WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 44: Tutorial on Two-Band Filter Bank

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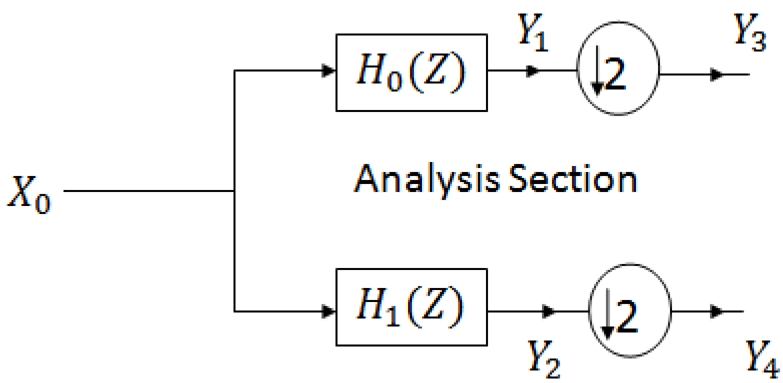
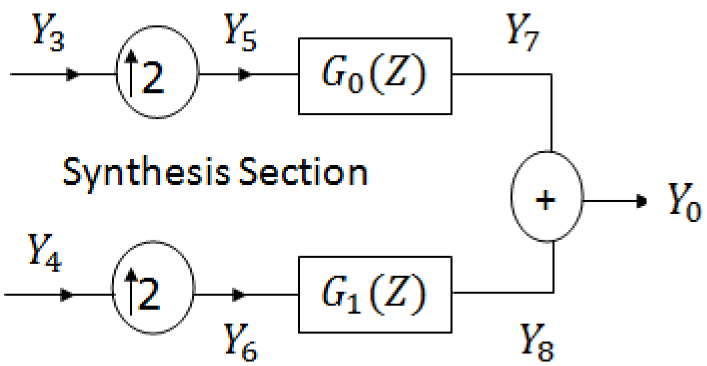
1 Two Band Filter Bank

The 2-band filter bank have two section. The analysis section and the synthesis section.



$H_0(Z)$ is a low pass filter with a cut off frequency $\frac{\pi}{2}$ and $H_1(Z)$ is a high pass filter with a cut off frequency $\frac{\pi}{2}$. Analysis section analyzes or breaks down the input in two components.



$G_0(Z)$ is a low pass filter with a cut off frequency $\frac{\pi}{2}$ and $G_1(Z)$ is a high pass filter with a cut off frequency $\frac{\pi}{2}$. Synthesis section re-synthesize output from the inputs.
It is impossible situation as we can never reach ideal low pass or high pass filter. Even so, it is possible to build perfect reconstruction structure. For *e.g.*, if we take Haar 2-band filter bank, we have set of filters H_0, H_1, G_0 and G_1 all of them have impulse response of length 2 which can create perfect reconstruction situation *i.e.*, output Y_0 is same as input X_0 except for a constant multiplier and a shift.

2 Haar 2-Band filter bank

$$H_0(Z) = (1 + Z^{-1}) \tag{1}$$

$$H_1(Z) = (-1 + Z^{-1}) \tag{2}$$

$$G_0(Z) = \frac{(1 + Z^{-1})}{2} \tag{3}$$

$$G_1(Z) = \frac{(1 - Z^{-1})}{2} \tag{4}$$

The factor of $\frac{1}{2}$ can either be on the analysis or synthesis side.
Let us take $x[n]$ be the input to this Haar 2-band filter bank.

$$x[n] = \underset{\substack{\uparrow \\ 0}}{7} \ \ 5 \ \ -4 \ \ 6 \ \ 3 \ \ 8$$

Analysis side:
$x[n]$ denotes time domain, $X(Z)$ denotes complex frequency domain.
Now, $H_0(Z) = (1 + Z^{-1})$, therefore corresponding impulse response $h_0(n)$ is

$$h_0(n) = \underset{\substack{\uparrow \\ 0}}{1} \ \ 1$$

Therefore $y_1 = x * h_0$,

$$y_1(n) = \underset{\substack{\uparrow \\ 0}}{7} \ \ 5 \ \ -4 \ \ 6 \ \ 3 \ \ 8 \ \ * \ \ \underset{\substack{\uparrow \\ 0}}{1} \ \ 1$$

$$= \underset{\substack{\uparrow \\ 0}}{7} \ \ 12 \ \ 1 \ \ 2 \ \ 9 \ \ 11 \ \ 8 \qquad = x * h_0$$

Hence, output at point Y_1 is as expected of length 7.

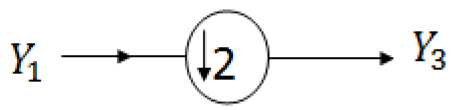
$y_2 = x * h_1$ and $H_1(Z) = (-1 + Z^{-1})$. Hence

$$h_1(n) = \underset{\substack{\uparrow \\ 0}}{-1} \ \ 1$$

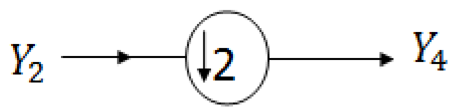
Therefore,

$$x * h_1 = \underset{\substack{\uparrow \\ 0}}{-7} \ \ 2 \ \ 9 \ \ -10 \ \ 3 \ \ -5 \ \ 8$$

Output at Y_2 is as expected of length 7.
After Downsampling by 2:



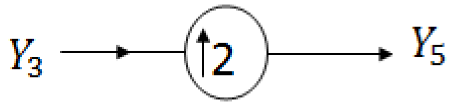
$$Y_3(n) = \underset{\substack{\uparrow \\ 0}}{7} \ \ 1 \ \ 9 \ \ 8$$



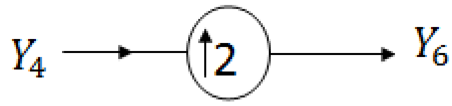
$$Y_4(n) = \quad \underset{\substack{\uparrow \\ 0}}{-7} \quad 9 \quad 3 \quad 8$$

As expected, the result after downsampling are of length 4.

Synthesis side:
After Upsampling by 2:

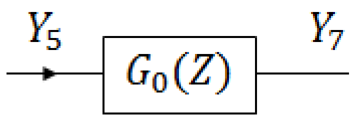


$$Y_5(n) = \quad \underset{\substack{\uparrow \\ 0}}{7} \quad 0 \quad 1 \quad 0 \quad 9 \quad 0 \quad 8$$



$$Y_6(n) = \quad \underset{\substack{\uparrow \\ 0}}{-7} \quad 0 \quad 9 \quad 0 \quad 3 \quad 0 \quad 8$$

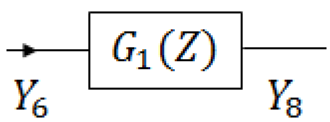
Again as expected the result after upsample are back to length 7.
Now, Y_5 is subjected to Low Pass Filter.



$$G_0(Z) = \frac{(1 + Z^{-1})}{2} \tag{5}$$

$$Y_7(n) = \quad \underset{\substack{\uparrow \\ 0}}{\frac{7}{2}} \quad \frac{7}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{9}{2} \quad \frac{9}{2} \quad \frac{8}{2} \quad \frac{8}{2}$$

Y_6 is subjected to High Pass Filter.



$$G_1(Z) = \frac{(1 - Z^{-1})}{2} \tag{6}$$

$$y_8[n] = \frac{y_6[n] - y_6[n-1]}{2} \tag{7}$$

$$Y_8(n) = \underset{\underset{0}{\uparrow}}{-\frac{7}{2}} \quad \frac{7}{2} \quad \frac{9}{2} \quad -\frac{9}{2} \quad \frac{3}{2} \quad -\frac{3}{2} \quad \frac{8}{2} \quad -\frac{8}{2}$$

Now, $Y_0 = Y_7 + Y_8$

$$Y_0(n) = \underset{\underset{0}{\uparrow}}{0} \quad 7 \quad 5 \quad -4 \quad 6 \quad 3 \quad 8 \quad 0$$

We can observe that the output sequence is same as input sequence shifted by one sample. We notice that $y_0[n] = x[n-1]$. The factor of $\frac{1}{2}$ has taken care of the scaling. Delay has occurred on account of causality need. We want filter to be casual. Causality is needed because if we do not allow some delay *i.e.*, time for the processing then we could not have real time processing. Causality is therefore required for a real time processing. We could have done without the delay if we do Non-Casual filtering in at least one of analysis or synthesis side.

3 Periodizing the Input

The periodic input $\tilde{x}[n]$ is:

$$\tilde{x}[n] = \sum_{k=-\infty}^{\infty} x[n+kN], \qquad N \geq 6 \tag{8}$$

For simplicity we will take N=6, so

$$\tilde{x}[n] = \ldots \quad 3 \quad 8 \quad \underset{\underset{0}{\uparrow}}{7} \quad 5 \quad -4 \quad 6 \quad 3 \quad 8 \quad 7 \quad 5 \quad -4 \ldots$$

We will analyze the output only in the range (0-5).

$$Y_1 = \ldots \quad \underset{\underset{0}{\uparrow}}{15} \quad 12 \quad 1 \quad 2 \quad 9 \quad 11 \quad 15 \ldots$$

$$Y_2 = \ldots \quad \underset{\underset{0}{\uparrow}}{1} \quad 2 \quad 9 \quad -10 \quad 3 \quad -5 \quad 1 \ldots$$

Now downsampling:

$$Y_3 = \ldots \quad \underset{\underset{0}{\uparrow}}{15} \quad 1 \quad 9 \quad 15 \ldots$$

$$Y_4 = \ldots \quad \underset{\underset{0}{\uparrow}}{1} \quad 9 \quad 3 \quad 1 \ldots$$

Period of Y_3 and Y_4 is '3'.
Now Upsampling:

$$Y_5 = \ldots \quad \underset{\underset{0}{\uparrow}}{15} \quad 0 \quad 1 \quad 0 \quad 9 \quad 0 \quad 15 \ldots$$

$$Y_6 = \ldots \quad \underset{\underset{0}{\uparrow}}{1} \quad 0 \quad 9 \quad 0 \quad 3 \quad 0 \quad 1 \ldots$$

Period of Y_5 and Y_6 is 6.

$$Y_7(n) = \dots \quad \frac{15}{2} \quad \frac{15}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{9}{2} \quad \frac{9}{2} \quad \frac{15}{2} \dots$$

$$Y_8(n) = \dots \quad \frac{1}{2} \quad \frac{-1}{2} \quad \frac{9}{2} \quad -\frac{9}{2} \quad \frac{3}{2} \quad -\frac{3}{2} \quad \frac{1}{2} \dots$$

$$Y_0 = \dots \quad \underset{\substack{\uparrow \\ 0}}{8} \quad 7 \quad 5 \quad -4 \quad 6 \quad 3 \quad 8 \dots$$

As expected output is delayed by one sample and is same as input. It it periodically repeated with a period of 6.