

$\uparrow$

1

Call it

$x[n]$

The square integrability of $x(t)$, i.e.

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt < +\infty$$

means:

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 < +\infty$$

$$x(t) \in \mathcal{L}_2(\mathbb{R})$$

$$\Rightarrow x[n] \in \underline{\underline{l_2(\mathbb{Z})}}$$

In general

$\ell_p(\mathbb{Z})$ is the linear space of sequences $x(n)$ ....

set of integers

such that

$$\sum_{n=-\infty}^{+\infty} |x(n)|^p$$

is <u>finite</u>.

We have just shown a correspondence:

$$x(t) \in V_0 \subset \mathcal{L}_2(\mathbb{R})$$

$$\hookrightarrow x[n] \in \ell_2(\mathbb{Z}).$$

$x[n]$ is the sequence of coefficients of expansion with respect to an ORTHONORMAL BASIS (of $x(t)$).

Suppose

$x(t), y(t) \in V_0$

corresponding to

$x[n], y[n]$

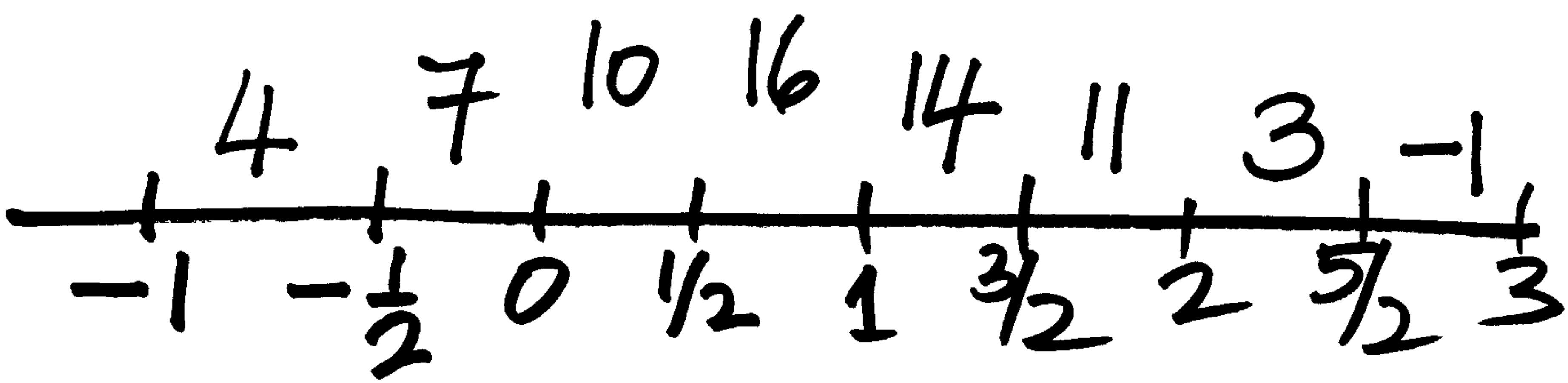
$$\langle x(t), y(t) \rangle$$

$$= \int_{-\infty}^{+\infty} x(t)\overline{y(t)}\, dt$$

$$= K_0 \sum_{n=-\infty}^{+\infty} x[n]\overline{y[n]}$$

Consider

$y(t) \in V_1$



Corresponding sequence:

$y[n]$
$= 4 \quad 7 \quad \underset{\substack{\uparrow \\ 0}}{10} \quad 16 \quad 14 \quad 11 \quad 3 \quad -1$

$$y(t) = \sum_{n \in \mathbb{Z}} y[n]\,\phi(2t-n)$$

$\phi(2t-n) = 1$

between

$$2t - n = 0$$
$$\Rightarrow t = n/2$$

and

$$2t - n = 1$$
$$\Rightarrow t = (n+1)/2$$

$$W_0 = \text{span}\{\psi(t-n)\}_{n \in \mathbb{Z}}$$

$$V_0 = \text{span}\{\phi(t-n)\}$$

$$V_1 = V_0 \oplus W_0$$

orthogonal sum

We can bring in the notion of "angle" between functions and corresponding sequences

$x(t),\ y(t) \in \mathcal{L}_2(\mathbb{R})$

"Angle" between $x(t),\ y(t) = \theta$

$$\cos\theta = \frac{\langle x, y \rangle}{\|x\|_2 \|y\|_2}$$

$$\frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1||\vec{v}_2|} = \cos(\widehat{v_1, v_2})$$

angle between v_1, v_2

<u>Exercise</u>: Establish a correspondence between the 'angle' between the functions and that between the corresponding <u>sequences</u>

Typical functions
in

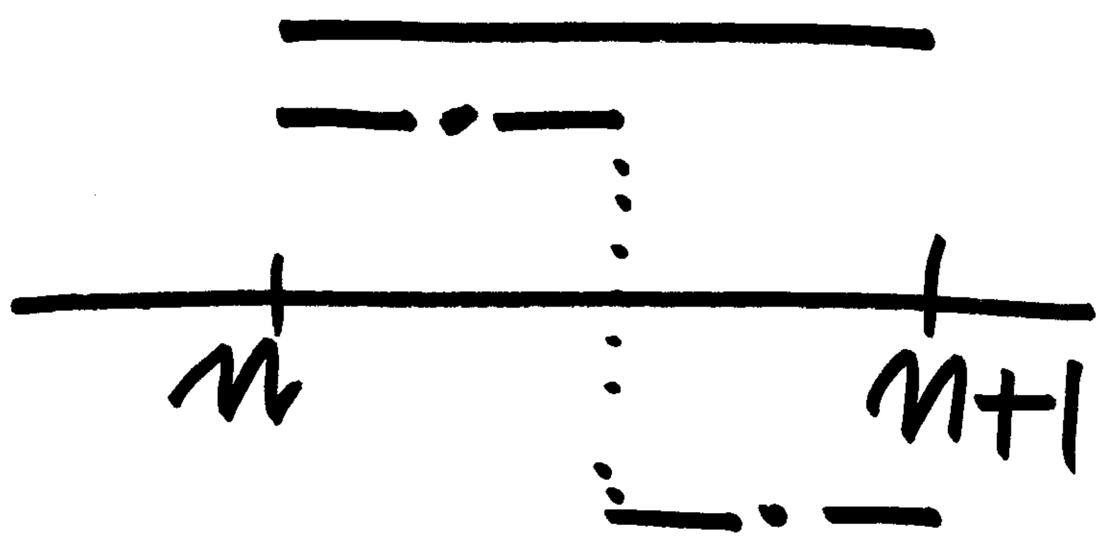
$V_1 \rightarrow$ piecewise const on $]\frac{n}{2}, \frac{n+1}{2}[$

$V_0 \rightarrow$ piecewise constant on $]n, n+1[$

$W_0 \rightarrow$ linear combinations of $\psi(t-n)$

Consider $x_1(t) \in V_0$

$x_2(t) \in W_0$



$\langle x_1, x_2 \rangle = 0$

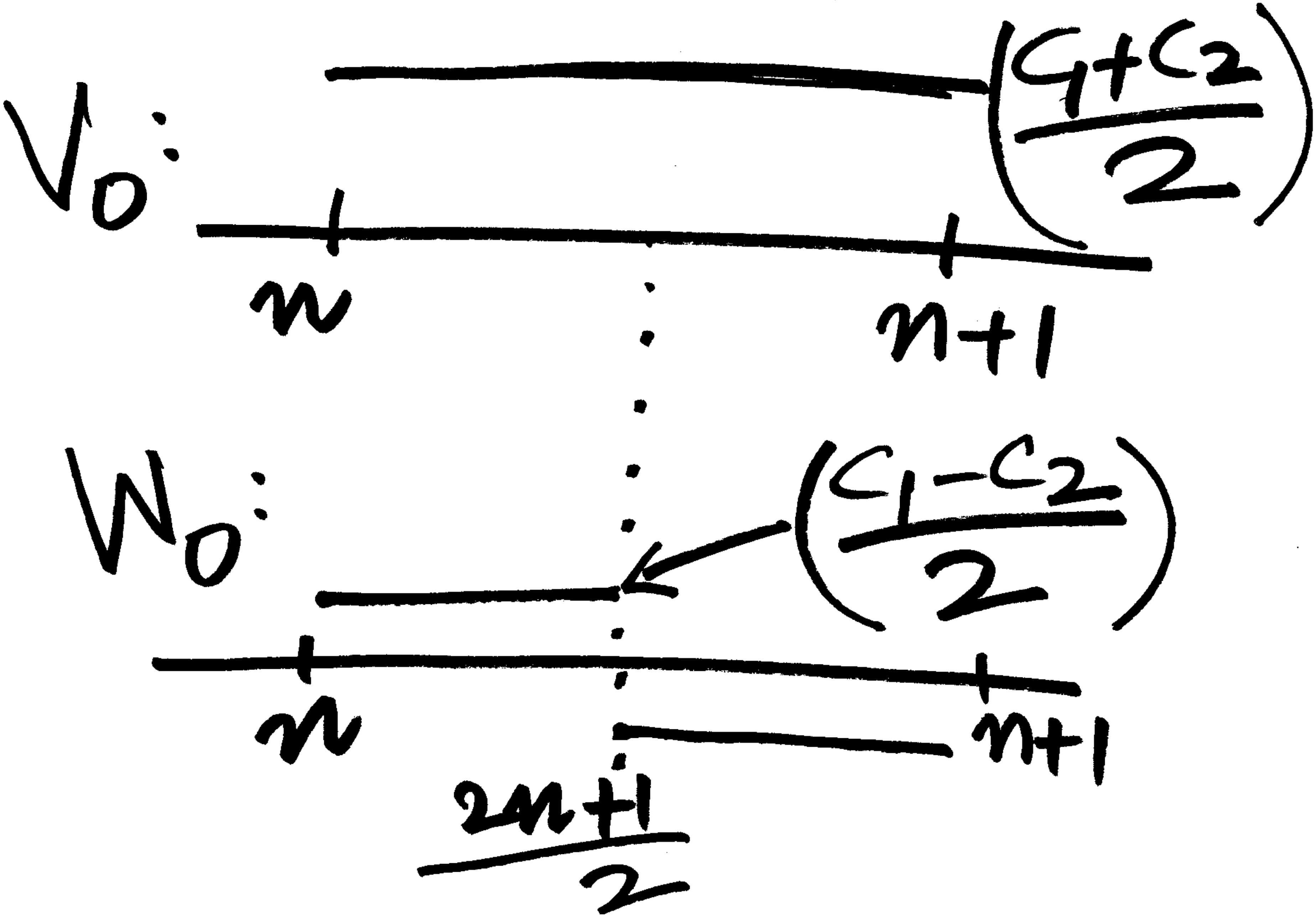
orthogonal

$$V_1 : p(t) \longrightarrow p[n]$$

$$V_0 : p_0(t) \longrightarrow p_0[n]$$

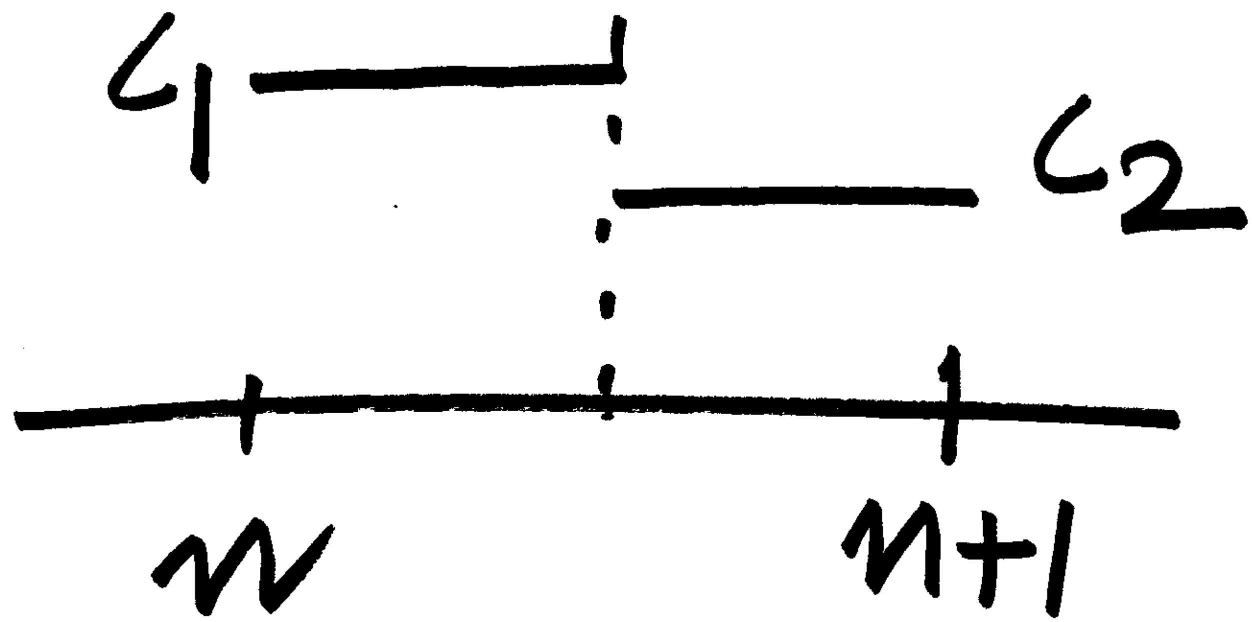
$$W_0 : q_0(t) \longrightarrow q_0[n]$$

$$p(t) = p_0(t) + q_0(t)$$



V_0:
$\left(\frac{c_1+c_2}{2}\right)$
n
$n+1$
W_0:
$\left(\frac{c_1-c_2}{2}\right)$
n
$n+1$
$\frac{2n+1}{2}$

V_1:



Next task:

Relate $p(n)$, $p_o(n)$, $q_o(n)$

$$p_0[n] = \frac{c_1 + c_2}{2}$$

$$q_0[n] = \frac{c_1 - c_2}{2}$$

$$p_0[n] = \frac{p[2n] + p[2n+1]}{2}$$

$$q_0[n] = \frac{p[2n] - p[2n+1]}{2}$$

Consider the following discrete time filter:

$$x[n] \rightarrow \boxed{\text{filter}} \rightarrow y[n]$$

$$y[n] = \frac{1}{2}\left\{x[n] + x[n+1]\right\}$$

$$p[n] \to \boxed{\text{filter}} \to \frac{1}{2}\{p[n] + p[n+1]\}$$

$$y[n] = \frac{1}{2}\{x[n] + x[n+1]\}$$

we want $2n$ not n!

$$x_{in}[n] \longrightarrow \boxed{\downarrow 2} \longrightarrow x_{out}[n]$$

Decimator

$$x_{out}[n] = x_{in}[2n] \;!$$

x_{in} · · · · · · · · · ·

$n \rightarrow$ −4 −3 −2 −1 0 1 2 3 4 5

$n \rightarrow$ · · −2 −1 0 1 2 · · ·

(retain even samples, halve

x_{out} · · · · · location

number!)