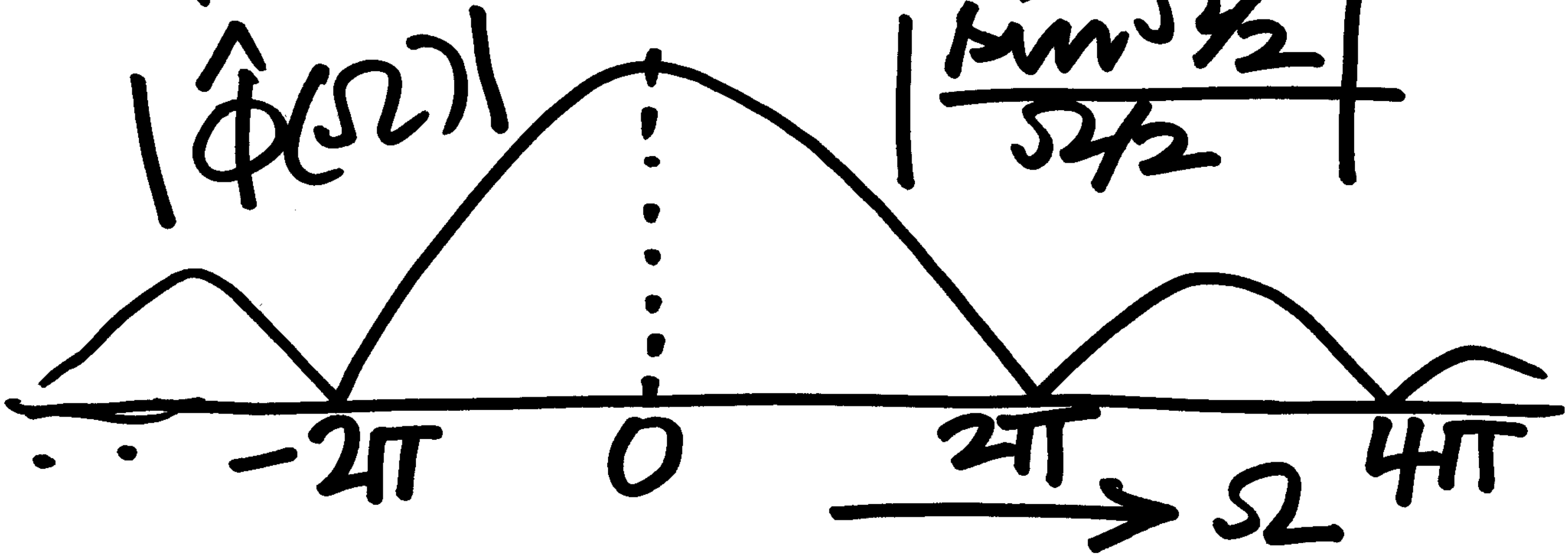
Prof. V. Gadre Dat

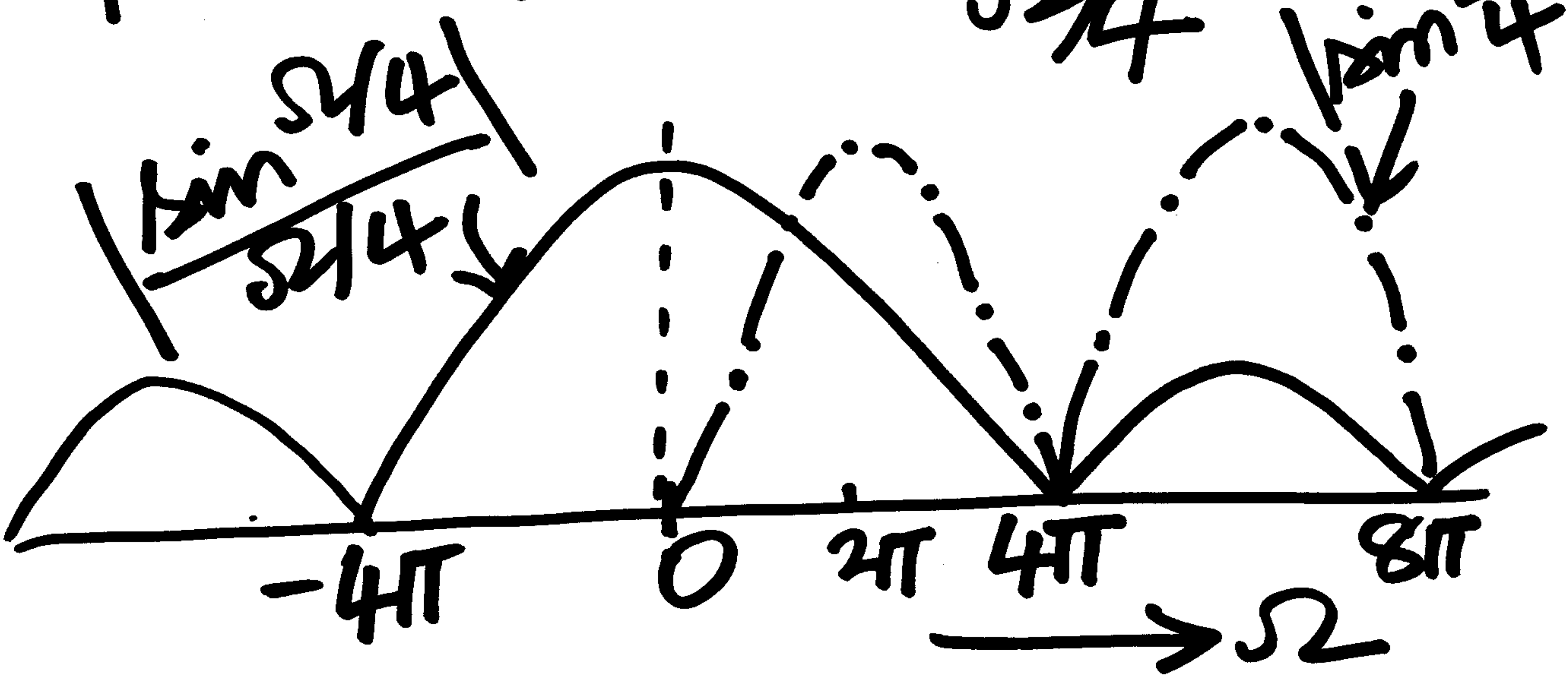
LECTURE 16

IDEAL TIME-FREQUENCY BEHAVIOUR

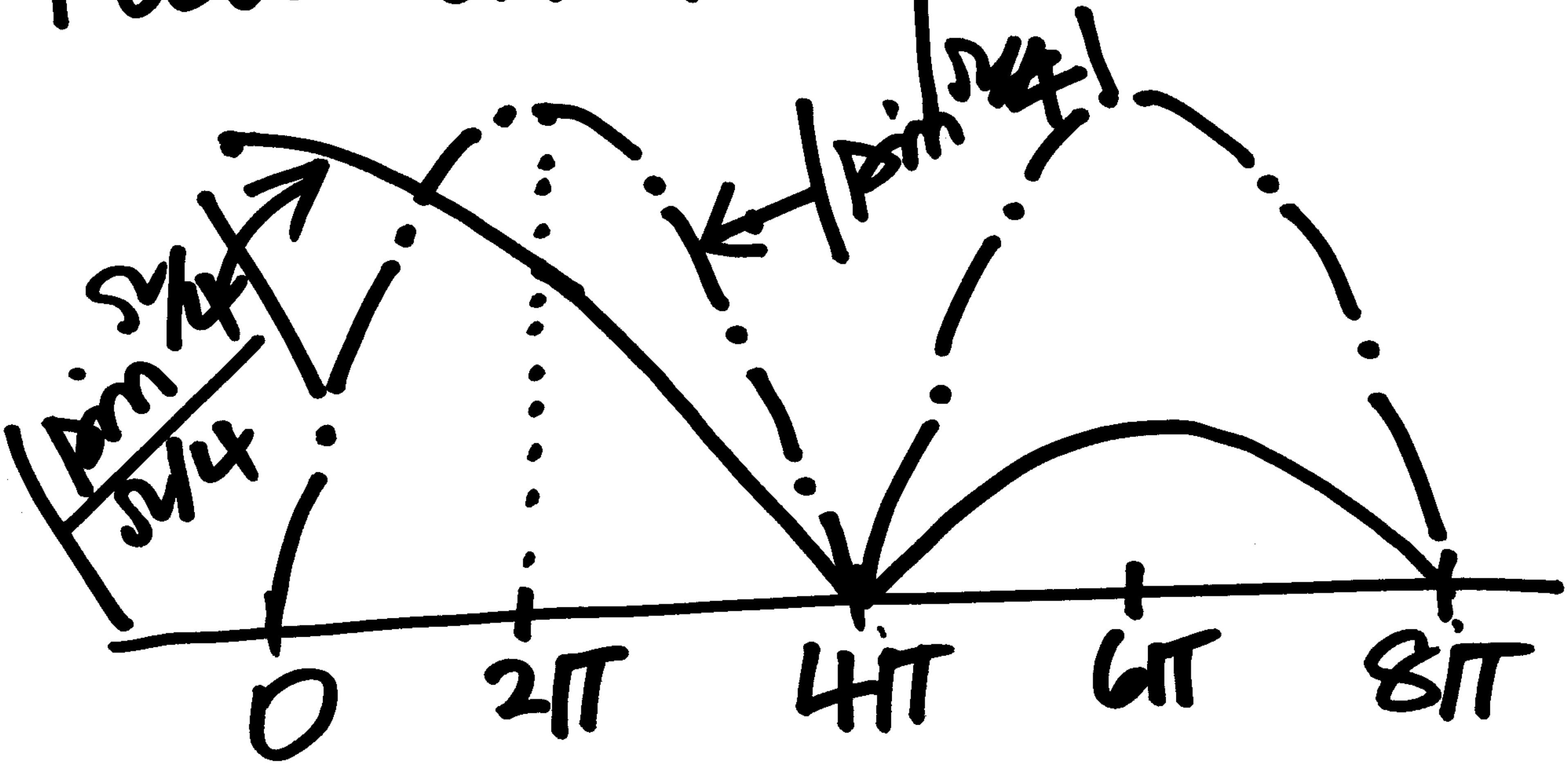
Form of magnitudes of Fourier Transforms

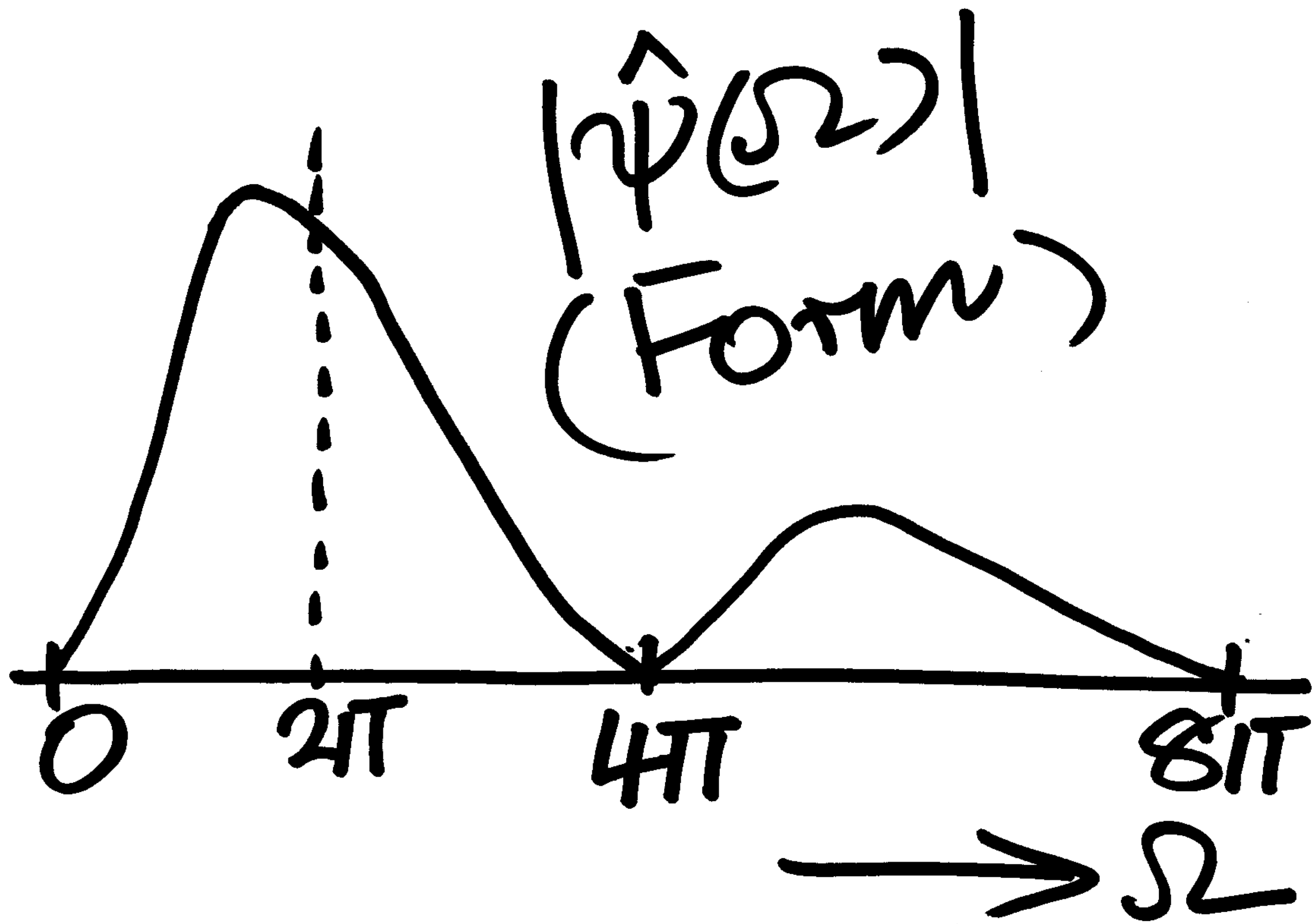


$$|\hat{\psi}(\Omega)| = \frac{\sin^2 \Omega/4}{\Omega/4}$$

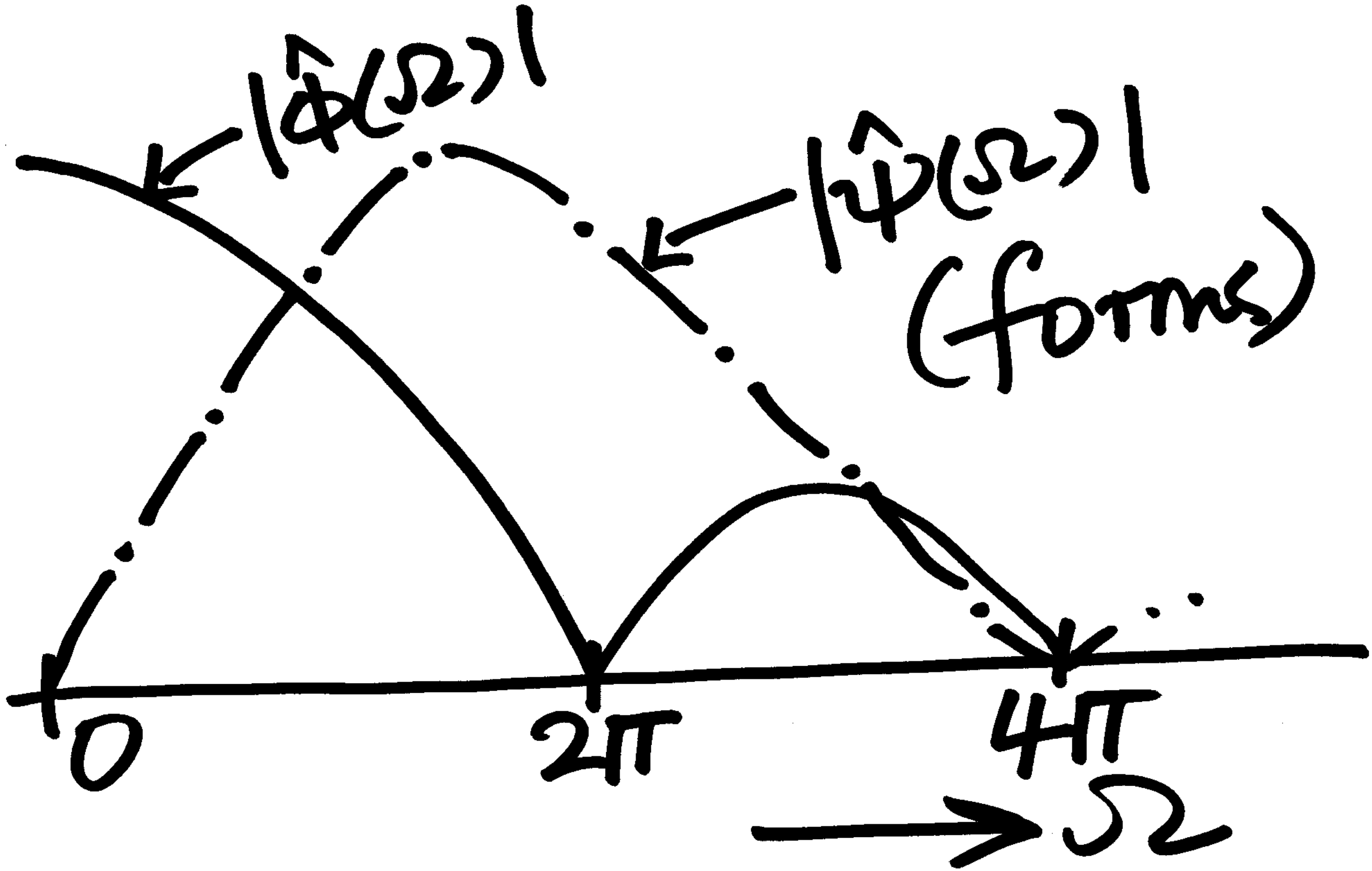


Focus on the positive:





$|\hat{\psi}(\Omega)|$
(Form)
0
2π
4π
8π
Ω



$|\hat{\Phi}(\Omega)|$
$|\hat{\Psi}(\Omega)|$
(forms)
0
2π
4π
Ω

$\hat{\Psi}(\cdot)$ in effect does "Constant Quality Factor" or "Constant-Q" Analysis

For bandpass filters or bandpass functions

$$\text{Quality factor } (Q) = \frac{\text{Centre frequency}}{\text{bandwidth}}$$

Effectively $\phi(\cdot)$ is doing a "lowpass" operation and...

$\psi(\cdot)$ is doing a bandpass operation.

$$\int_{-\infty}^{+\infty} \overline{x(t)}\, \phi(t+\tau)\, dt$$

(confine to real functions x).

$$\int_{-\infty}^{+\infty} \phi(t+\tau)\, x(t)\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{\phi(t+\tau)}\, \hat{x}(\Omega)\, d\Omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underbrace{\hat{x}(\Omega)\, \hat{\phi}(\Omega)}\, e^{j\Omega\tau}\, d\Omega$$

Inverse Fourier Transform of $\hat{\phi}(\Omega)\, \hat{x}(\Omega)$.

When we sample at $t = n$

$n \in \mathbb{Z}$,

$\dots$?

$$y(t),\ \hat{y}(\Omega) \longrightarrow \boxed{\text{Sample (ideally) at}} \longrightarrow\ ?$$

$n \in \mathbb{Z}$

$$\overset{\text{Constant}}{(v)}\underbrace{\sum_{k\in\mathbb{Z}} \hat{y}\left(\Omega+\frac{2\pi k}{1}\right)}$$

Had $\hat{\Phi}(\Omega)$ been an ideal lowpass function, cutoff π, ...

then this 'aliasing'

$$\sum_{k\in\mathbb{Z}} \hat{y}(\Omega + 2\pi k)$$

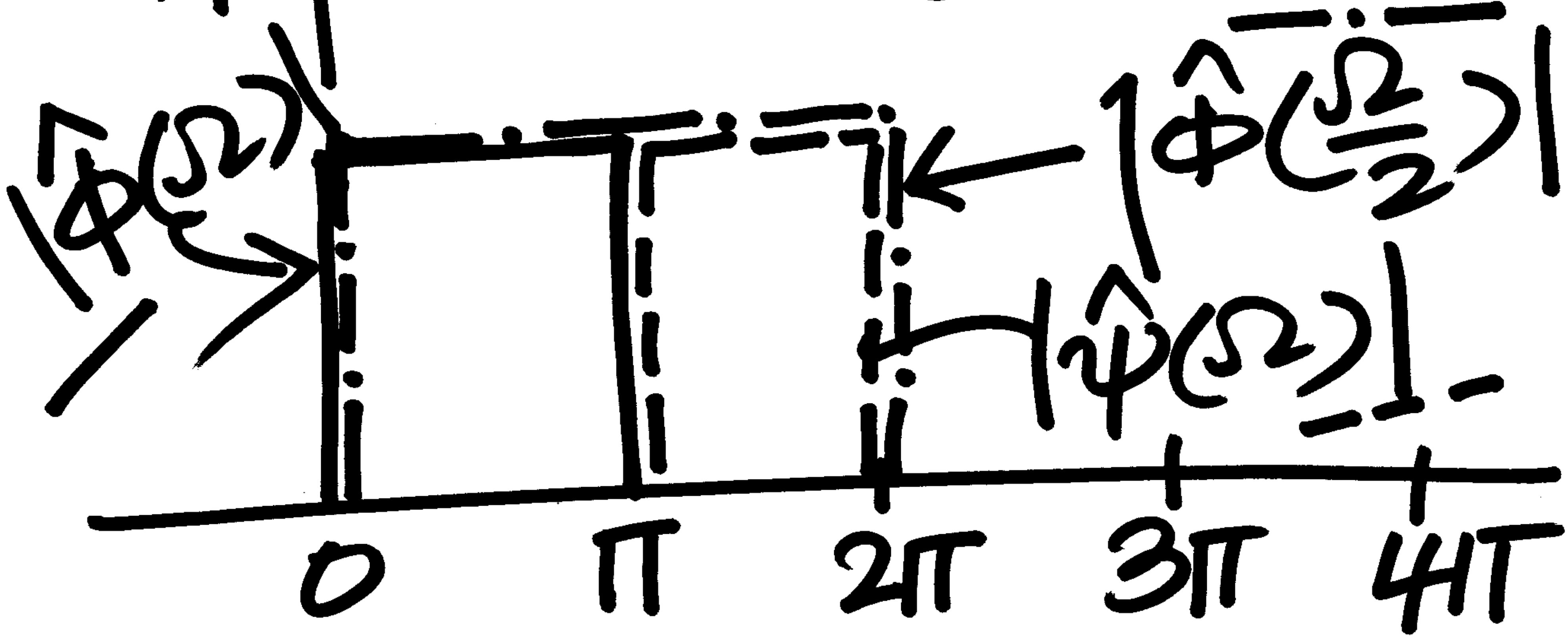
would leave $\hat{y}(\Omega)$ unaffected.

For V_1 (MRA ladder)
Haar
we expand by 2
in frequency.

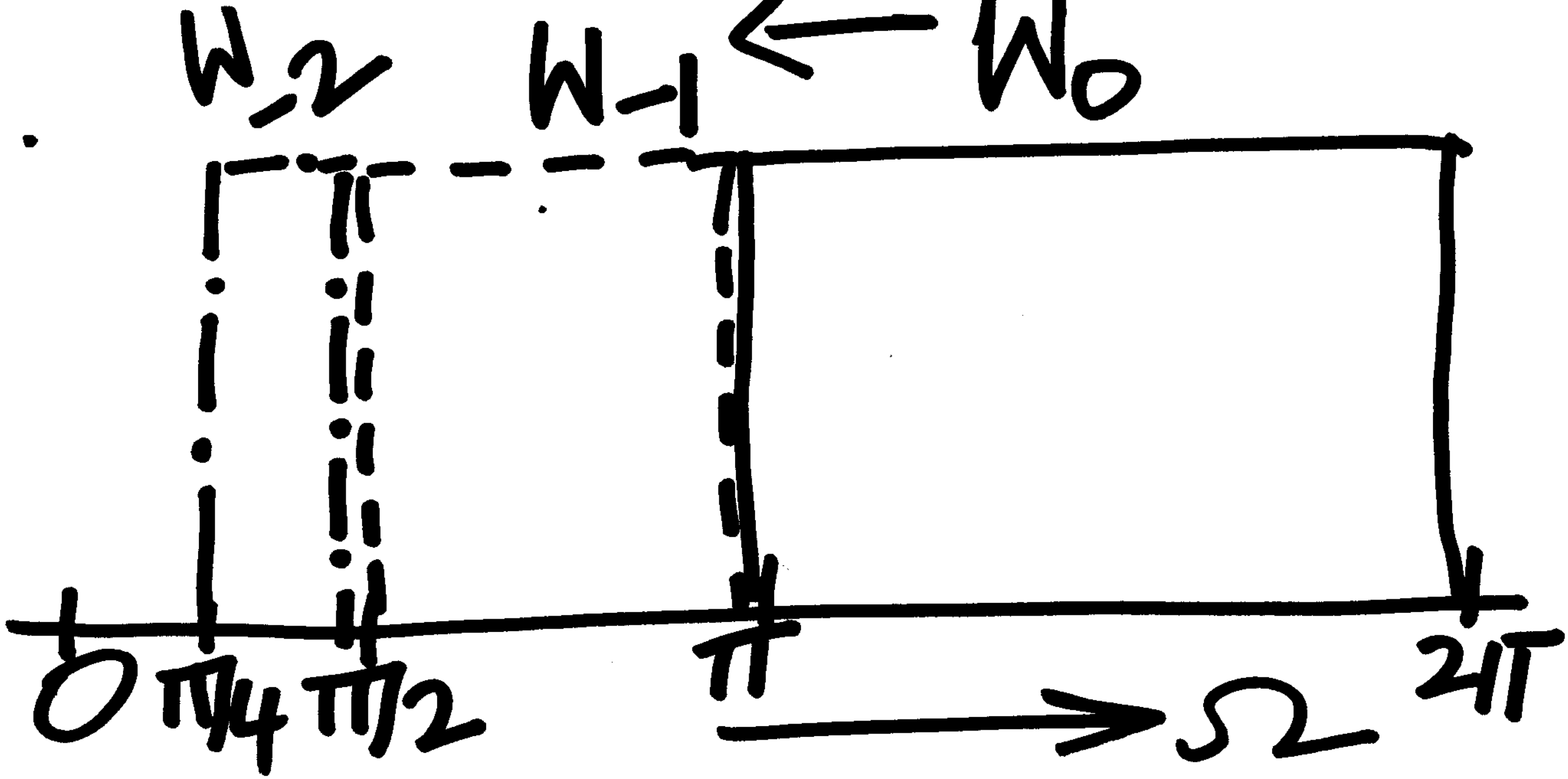
That means we are asking for a lowpass filter with cutoff 2π (instead of π)!

$\psi(\cdot)$ is aspiring to be a bandpass function between π and 2π!

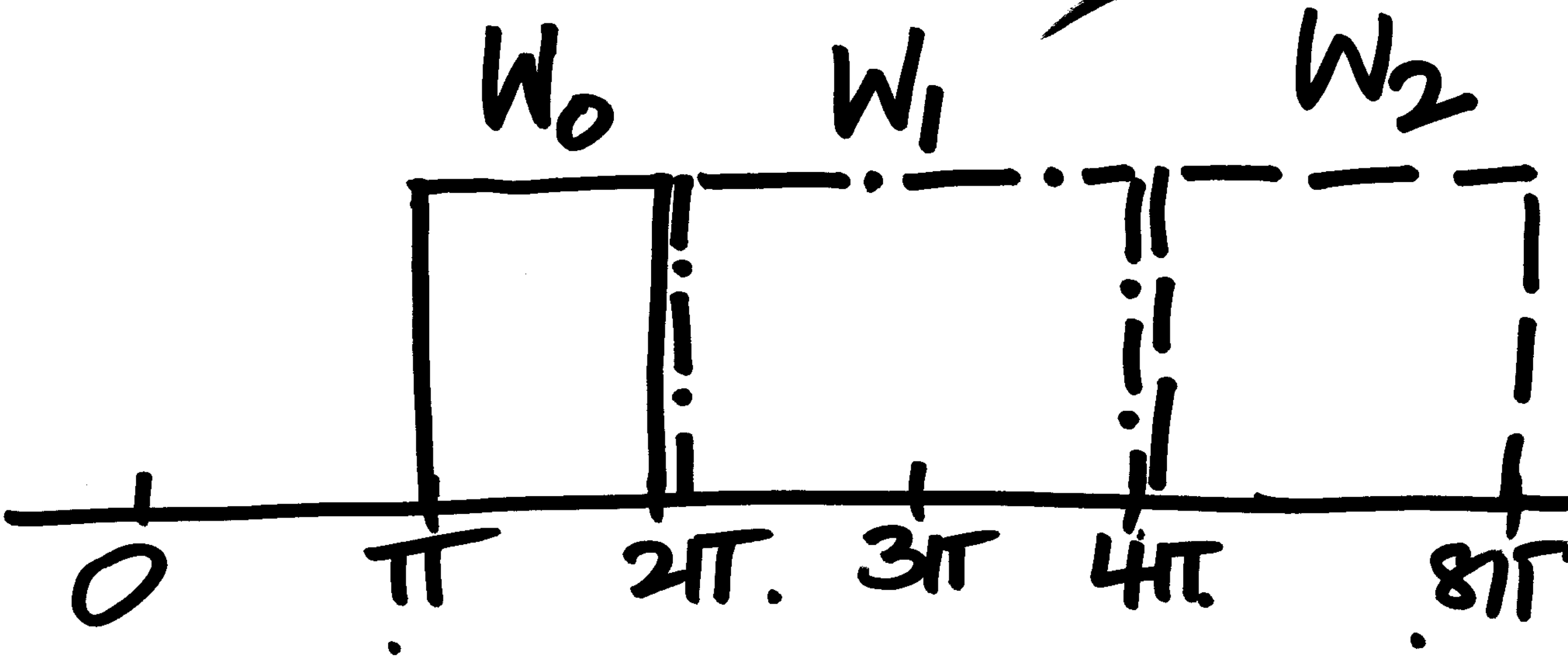
'Aspiration' (Ideal)

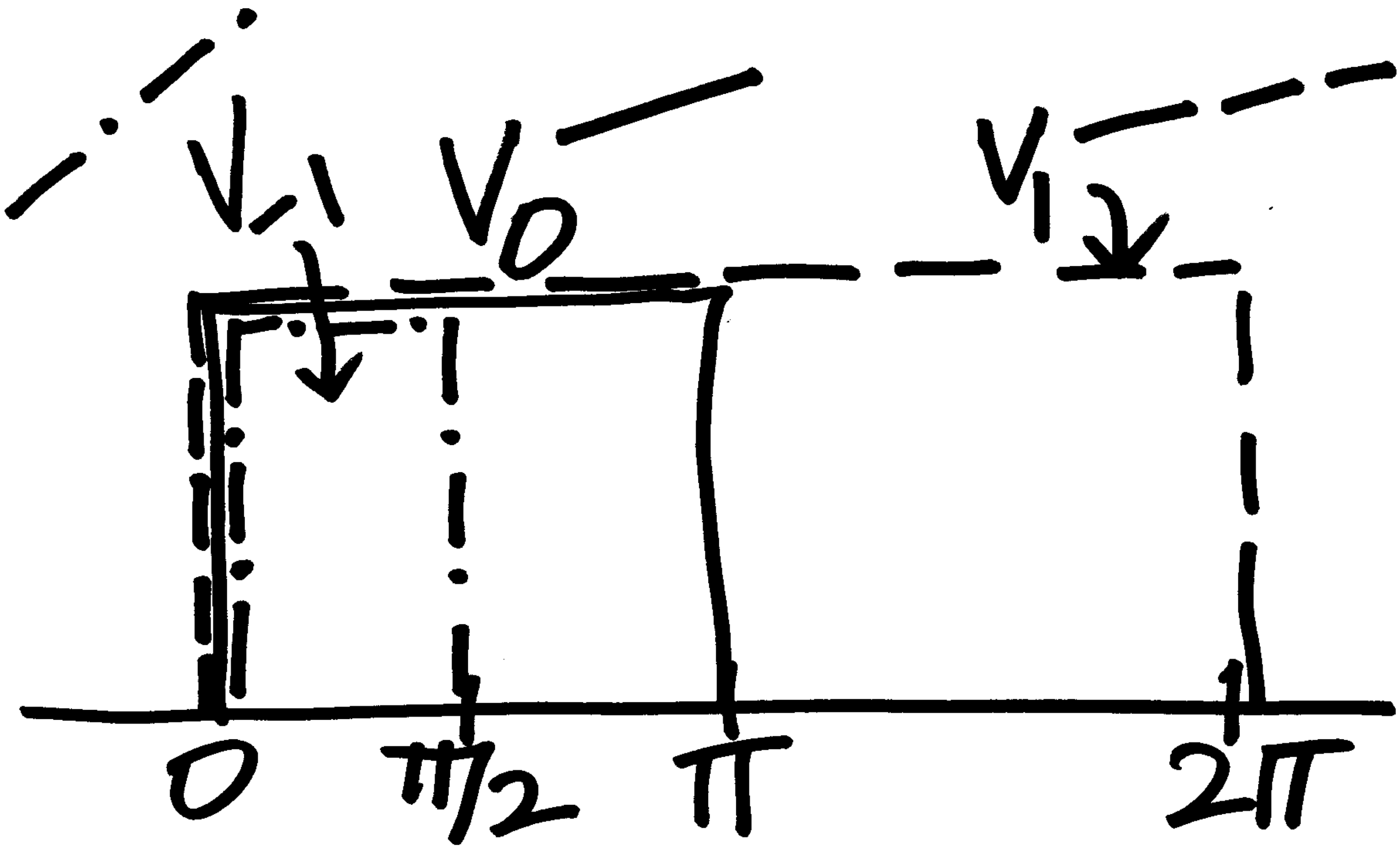


'down' the ladder



'up the ladder'





V_{-1}
V_0
V_1
0
π/2
π
2π

Can we be compactly supported (nonzero strictly on a finite interval) simultaneously in TIME AND FREQUENCY?

Answer: <u>No</u>

A function and its Fourier Transform <u>cannot</u> both be compactly supported.

Why not?

Suppose

$$x(t) \xrightarrow[\text{Fourier Transform}]{} \hat{x}(\Omega)$$

Let $\hat{x}(\Omega)$ be compactly supported

Let specifically $\hat{x}(\Omega)$ be nonzero only between $0 \leq \Omega_1 \leq |\Omega| \leq \Omega_2$

$$x(t) = \frac{1}{2\pi}\int_{\Omega_1}^{\Omega_2} \hat{x}(\Omega)\, e^{j\Omega t}\, d\Omega$$

$$+ \int_{-\Omega_2}^{-\Omega_1} \cdots \text{(same integrand)}$$