WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 39: Inability of Simultaneous Time and Band Limitedness

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1 Introduction

In this lecture we will try to prove a theorem based on the Uncertainty principle of Fourier transform which states that a non-zero function cannot be both time limited and band limited at the same instance. Basically this means that a non-zero function cannot be compactly supported in both the domains simultaneously. Now a function is said to be compactly supported if it has a finite support; for example a rectangular pulse ranging from point a to b on real line is compactly supported. On contrary, the Gaussian function which extends throughout the real line is not a compactly supported function. We will prove the result for a class of functions in L_1 space and then extend the proof for a general class L_P where $1 < P < \infty$.

2 Background of the proof

Let us first recall some basics before we study other tools used in the proof and the proof itself.

2.1 Fourier transform

Fourier transform provides a way to look at the signal from its frequency domain. Fourier transform rely on the principle that a signal can be represented by the linear combination of sine waves, and their respective frequencies together constitute the frequency domain of the signal. Further we can move to complex domain since $\sin(t) = \dfrac{e^{jt} - e^{-jt}}{2j}$. Mathematically, Fourier transform projects the function onto the complex exponentials. Thus, Fourier transform of a function $x(t)$ is given as:

$$X(\Omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\Omega t}dt$$

NOTE:
For a function, their exists a Fourier transform only if the function is absolutely integrable i.e. $\mathbf{x(t)}$ is said to have a Fourier transform only if $\int_{-\infty}^{+\infty} |\mathbf{x(t)}|\mathbf{dt}$ is finite.

2.2 Space $L_P(\mathbb{R})$ and Norm

A function $x(t)$ lies in space $L_P(\mathbb{R})$ if $\left\{ \int_{-\infty}^{+\infty} |x(t)|^P dt \right\}^{\frac{1}{P}}$ is finite, and the term is called as the L_P norm of the function $x(t)$ where $1 \leq P \leq \infty$. With the context of our discussion let us see an example of function with finite L_1 norm:

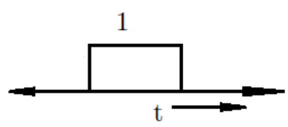

Figure 1: Function belonging to $L_1(\mathbb{R})$

We see that, for above given function the absolute integral of the function is equal to 1 and thus it belong to space L_1. Also, it can be proved that the function exists in space L_P in general for $1 \leq P < \infty$. On the other hand *sinc* function does not have a finite absolute integral and therefore does not belong to space L_1.

2.3 Hölders Inequality

Aforesaid, in this proof we will consider the functions in L_1 space and then extend the proof more general class of functions in spaces L_P where $1 < P \leq \infty$. Now, for time limited functions it can be proved that if a function belongs to L_P where $1 < P \leq \infty$ then it also belongs to space L_1 using Hölders Inequality.
Mathematically,

$$\begin{aligned} f &\in L_P, \quad for \quad p \in (1, \infty) \\ &\Rightarrow f \in L_1 \end{aligned}$$

Hölders Inequality states that for two functions $f(t)$ and $g(t)$;

$$\int_{-\infty}^{+\infty} f(t)g(t)dt \leq \left(\int_{-\infty}^{+\infty} |f(t)|^p dt\right)^{\frac{1}{p}} \left(\int_{-\infty}^{+\infty} |g(t)|^q dt\right)^{\frac{1}{q}} \tag{1}$$

The above condition holds true only if $\frac{1}{p} + \frac{1}{q} = 1$ and $p, q > 0$. The above result of Hölders Inequality can be obtained using Young's Inequality and generalized AM-GM Inequality stated as below.

2.4 Young's Inequality

Young's inequality states that, given two positive real numbers a and b,

$$\begin{aligned} a, b &\in \mathbb{R}^+ \\ ab &\leq \frac{a^p}{p} + \frac{b^q}{q} \end{aligned} \tag{2}$$

Given,

$$\frac{1}{p} + \frac{1}{q} = 1$$

This can be proved using the generalized AM-GM Inequality as explained below.

2.5 Arithmetic Mean- Geometric Mean (AM-GM) Inequality

AM-GM inequality states that, given two positive real numbers a and b with weights α_1 and α_2, following equations holds.

$$\frac{\alpha_1 a + \alpha_2 b}{\alpha_1 + \alpha_2} \geq \left(a^{\alpha_1} b^{\alpha_2}\right)^{\frac{1}{\alpha_1 + \alpha_2}}$$

Now, substituting $a = a^p$, $b = b^q$, $\alpha_1 = p$ and $\alpha_2 = q$ and knowing that $\frac{1}{p} + \frac{1}{q} = 1$, we get equation (2) i.e. Young's inequality.
Further we will use the result of Young's inequality to get the result (1) i.e. the Hölders Inequality. To do this, consider two real functions $f(t)$ and $g(t)$ in space $L_p(\mathbb{R})$ and $L_q(\mathbb{R})$ respectively. We first normalize these functions by dividing each of them with their respective norms.
Mathematically,

$$\begin{aligned} f(t) &\in L_p(\mathbb{R}) \\ g(t) &\in L_q(\mathbb{R}) \end{aligned}$$

given that $\frac{1}{p}+\frac{1}{q}=1$. Now the normalized functions are given as:

$$\begin{aligned} F(t) &= \frac{f(t)}{\parallel f \parallel_p} \\ G(t) &= \frac{g(t)}{\parallel g \parallel_q} \end{aligned}$$

Where,

$$\begin{aligned} \parallel f \parallel_p &= \left(\int_{-\infty}^{+\infty} |f(t)|^p dt\right)^{\frac{1}{p}} \\ \parallel g \parallel_q &= \left(\int_{-\infty}^{+\infty} |g(t)|^q dt\right)^{\frac{1}{q}} \end{aligned}$$

Now using Young's inequality we can point wise state that,

$$\frac{|F^p(t)|}{p}+\frac{|G^q(t)|}{q} \geq |F(t)G(t)|$$

Integrating above equation over real line we get,

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{|F(t)|^p}{p} dt+\int_{-\infty}^{+\infty} \frac{|G(t)|^q}{q} dt &\geq \int_{-\infty}^{+\infty} |F(t)G(t)| dt \\ \int_{-\infty}^{+\infty} \frac{|f(t)|^p}{\|f\|_p^p p} dt+\int_{-\infty}^{+\infty} \frac{|g(t)|^q}{\|g\|_q^q q} dt &\geq \int_{-\infty}^{+\infty} \frac{|f(t)g(t)|}{\|f\|_p\|g\|_q} dt \\ \frac{1}{p} \frac{\int_{-\infty}^{+\infty} |f(t)|^p dt}{\|f\|_p^p}+\frac{1}{q} \frac{\int_{-\infty}^{+\infty} |g(t)|^q dt}{\|g\|_q^q} &\geq \frac{\int_{-\infty}^{+\infty} |f(t)g(t)| dt}{\|f\|_p\|g\|_q} \\ \frac{1}{p}+\frac{1}{q} &\geq \frac{\int_{-\infty}^{+\infty} |f(t)g(t)| dt}{\|f\|_p\|g\|_q} \end{aligned}$$

Therefore,

$$\|f\|_p\|g\|_q \geq \int_{-\infty}^{+\infty} |f(t)g(t)| dt \tag{3}$$

Since,

$$\frac{1}{p}+\frac{1}{q} = 1$$

Thus, using the Hölders Inequality as stated in equation (3), by selecting appropriate function $g(t)$ we can prove that function $f(t)$ which belongs to $L_p(\mathbb{R})$ also belongs to $L_1(\mathbb{R})$. Now, to prove this consider a function spread over finite interval C which belongs to space L_p, $1<p<\infty$. Since the function $f(t)$ belongs to space L_p its p^{th} norm is finite i.e. $\left(\int_{-\infty}^{+\infty} |f(t)|^p dt\right)^{\frac{1}{p}}$ is finite. The function $f(t)$ can be similar to that shown in figure 2. We select $g(t)$ as shown in figure 3 to get the result. Now putting down the Hölders Inequality for these two functions as follows:

$$\begin{aligned} \|f\|_p\|g\|_q &\geq \int_C |f(t)g(t)| dt \\ \int_C |f(t)| dt &\leq \left(\int_C |f(t)|^p dt\right)^{\frac{1}{p}}\left(\int_C dt\right)^{\frac{1}{q}} \\ \int_C |f(t)| dt &\leq \left(\int_C |f(t)|^p dt\right)^{\frac{1}{p}} |C|^{\frac{1}{q}} \end{aligned}$$

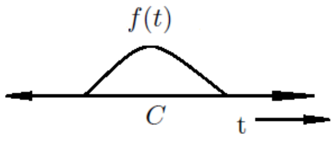

Figure 2: function $f(t)$

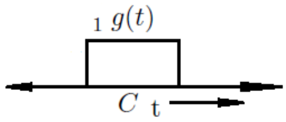

Figure 3: function $g(t)$

Since the right side of the above equation is the product of L_p norm which is finite and another finite quantity the net result on the RHS is finite. Also since the LHS is less than or equal to RHS we can infer that the L_1 norm of $f(t)$ is finite. Thus, we have proved our statement that if a function belong to space $L_p(\mathbb{R})$, where $1 < p < \infty$ then it also belongs to space L_1.
With this background let us now know some tools which will help us in proving the final result.

3 Tools for the proof

3.1 Fourier transform of $rect(a)$

The Fourier transform of the $rect(a)$ function shown in figure 4 is

$$\begin{aligned}
\mathbb{F}(rect(a)) &= \frac{2}{\Omega}\sin\left(\frac{a\Omega}{2}\right) \\
&= \frac{1}{\pi f}\sin(\pi a f) \\
&= asinc(af)
\end{aligned}$$

Where, Ω= angular frequency and f= Hertz frequency.

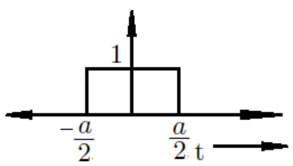

Figure 4: function $rect(a)$

3.2 Vandermonde Matrix

A Vandermonde matrix is defined as

$$V_{i,j} = a_i^{j-1}$$

$$V = \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_m & a_m^2 & \cdots & a_m^{n-1} \end{bmatrix}$$

Where, m and n are the number of rows and columns respectively.
Further, whenever we need to solve any linear equation of the form$A\bar{x} = \bar{y}$, we always look for the invertibility of matrix A so that we can solve for $\bar{x}$ as $\bar{x} = A^{-1}\bar{y}$. Now the Vandermonde matrix have very good properties by which it can be shown to be invertible once we know the entries. This is because the determinant of a Vandermonde matrix can be proved to be of the form

$$det(V) = (-1)^{m\frac{m-1}{2}} \cdot \prod_{1\leq i<j\leq m} (a_i - a_j)$$

Therefore until all the entries of a particular row are distinct the value of $det(V)$ will not be equal to zero and V will be invertible.

4 Review of existing proof

Consider a very smooth function which is compactly supported. For example we can have the following function:

$$\begin{aligned} f(x) &= e^{\frac{-1}{1-x^2}} \qquad |x| \leq 1 \\ &= 0 \qquad\quad\;\; else \end{aligned}$$

The function has the structure as shown in figure 5. It can be shown that all the derivatives of $f(x)$

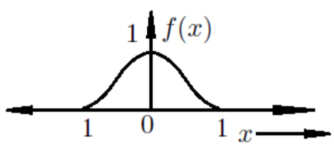

Figure 5: Example of a smooth and compactly supported function

exists in the interval $-1 \leq x \leq 1$ with all derivatives $f^p(x) \rightarrow 0$ as $|x| \rightarrow 1$. Now we need to find if we can have a compactly supported spectrum for such well behaved time limited function. Let us look at an existing proof.
For any general well behaved function in space $L_1(\mathbb{R})$ similar to above example we have the Fourier transform given by:

$$F(\Omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\Omega t}dt$$

Now we will also assume that the function is band limited and show that our assumption will hold only for a zero function. Therefore, with the assumption we have

$$\begin{aligned} |F(\Omega)| &\leq \int_{-\infty}^{+\infty} |f(t)||e^{-j\Omega t|}dt \\ &\leq \int_{-\infty}^{+\infty} |f(t)|dt \\ &\leq L_1 \; norm \; of \; f(t) \end{aligned}$$

Now, if we take the derivative of $f(t)$, it can be represented as,

$$f'(t) \quad = \quad j\Omega \int_{-\infty}^{+\infty} f(t)e^{-j\Omega t}dt$$

In above equation, since function is bounded in both time and frequency, we get a finite bound on the magnitude of $f'(t)$. Similarly we can show that all the derivatives of $f(t)$ have finite bound on their magnitude. Further we can express function $f(t)$ in the form of a McLauren series as:

$$f(t) \quad = \quad f(0) + tf'(0) + \frac{t^2}{2!}f''(0) + \cdots$$

Now as per our assumption all the above derivatives of $f(t)$ are finite and therefore above equation is essentially a power series expansion i.e.

$$f(t) \quad = \quad \sum_{i=0}^{\infty} a_i t^i \qquad \forall\ t$$

Now by 'Identity theorem', if the power series is zero in any open interval, then it is going to be zero everywhere on the real line. In our case we are essentially dealing with functions which are compactly supported i.e. having zero value outside the support. Therefore as per the Identity theorem whole of the function must be zero. Thus we can state that a non-zero function cannot be both time and band limited simultaneously.
We see that above proof requires some knowledge of complex analysis and we will therefore follow another approach which we will discuss now.

5 The Proof

The proof of the theorem is based on the by the Whittaker Shannon interpolation formula which states that under certain limiting conditions, a function x(t) can be recovered exactly from its samples. The formula is as follows:

$$x(t) \quad = \quad \sum_{n=-\infty}^{\infty} x[n] \cdot \mathrm{sinc}\left(\frac{t-nT}{T}\right)$$

where,

$$\begin{aligned} x[n] \quad &= \quad \textit{samples} \\ T \quad &= \quad \textit{Sampling period} \end{aligned}$$

Now, consider a function which is band limited. When we multiply the function with a train of impulse we get the discrete version of signal. In frequency domain we are essentially convolving the fourier transform of the signal and the fourier transform of the train of impulse. Now, if the sampling time is taken to be say T, then the fourier transform of the train of impulses is another train of impulse with a separation of $\frac{1}{T}$ between subsequent samples on the frequency axis. Further as per the property of delta function, its convolution with any other function gives back the original function and thus we get a repeated structure in the fourier domain wherein the spectrum of the function is repeated after an interval of $\frac{1}{T}$ on frequency axis. Now for perfect reconstruction of the original signal in time domain, the repetition rate should be such that the spectrum of original signal doesn't overlap. This is ensured by maintaining a sufficiently large sampling rate in time domain which is given by Shannon-Nyquist sampling rate which is $T \leq \frac{1}{2f}$, where f is the bandwidth of the signal.
Now let us consider functions which are time and bandlimited. Let us assume some arbitrary structure to represent the sinal in two domains as in figure 6 and 7. The function when discretized by multiplying with a train of impulse will have its time and frequency domain representation as shown

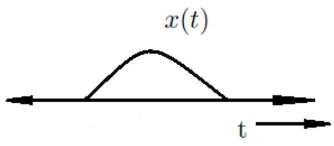

Figure 6: Example of a time limited function

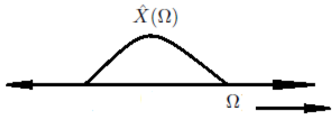

Figure 7: Frequency domain representation of an arbitrary time limited function shown in figure (6)

in figure 8 and 9.
Now suppose that we have allowed some freedom while using the low pass filtering which has a cutoff of Ω' such that $\Omega' > \Omega$. We than define a parameter $m = 2T\Omega'$. On the other hand in time domain due to the corresponding operations we get the series to be as follows:

$$x(t) = \sum_{n=-\infty}^{\infty} mx[n] \cdot \text{sinc}\left(m\frac{t-nT}{T}\right)$$

But since the signal signal has a finite interval the value of n will run for a finite duration, say from $+k$ to $-k$. Therefore the series will be of the form

$$\begin{aligned} x(t) &= \sum_{n=-k}^{k} mx[n] \cdot \text{sinc}\left(m\frac{t-nT}{T}\right) \\ &= \sum_{n=-k}^{k} mx[n] \cdot \frac{\sin \pi m \frac{t-nT}{T}}{\pi m \frac{t-nT}{T}} \end{aligned}$$

The reconstruction will occur provided $1 \geq m \geq 2T\Omega$. Now as long as the condition on m is satisfied $x(t)$ will be a function of m and t $x(t) = f(m,t)$. For a fixed t $x(t)$ becomes a function of m alone i.e. $x(t) = f(m)$. Taking the derivative with respect to m we get,

$$f'(m) = \sum_{n=-k}^{k} x[n] \cdot \cos \pi m \frac{t-nT}{T}$$

Since $x(t)$ was already fixed $f'(m) = 0$. Thus, in general higher derivatives of $f(m)$ can be represented as,

$$f^{2i+1'}(m) = \sum_{n=-k}^{k} x[n] \cdot \left(\pi\frac{t-nT}{T}\right)^{2i} \cdot \cos \pi m \frac{t-nT}{T}$$

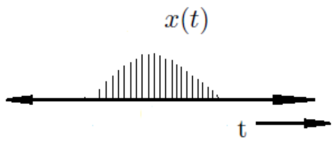

Figure 8: Discretized signal

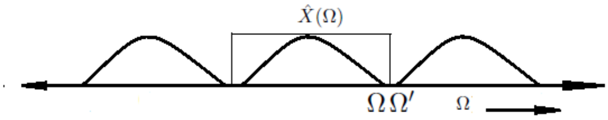

Figure 9: Frequency domain representation of an arbitrary time limited function shown in figure (8)

$$= 0$$

Further we can write this whole equations in matrix form where,

$$\begin{aligned}
V_{in} &= \left(\pi \frac{t-nT}{T}\right)^{2i} \\
C_n &= x[n] \cos \pi m \frac{t-nT}{T} \\
VC &= 0
\end{aligned}$$

Now, the V matrix has a Vandermonde sort of structure and is going to be invertible as long as $(t - Tn_i)^2 \neq (t - Tn_j)^2$. Thus, the matrix is non invertible only for finite values of t, say a set B given by

$$t = \frac{n_i + n_j}{2} T$$

For all other values of t we have

$$\begin{aligned}
C &= V^{-1} \cdot [0] \\
&= 0
\end{aligned}$$

i.e. C_n is zero for all n. This is possible if $x[n] = 0$ or $\cos \pi m \frac{t-nT}{T} = 0$ or both of them are zero. Now the *cos* part is zero only when $m \frac{t-nT}{T} = (2i+1)\frac{\pi}{2}$. But since the function is zero for other values of m as well, we can interpret that $x[n] = 0$ for all other m. Further it can be shown that time limited and band limited functions are very smooth and if this is true our cardinal series exactly converges to $x(t)$. And since $x[n] = 0$, $x(t) = 0$. This is true only for the values of t which do not belong to set B, but since we have assumed a continuous function it cannot be so that the function is non zero over some set of values and thus $x(t) = 0$ for all t. Thus only possible function to be limited in both time and frequency domain simultaneously is a zero function.