WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 43: Tutorial on Uncertainty Product

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1 Uncertainty Product

There is a bound on simultaneous time and frequency localization. So essentially one cannot localize as much as one wants simultaneously in time and frequency.
One can define for time centered function $x(t)$, time variance

$$\sigma_t^2(x) = \frac{\|tx(t)\|_2^2}{\|x(t)\|_2^2}$$

Recall meaning of time center, time center

$$t_0 = \int \frac{t|x(t)|^2}{\|x(t)\|_2^2} dt$$

By time centered, we mean $t_0 = 0$.
Similarly for frequency, frequency center or frequency mean

$$\Omega_0 = \int \frac{\Omega|\widehat{x}(\Omega)|^2}{\|\widehat{x}(\Omega)\|_2^2} d\Omega$$

By frequency centered, we mean $\Omega_0 = 0$.
Analogous to time variance, for frequency centered function $\widehat{x}(\Omega)$ frequency variance

$$\sigma_\Omega^2(x) = \frac{\|\Omega\widehat{x}(\Omega)\|_2^2}{\|\widehat{x}(\Omega)\|_2^2}$$

If function is not time centered and frequency centered then one need to take second moment around respective centers.
For an $L_2(\mathbb{R})$ function the uncertainty product i.e. product of time and frequency variance is lower bounded by 0.25.

$$\sigma_t^2(x).\sigma_\Omega^2(x) \geq \frac{1}{4}$$

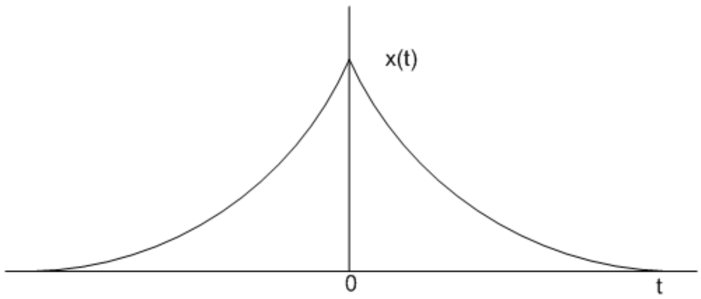
Example 1. Calculate uncertainty product of $e^{-|t|}$ for all t.
Sol. First we need to verify if the function is $L_2(\mathbb{R})$ and check if it is centered in time and frequency.

$$\|x(t)\|_2^2 = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

$$\|x(t)\|_2^2 = 2\int_0^{+\infty} e^{-2t} dt$$

$$\|x(t)\|_2^2 = 1$$

From the sketch of $x(t)$ one can clearly say that it is symmetric about $t = 0$ and is a real and even function of t. Since it real and even function in time domain it should have real and even fourier transform too.
So obviously the function $x(t)$ is time and frequency centered.



Time variance σ_t^2:

$$\sigma_t^2(x) = \frac{\int_{-\infty}^{+\infty} t^2|x(t)|^2 dt}{\int_{-\infty}^{+\infty} |x(t)|^2 dt} = 2\int_0^{+\infty} t^2 e^{-2t} dt = \frac{1}{2}$$

Frequency variance σ_Ω^2:

$$\sigma_\Omega^2(x) = \frac{\|\Omega\widehat{x}(\Omega)\|_2^2}{\|\widehat{x}(\Omega)\|_2^2}$$

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By applying Parseval's Theorem it becomes

$$\sigma_\Omega^2(x) = \frac{\left\|\frac{dx(t)}{dt}\right\|_2^2}{\|x(t)\|_2^2}$$

Now

$$x(t) = e^{-|t|}$$

So

$$\left\|\frac{dx(t)}{dt}\right\|_2^2 = \|x(t)\|_2^2$$

$$\sigma_\Omega^2(x) = 1$$

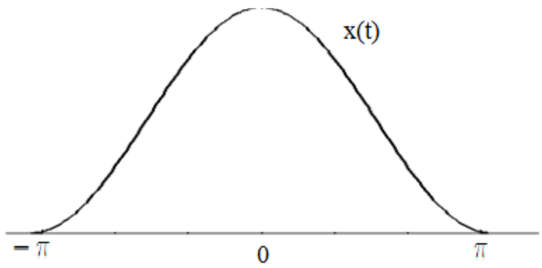
Uncetrtainty Product:

$$\sigma_t^2(x).\sigma_\Omega^2(x) = \frac{1}{2} * 1$$

$$\sigma_t^2(x).\sigma_\Omega^2(x) = 0.5(> 0.25)$$

Example 2. Calculate uncertainty product of raised cosine function.
Sol.

$$\begin{aligned} x(t) &= 1+\cos t \qquad -\pi < t < \pi \\ &= 0 \qquad\qquad otherwise \end{aligned}$$



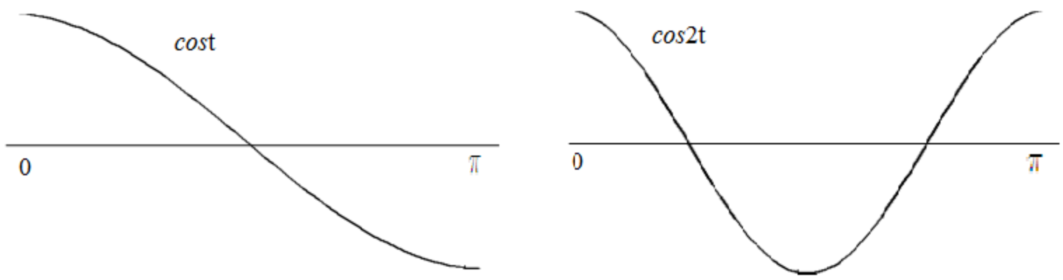
From the above figure it is clear that the function $x(t)$ is real and even, so it is already time and frequency centered. Since the function is real and even, hence

$$\begin{aligned} \|x(t)\|_2^2 &= \int_{-\infty}^{+\infty} |x(t)|^2 dt \\ &= 2\int_0^{+\infty} |x(t)|^2 dt \\ &= 2\int_0^{\pi} (1+\cos t)^2 dt \\ &= 2\int_0^{\pi} (1+\cos^2 t + 2\cos t) dt \end{aligned}$$

Consider

$$\int (1+\cos^2 t + 2\cos t)dt = \int (1 + \frac{1+\cos 2t}{2} + 2\cos t)dt$$

Now let us sketch $\cos t$ and $\cos 2t$



From the above figure it is clear that they have zero integral over $]0,\pi[$
So the above integral becomes

$$\|x(t)\|_2^2 = 2\int_0^{\pi} (1+\frac{1}{2})dt = 3\pi$$

Frequency variance σ_Ω^2:

$$\sigma_\Omega^2(x) = \frac{\left\| \frac{dx(t)}{dt} \right\|_2^2}{\|x(t)_2^2\|}$$

Now

$$\frac{d}{dt}x(t) = \frac{d}{dt}(1+\cos t) = -\sin t$$

$$\begin{aligned}\left\| \frac{dx(t)}{dt} \right\|_2^2 &= 2\int_0^\pi \sin^2 t dt \\ &= \int_0^\pi (1-\cos 2t)dt \\ &= \pi\end{aligned}$$

So frequency variance $\sigma_\Omega^2 = \frac{1}{3}$

Time variance σ_t^2:

We need

$$\begin{aligned}\int t^2|x(t)|^2 dt &= \int t^2(1+\cos t)^2 dt \\ &= \int t^2 \left(1 + \frac{1+\cos 2t}{2} + 2\cos t\right) dt\end{aligned}$$

Let us consider the term

$$\int t^2 \cos mt dt$$

$$= t^2\frac{\sin mt}{m} + 2t\frac{\cos mt}{m^2} - 2\frac{\sin mt}{m^3}$$

Now our limit is 0 to π, therefore we do not need to look at the 'sin' term which are zero at $t = 0$ and $t = \pi$. Again we do need terms containing 't' at $t = 0$. We need only consider the term $2t\frac{\cos mt}{m^2}\Big|_0^\pi$

For m=1, $2t\cos t\Big|_0^\pi = 2\pi\cos\pi = -2\pi$

For m=2, $2t\frac{\cos t}{4}\Big|_0^\pi = 2\pi\frac{\cos 2\pi}{4} = 0.5\pi$

Putting these values in above equation

$$\begin{aligned}\int_0^\pi t^2\left(1+\frac{1+\cos 2t}{2} + 2\cos t\right) dt &= \int_0^\pi 1.5t^2 dt + \int_0^\pi t^2\frac{\cos 2t}{2}dt + 2\int_0^\pi t^2\cos t dt \\ &= \frac{\pi^3}{2} + \frac{\pi}{4} - 4\pi\end{aligned}$$

Time variance σ_t^2

$$\sigma_t^2 = \frac{2\int_0^\pi t^2|x(t)|^2 dt}{2\int_0^\pi (1+\cos t)^2 dt}$$

$$\sigma_t^2 = \frac{\pi^2}{3} - \frac{5}{2}$$

<u>Uncertainty Product</u>:

$$\sigma_t^2(x).\sigma_\Omega^2(x) = (\frac{\pi^2}{3} - \frac{5}{2}) * \frac{1}{3}$$

$$\sigma_t^2(x).\sigma_\Omega^2(x) = \frac{\pi^2}{9} - \frac{5}{6} \, (> 0.25)$$