WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 41: Tutorial-1

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1 Tutorial Exercises

Consider the following two functions:

$$lclx_1(t) = 1 - |t| \qquad -1 \leq t \leq 1 \tag{1}$$

$$= 0 \qquad otherwise \tag{2}$$

$$lclx_2(t) = e^{-t} \qquad t \geq 0 \tag{3}$$

$$= 0 \qquad otherwise \tag{4}$$

Q 1. Verify that $x_1(t)$ and $x_2(t)$ belong to $L_2(\mathbb{R})$. Also find their norms.
Ans. We will find norms of $x_1(t)$ and $x_2(t)$ and show that they are finite.

norm squared of x_1 in $L_2(\mathbb{R})$ =

$$\int_{-\infty}^{\infty} |x_1(t)|^2 dt$$

from symmetry,

$$||x_1||_2^2 = 2\int_0^{\infty} (1-t)^2 dt = \frac{2}{3}$$

$$||x_1||_2 = \sqrt{\frac{2}{3}}$$

Similarly,

$$||x_2||_2^2 = \int_0^{\infty} (e^{-t})^2 dt = \frac{1}{2}$$

$$||x_1||_2 = \sqrt{\frac{1}{2}}$$

Since L_2 norm is finite for both functions, they belong to $L_2(\mathbb{R})$.

Q 2. Obtain the projections of x_1 and x_2 in the space V_0 in the Haar MRA.
Ans. First, let us do the exercise for function x_1.
It is easy to see that non zero projections will only be there in]-1,1[and by symmetry, projection of $x_1(t)$ in]-1,0[= projection of $x_1(t)$ in]0,1[= average of function in each of the intervals which is equal to

$$\int_0^1 (1-t)dt = 0.5$$

We can plot this projection as shown in fig-1. We will denote it by $Proj_{V_0}x_1$.

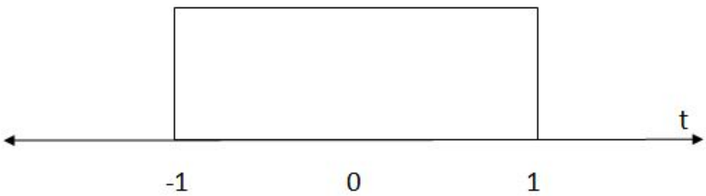

Figure 1: Projection of x_1 in V_0

Now let us do the same exercise for function x_2. Its projection will be non-zero in only positive half of real axis.
Consider the standard intervals of unit length]n,n+1[. Projection of x_2 in this interval will be

$$\int_n^{n+1} e^{-t}dt = e^{-n}(1-e^{-1})$$

Thus, we get exponentially decaying series of constants as depicted in fig-2.

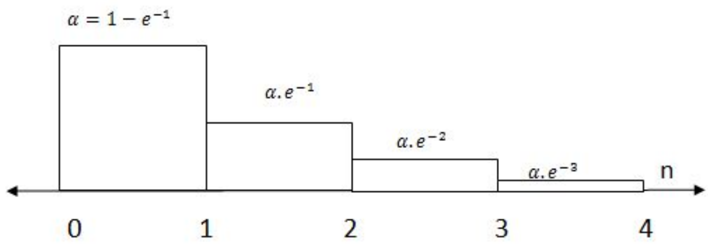

Figure 2: Projection of x_2 in V_0

To verify that this projection also belongs to $L_2(\mathbb{R})$, we will show finite value of $\int_{-\infty}^{\infty} |Proj_{V_0}x_1|^2dt$

$$\int_{-\infty}^{\infty} |Proj_{V_0}x_1|^2dt = \sum_{n=0}^{\infty}(e^{-n}(1-e^{-1}))^2 \tag{5}$$

$$\tag{6}$$

$$= \quad (1-e^{-1})^2 \sum_{n=0}^{\infty} e^{-2n} \tag{7}$$

$$\tag{8}$$

$$= \quad \frac{(1-e^{-1})^2}{(1-e^{-2}}) \tag{9}$$

Hence, the projection also belongs to $L_2(\mathbb{R})$.

Q 3. Obtain the projections of the functions x_1 and x_2 on the space V_1 in the Haar MRA.
Ans. We need standard intervals of lenth $2^{-1} = 0.5$ to get projections in space V_1.
By symmetry, We can evaulate only in]0,1[.
In interval]0,0.5[

$$\frac{1}{\frac{1}{2}} \int_{0}^{0.5} (1-t)dt = 0.75$$

In interval]0.5,1[

$$\frac{1}{\frac{1}{2}} \int_{0.5}^{1} (1-t)dt = 0.25$$

This is denoted by $Proj_{V_1}x_1$ and is depicted in fig-3.

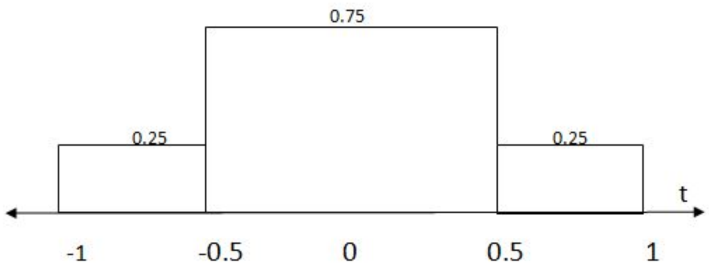

Figure 3: Projection of x_1 in V_1

To get the ideas of projections clear, we draw both $Proj_{V_1}x_1$ and $Proj_{V_0}x_1$ (shown in thick red line) on the same graph in fig-4.

Now, we can find the projection of x_1 in incremental subspace W_0:

$$Proj_{W_0}x_1 = Proj_{V_1}x_1 - Proj_{V_0}x_1$$

This shown in fig-5.
We can observe that

$$Proj_{W_0}x_1 = 0.25\psi(t) - 0.25\psi(t+1)$$

where $\psi(t)$ is Haar Wavelet function.

Now let's do the same for function x_2.

In the interval]0.5n,0.5(n+1)[where $n\epsilon\mathbb{Z}$ and $n \geq 0$,

$$Proj_{V_1}x_2 = \frac{1}{2} \int_{0.5n}^{0.5(n+1)} e^{-t}dt$$

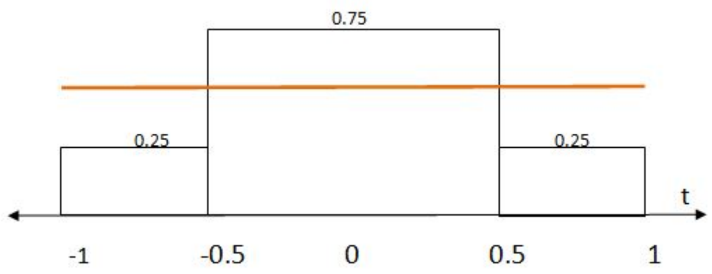

Figure 4: Projection of x_1 in V_1 and V_0 (shown in thick red line) in the same graph

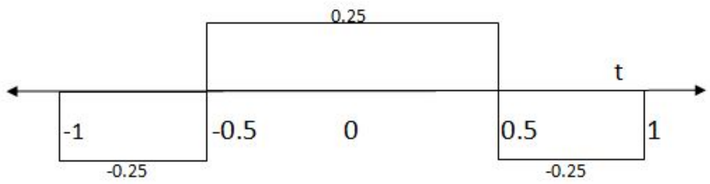

Figure 5: Projection of x_1 in W_0

$$= 2e^{-\frac{n}{2}}(1 - e^{\frac{-1}{2}})$$

which is an exponential sequence. We can see that exponnential nature of function replicates itself in the projection.
Now we will find $Proj_{W_0}x_2$ in]n,n+1[. It will be a multiple of $\psi(t-n)$. The constant by which $\psi(t-n)$ denoted by d_n can be found as following:
d_n = average of x_2 over]n,n+0.5[- average of x_2 over]n,n+1[

$$d_n = e^{-n}(1 - e^{\frac{-1}{2}}) - e^{-n}(1 - e^{-1})$$

$$d_n = e^{-n}(e^{-1} - e^{\frac{-1}{2}})$$

Therefore,

$$Proj_{W_0}x_2 = \sum_{n=0}^{\infty} d_n \psi(t-n)$$

For exponentially decaying functions, the projections on V_m ($m\epsilon\mathbb{Z}$) and the projections on W_m ($m\epsilon\mathbb{Z}$) are all exponentially decaying piecewise constants.

2 Self Evaluation Quizzes

Q 1. Show that d_n can also be obtained by $< x_2, \psi(t-n) >$.
Ans.

$$
\begin{aligned}
< x_2, \psi(t-n) > &= \int_{n}^{n+0.5} e^{-t} dt - \int_{n+0.5}^{n+1} e^{-t} dt \\
&= e^{-n} - 2e^{-(n+0.5)} + e^{-(n+1)}
\end{aligned}
$$

On rearrangement, we get the same value for d_n as obtained above in Question-3.