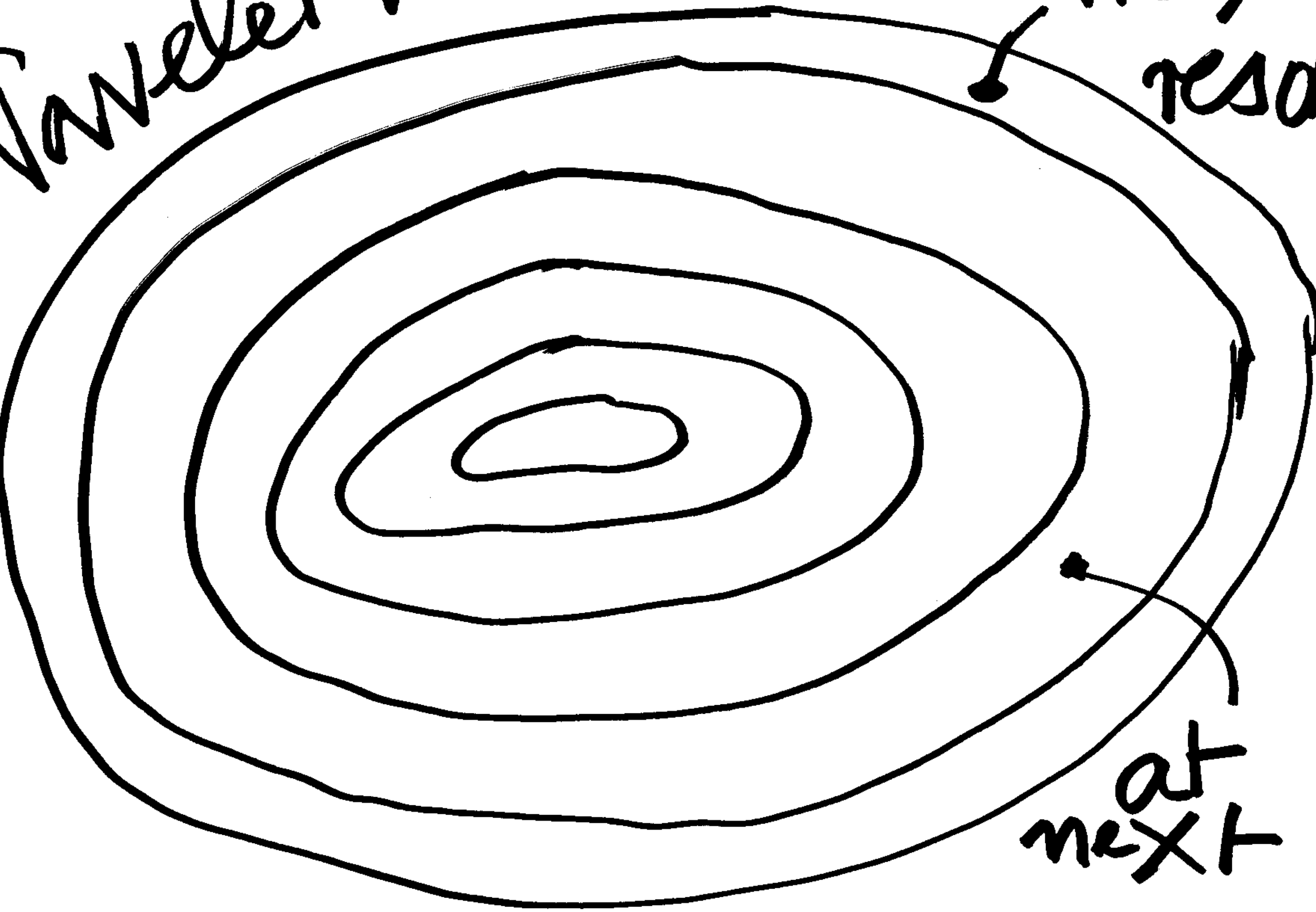
Prof. U M Gadre
Lec. 3
8|1|10

LECTURE 3

THE HAAR MULTIRESOLUTION ANALYSIS (MRA)

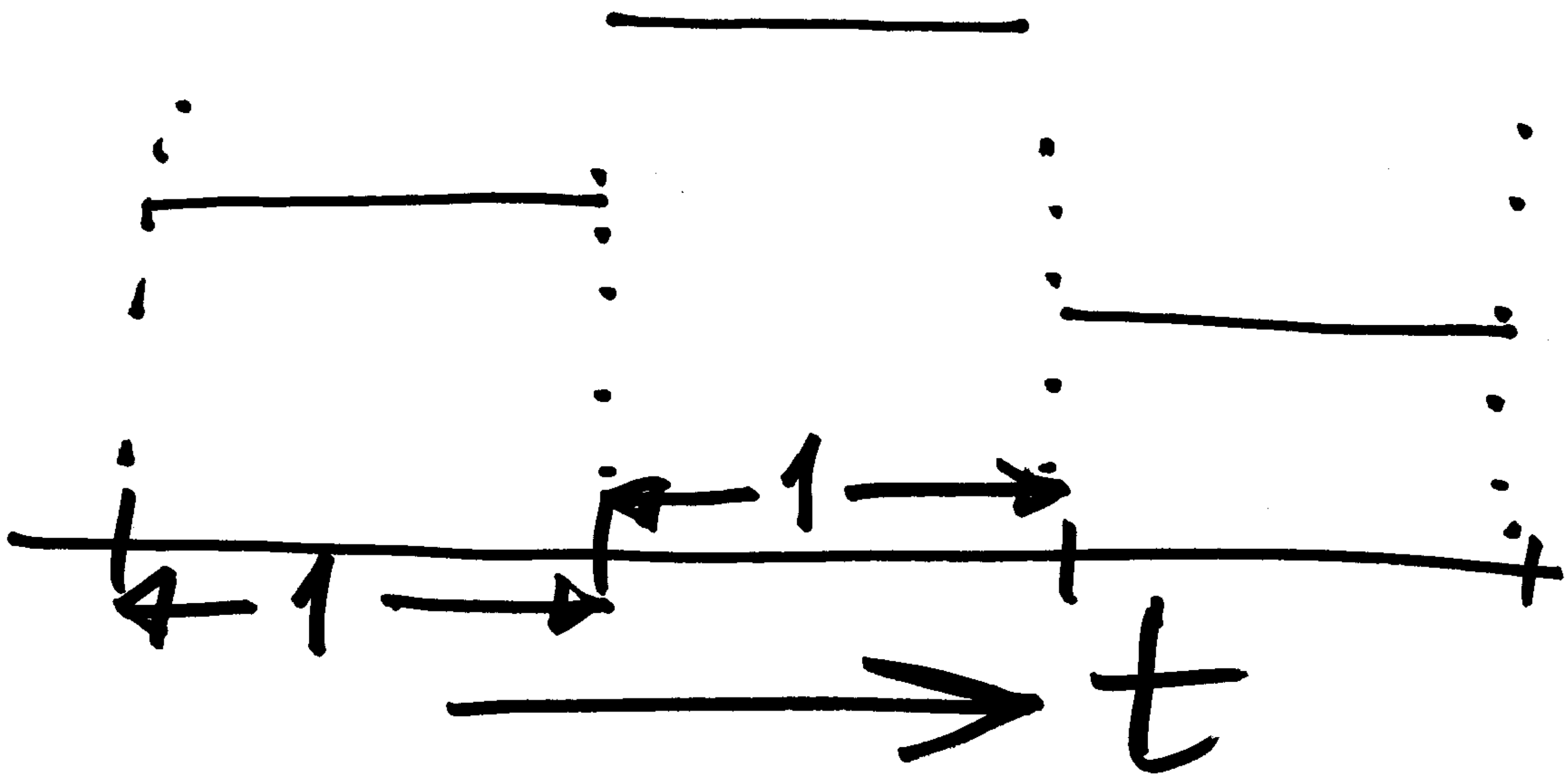
Wavelet translates at max resolution



at next

We are essentially "peeling off" shell by shell using different dilates and translates!

Focal point:
Piecewise constant
approximation on
unit intervals


1
1
t

What function $\phi(t)$ is such that its integer translates can span this space?

$\mathbb{Z}$: set of integers

$\forall$: 'for all'

linear space of functions:
a set of functions, such
that their linear combinations
fall in the same set.

The space of piecewise constant functions on the standard unit intervals!

$\forall n \in \mathbb{Z} \quad]n, n+1[$

V_0 : $\Big\{ x(t)$, such that

$x(\cdot) \in \mathcal{L}_2(\mathbb{R})$

'belongs to'

and $x(\cdot)$ is piecewise

constant on all $]n, n+1[$ $\Big\}$

n integer

We say V_0, because of piecewise constancy on intervals of size $2^{-0} = 1$

Similarly

$V_1 : \{ x(t),\ x \in \mathcal{L}_2(\mathbb{R})$

and $x(\cdot)$ is piecewise constant on standard 2^{-1} intervals $]n.2^{-1}, (n+1)2^{-1}[$ $\forall n \in \mathbb{Z} \}$

In general V_m

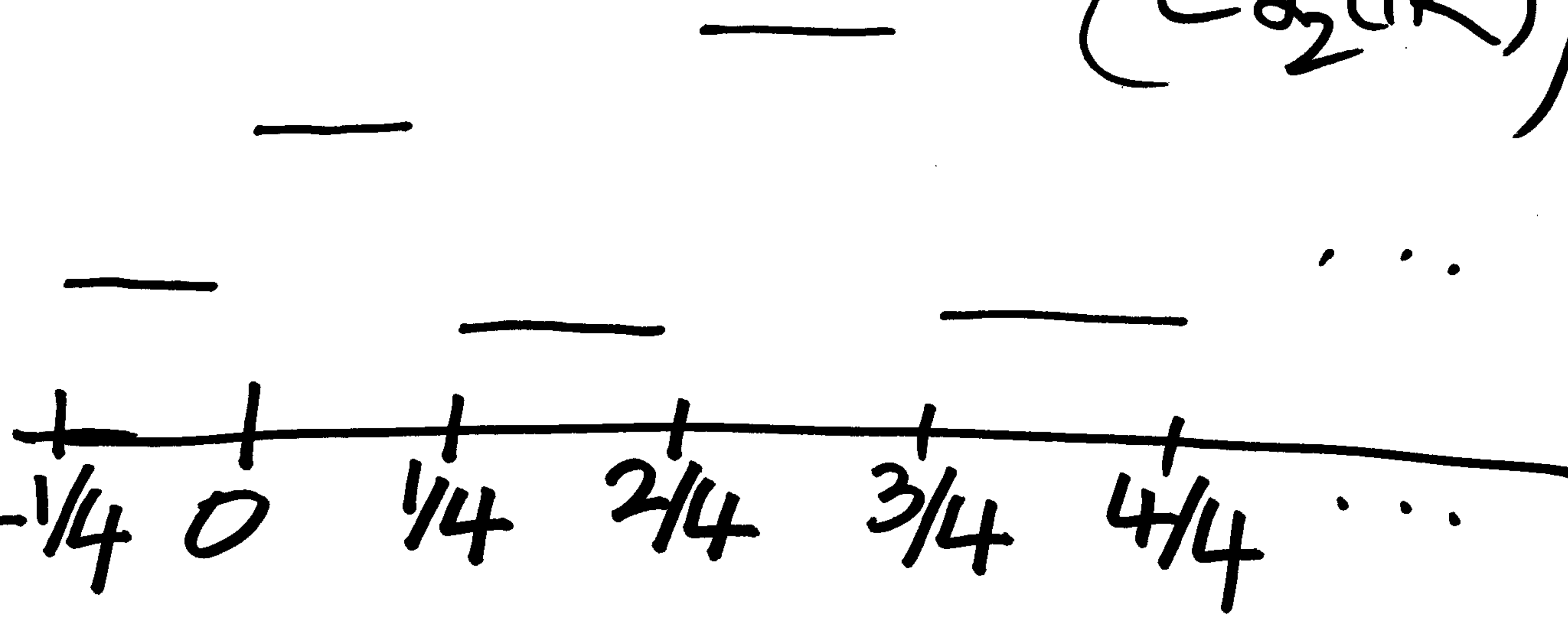
$= \{ x(t),\ x \in \mathcal{L}_2(\mathbb{R})$

and $x(\cdot)$ is piecewise constant on all $]n \cdot 2^{-m}, (n+1) \cdot 2^{-m}[$

$n \in \mathbb{Z} \}$

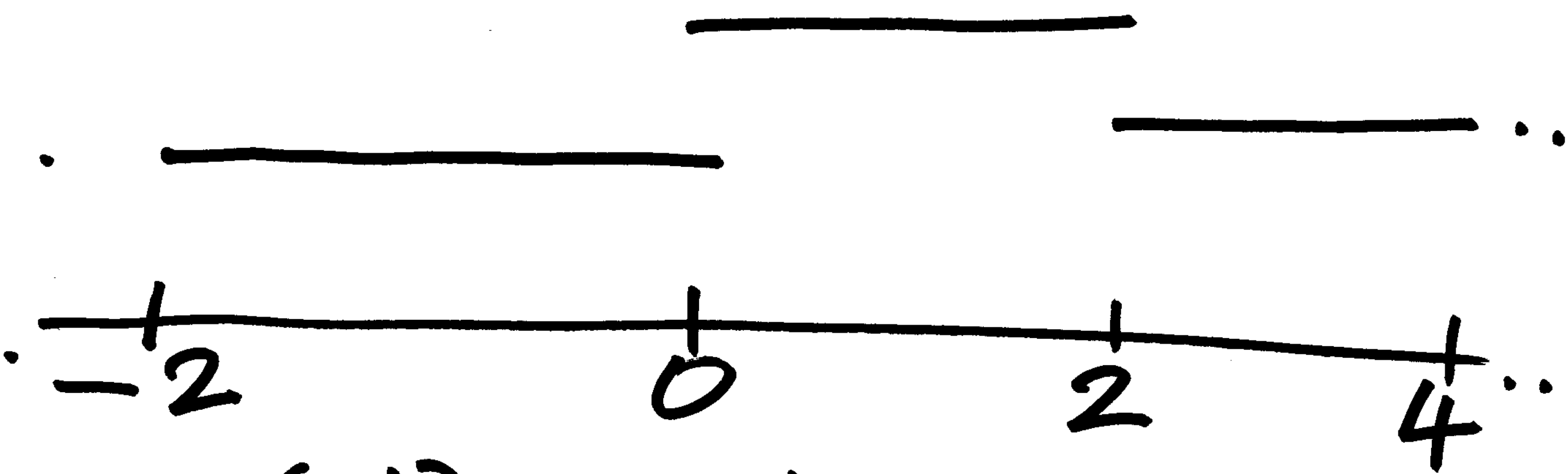
Example of $x(\cdot) \in V_2$

$(\in \mathcal{L}_2(\mathbb{R}))$



The absolute squared sum of the piecewise constant values, in all V_m, must be convergent $(\in \mathcal{L}_2(\mathbb{R}))$

Example of $x(\cdot) \in V_{-1}$



$$2^{-(-1)} = 2^1 = 2$$

"A ladder of subspaces" implied:

$$\cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots$$

Intuitively $\longrightarrow$ towards $\mathcal{L}_2(\mathbb{R})$!

What happens when we go leftwards?

$$\cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots$$

$$\longleftarrow \; ?$$

Piecewise constants on LARGER INTERVALS

$\mathcal{L}_2$ norm of functions going leftward

$$= \sum_n |c_n|^2 \, 2^{-m}$$

$$= 2^{|m|} \sum_n |c_n|^2 \quad m \text{ -ve} \quad m \to -\infty$$

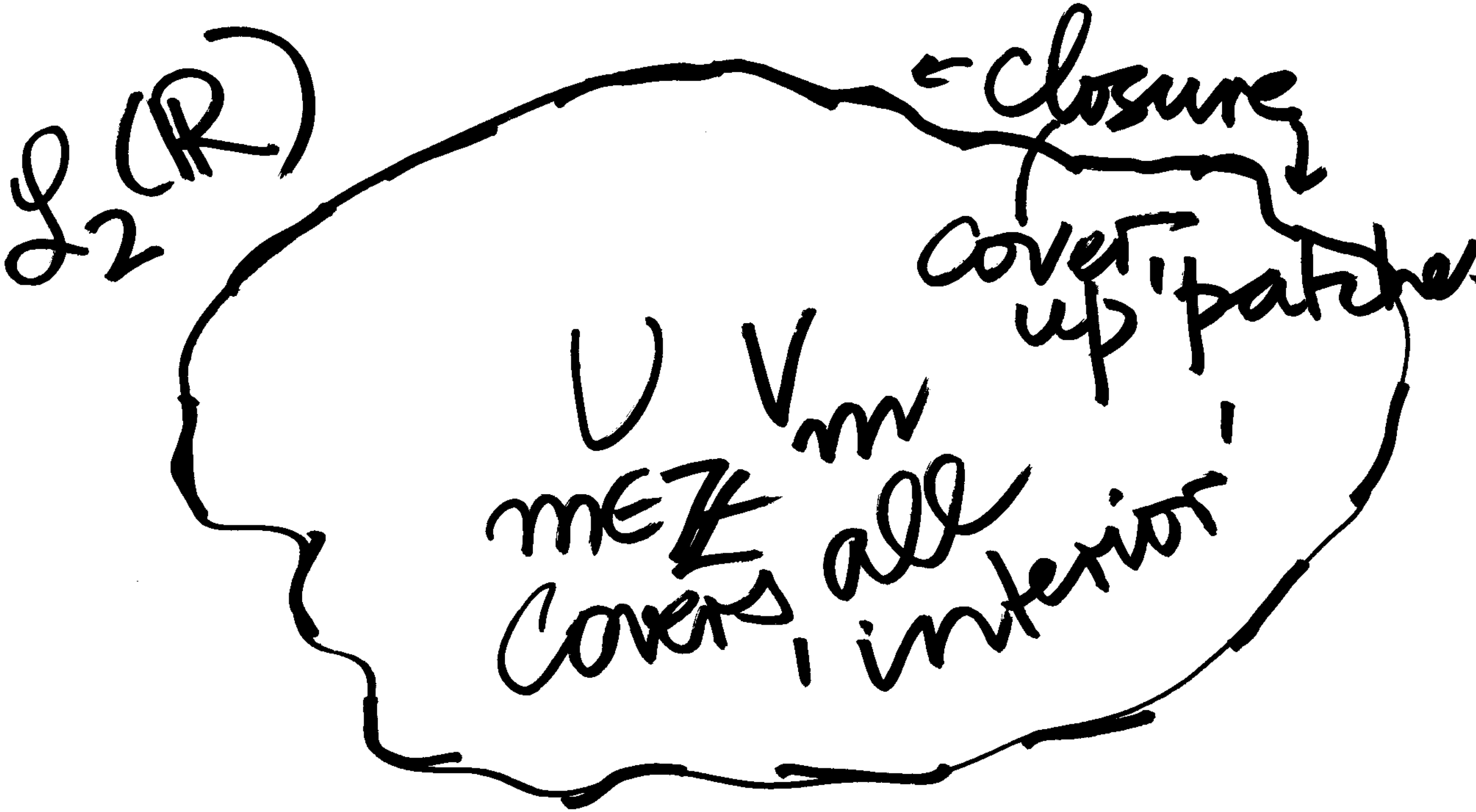
If $2^{|m|}\sum_n |c_n|^2$ must converge, no matter how large $|m|$ is, then $\underline{\sum_n |c_n|^2 \longrightarrow 0}$

Moving upwards:

Union

$$\overline{\bigcup_{m \in \mathbb{Z}} V_m} = L_2(\mathbb{R})$$

closure



$\mathcal{L}_2(\mathbb{R})$
← closure
cover up 'patches'
$\bigcup_{m \in \mathbb{Z}} V_m$
covers all 'interior'

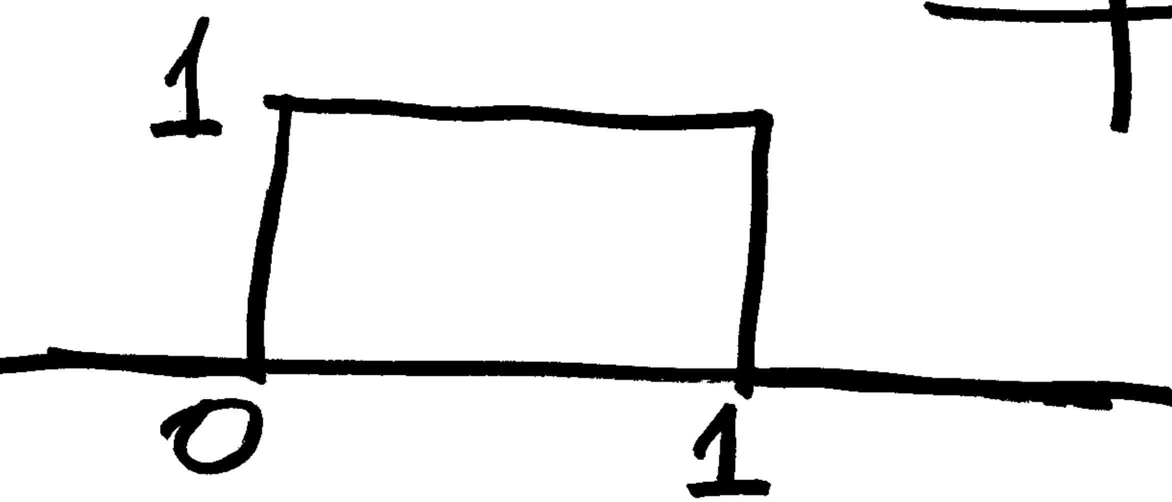
Moving 'downwards'

Intersection !

$$\bigcap_{m \in \mathbb{Z}} V_m = \underbrace{\{0\}}_{\text{trivial subspace}}$$

We say $\{f_1, \dots, f_k \dots\}$
span a linear space
if any function in the
space can be generated
by linear combinations
of this <u>set</u>

What function $\Phi(t)$ and its integer translates span V_0?

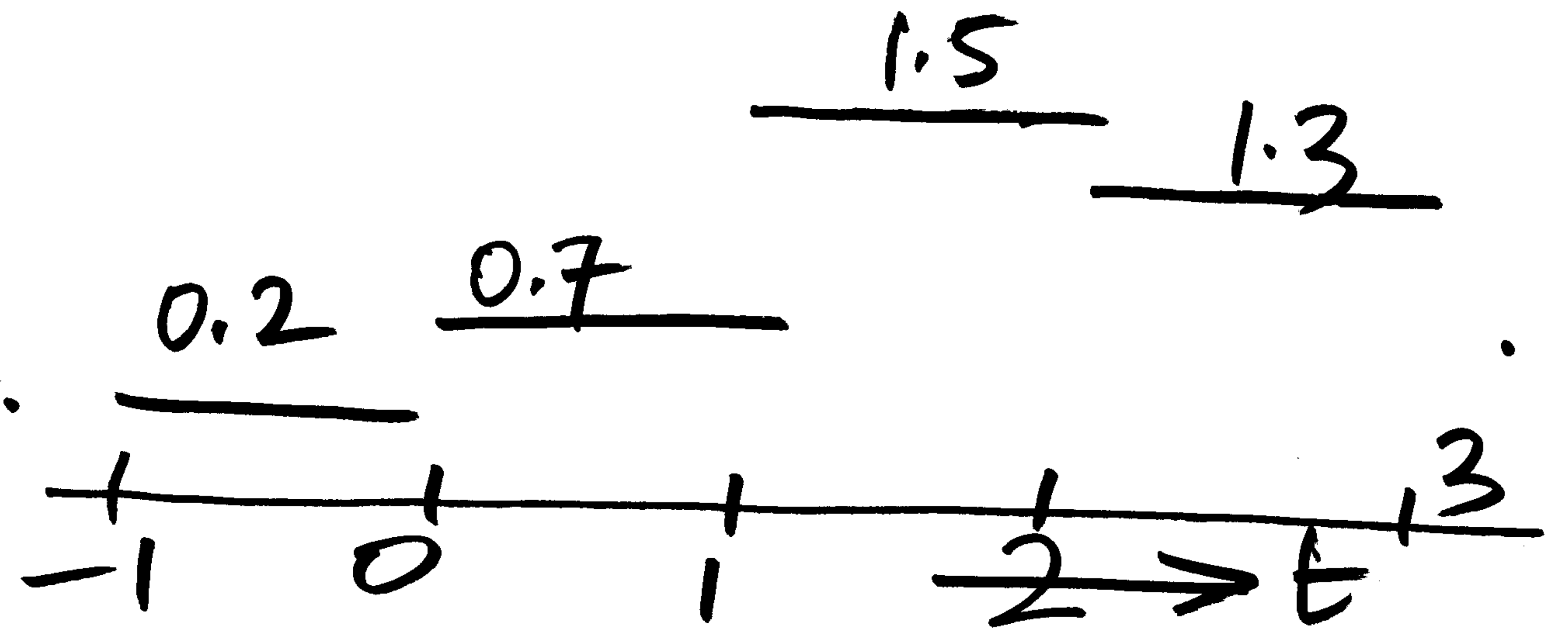


Any function in V_0 can be written

$$\sum_{n \in \mathbb{Z}} c_n \, \phi(t-n)$$

piecewise constants

integer translates of $\phi(\cdot)$



$$= \cdots + 0.2\,\phi(t+1) + 0.7\,\phi(t) + 1.5\,\phi(t-1) + 1.3\,\phi(t-2) + \cdots$$

Any space V_m can be similarly constructed

$$V_m = \operatorname*{span}_{n \in \mathbb{Z}} \{\phi(2^m t - n)\}$$

$\phi(t)$ is called the Scaling Function (in this case of the Haar <u>Multiresolution Analysis</u>!)

??

$$\cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots$$

with these properties ??

is called a

Multiresolution Analysis

(MRA)

Axioms of a Multiresolution Analysis

Ladder of subspaces of $\mathcal{L}_2(\mathbb{R})$: $\cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots$

such that

1. $\overline{\bigcup_{m \in \mathbb{Z}} V_m} = \mathcal{L}_2(\mathbb{R})$

2. $\bigcap_{m \in \mathbb{Z}} V_m = \{0\}$

Contd...

3. There exists $\phi(t)$ such that

$$V_0 = \operatorname*{span}_{n \in \mathbb{Z}} \{\phi(t-n)\}$$

4. $\{\phi(t-n)\}_{n \in \mathbb{Z}}$ is an ORTHOGONAL SET!

5. If $f(t) \in V_m$,
then $f(2^{-m} t) \in V_0$
$\forall m \in \mathbb{Z}$

6. If $f(t) \in V_0$, $\forall n \in \mathbb{Z}$
then $f(t-n) \in V_0$

Theorem: Given these axioms, there exists $\psi(\cdot) \in \mathcal{L}_2(\mathbb{R})$ so that

$$\{\psi(2^m t - n)\}_{m \in \mathbb{Z},\, n \in \mathbb{Z}} \text{ span } \mathcal{L}_2(\mathbb{R})$$