WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 36: The Polyphase and the Modulation Approach

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1 Introduction

In the previous lecture we have discussed briefly with the idea of polyphase decomposition. We will use it to construct different kinds of structure to carry out computation efficiency in a filter bank. In this lecture we will put down formally the approach based on polyphase components for perfect reconstruction.

We will discuss two approaches for perfect reconstruction:
(1) The Polyphase Approach
(2) The Modulation Approach.
And we will generalized it for M-bank filter bank.

2 The Polyphase Approach

Consider two bank filter bank.

$$x[n] \underset{Z-Transform}{\longrightarrow} X(Z)$$

The Polyphase decomposition of order two is given as,

$$\begin{aligned} n &= 2m \\ &= 2m+1, \text{ for all integer m} \end{aligned}$$

To generalize it, consider the order 3.

$$\begin{aligned} n &= 3m \\ &= 3m+1 \\ &= 3m+2, \text{ for all integer m} \end{aligned}$$

In general for order M

$$\begin{aligned} n &= Mm \\ &= Mm+1 \\ &\vdots \\ &= Mm+(M-1), \text{ for all integer m} \end{aligned}$$

So now let us put down explicitly the mechanism for the decomposition of $X(Z)$, the Z-transform of $x[n]$, into the Z-transform of the polyphase component of the order M. For that we need to spilt the index M.

To decompose $X(Z)$ using polyphase decomposition of the order M.

$$X(Z) = \sum_{n=-\infty}^{+\infty} x[n]Z^{-n}$$

$$\sum_{n=-\infty}^{+\infty} \cdots (n) \text{ variation as a function of } n$$

Now splitting $\sum_n$ into

$$\sum_{l=0}^{M-1} \sum_{m=-\infty}^{+\infty} \cdots (Mm+l) \text{ variation as a function of } (Mm+l)$$

$$\begin{aligned} X(Z) &= \sum_{l=0}^{M-1} \sum_{m=-\infty}^{+\infty} x[Mm+l]Z^{-(Mm+l)} \\ X(Z) &= \sum_{l=0}^{M-1} Z^{-l} \left\{ \sum_{m=-\infty}^{+\infty} x[Mm+l]Z^{-(Mm)} \right\} \end{aligned}$$

The term $\sum_{m=-\infty}^{+\infty} x[Mm+l]Z^{-(Mm)}$ is the Z-transform of all those points which lies at multiples of $(M+l)$.

For example $l=0$ this represents the Z-transform of all those points which lie at multiples of M. $l=1$ this refers to all those points which lie at multiples of $(M+1)$ displaced by 1 from multiples of M. And so on this can go up to $(M-1)$ after this l again becomes zero.

We break $X(Z)$ into M disjoint parts.

$$X(Z) = \sum_{l=0}^{M-1} Z^{-l} \left\{ \sum_{m=-\infty}^{+\infty} x[Mm+l]Z^{-(Mm)} \right\}$$

The term $\sum_{m=-\infty}^{+\infty} x[Mm+l]Z^{-(Mm)}$ can be represented by $X_{l,M}(Z^M)$. M^{th} order polyphase component and l^{th} of those component with the argument given by Z^M is,

$$X_{l,M}(Z) = \sum_{m=-\infty}^{+\infty} x[Mm+l]Z^{-m}$$

$X_{l,M}(Z)$ is essentially the Z-transform of l^{th} polyphase component of sequence $x[\cdot]$, order M.

Two important points order of decomposition and the component number. There will be as many components as the order.

When $M=2$ then $l=0$ or 1 When $M=3$ then $l=0,1$ or 2 and so on.

Relationship between Z-transform of the original sequence and Z-transform of its polyphase component is given as,

$$X(Z) = \sum_{l=0}^{M-1} Z^{-l} X_{l,M}(Z^M)$$

This is the manifestation of the polyphase decomposition in the Z-domain.

Now we would like to see how polyphase decomposition works when we have analysis and synthesis side. We would like to put down a general relationship for analysis and synthesis polyphase components and how they interact to give perfect reconstruction.

General analysis branch in M-Bank filter bank.

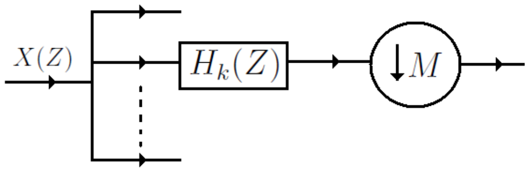


Figure 1: Analysis Branch

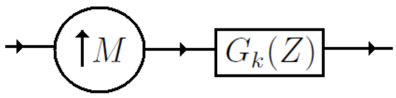


Figure 2: Synthesis Branch

In a given M-Bank filter bank the number of analysis branches and synthesis branches must be same.

B is the number of branches and its value can be different from M.

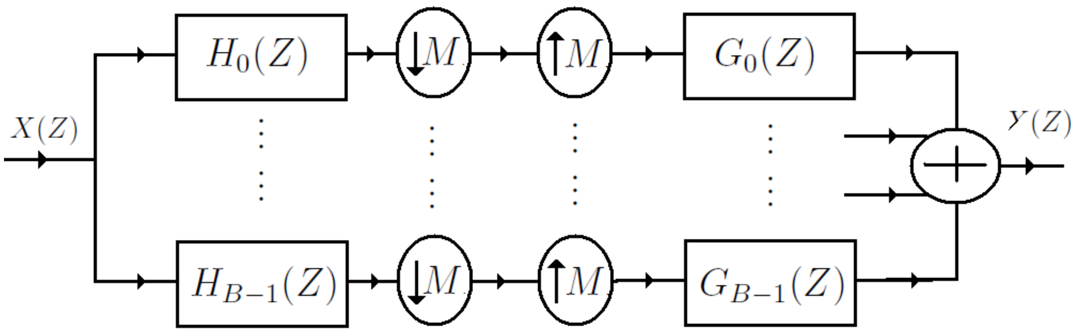


Figure 3: M-Bank Filter Bank Structure

$$
\begin{aligned}
B &= M: \text{ critically sampled M-Bank filter bank} \\
B &< M: \text{ under sampled M-Bank filter bank} \\
B &> M: \text{ over sampled M-Bank filter bank}
\end{aligned}
$$

3 Polyphase Approach

Decompose the filters, both analysis and synthesis into polyphase components and decompose the input and output also into its polyphase component of order M. Order of the polyphase decomposition is the same as the down and up-sampling factors.
Consider the k^{th} branch.

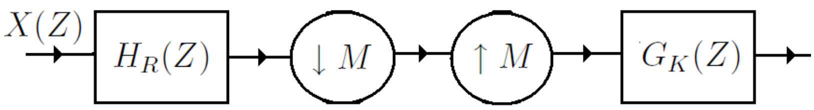


Figure 4: k^{th} branch in M-Bank Filter Bank

Z domain analysis,

$$X(Z) = \sum_{l=0}^{M-1} Z^{-l} X_{l,M}(Z^M)$$

Similarly,

$$H_k(Z) = \sum_{l=0}^{M-1} Z^{-l} H_{k,l,M}(Z^M)$$

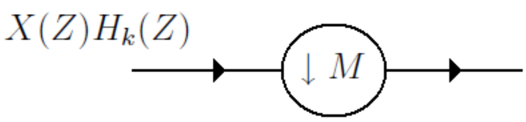


Figure 5: 0^{th} Polyphase component of order M

$$X(Z)H_k(Z) = \sum_{l_1=0}^{M-1} \sum_{l_2=0}^{M-1} Z^{-l_1} Z^{-l_2} X_{l_1,M}(Z^M) H_{k,l_2,M}(Z^M)$$

The 0^{th} polyphase components results when $Z^{-l_1}.Z^{-l_2} = Z^{-(l_1+l_2)}$ contributes $(Z^M)^{l_0}$, $l_0 \in \mathbb{Z}$.

Considering l_1 and l_2, when l_1 is zero, l_2 is also zero. When l_1 is 1, l_2 is $M-1$. And so on when l_1 is $M-1$ l_2 is 1.

With one l_1 there is one unique l_2. The relation for the l_2 is,

$$l_2 = (M - l_1) \text{ modulo M}$$

So when $X(Z)H_k(M)$ is down sampled with M, we will get

$$X_{0,M}(Z)H_{k,0,M}(Z) + \left\{ \sum_{l=1}^{M-1} X_{l,M}(Z) H_{k,M-1,M}(Z) \right\} Z^{-1}$$

Except for the first case (i.e. when l_1 and l_2 both are zero)$(l_1 + l_2)$ gives M as so after down-sampling by M, we get Z^{-M} as Z^{-1}.

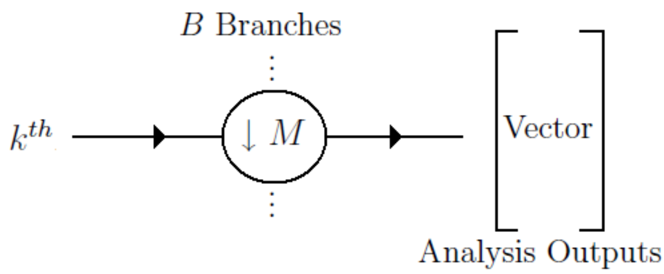


Figure 6: Analysis Outputs of k^{th} branches

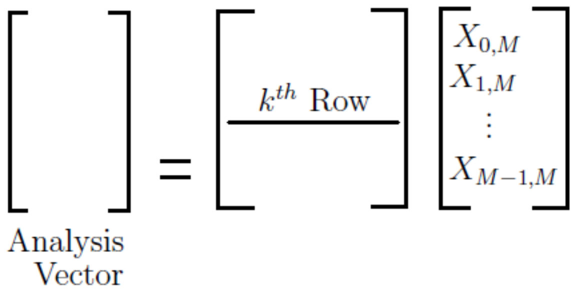


Figure 7: Analysis Polyphase Matrix

The k^{th} row in the matrix (Figure.7) is

$$k^{th} \text{ row} = H_{k,0,M}(\cdot).Z^{-1}.H_{k,M-1,M}(\cdot) \ldots Z^{-1}H_{k,1,M}(Z)$$

The size of the polyphase matrix will be as many branches times the polyphase decomposition $B \times M$.

Lets consider what happens to the k^{th} branch after up sampling by M followed by filtering by G.

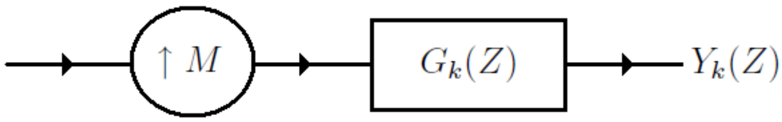


Figure 8: k^{th} Synthesis Branch

Decomposing $Y_k(Z)$ into its polyphase components.

$$Y_k(Z) = \sum_{l=0}^{M-1} Z^{-l} Y_{k,l,M}(Z^M)$$

$$G_k(Z) = \sum_{l=0}^{M-1} Z^{-l} G_{k,l,M}(Z^M)$$

Therefore,

$$Y_{k,l,M}(Z^M) = G_{k,l,M}(Z^M).\{\text{ Output of the } k^{th} \text{ up sampler}\}$$

Output of the k^{th} up-sampler can be given as,

$$\begin{bmatrix} k^{th} \text{ row of analysis} \\ \text{polyphase matrix} \end{bmatrix} \begin{bmatrix} X_{0,M} \\ X_{1,M} \\ \vdots \\ X_{M-1,M} \end{bmatrix}$$

Figure 9: Output of the k^{th} Up Sampler

Because of the up-sampler Z has been replaced by Z^M.

This is the polyphase approach to analyze the overall M-Bank filter Bank. We have seen the k^{th} branch output is

$$Y(Z) = \sum_{k=1}^{B} Y_k(Z) \qquad \text{where, B is the Number of branches}$$

$$Y_{l,M}(Z^M) = \sum_{k=1}^{B} G_{k,l,M}(Z^M)(\text{Output of the } k^{th} \text{ up sampler})$$

We can write down the output of the polyphase vector component and input polyphase component and relate them.

$$\underbrace{\begin{bmatrix} Y_{0,M} \\ Y_{1,M} \\ \vdots \\ Y_{M-1,M} \end{bmatrix}}_{\text{Output Polyphase Components}} = \underbrace{\begin{bmatrix} M \text{ x } B \quad \text{Polyphase} \\ \text{Components of } G_k \end{bmatrix}}_{\text{Synthesis Polyphase Matrix}}$$

Figure 10:

l^{th} Row of synthesis polyphase matrix is

$$[G_{0,l,M}(Z^M) \cdots G_{B-1,l,M}(Z^M)]$$

Now we have M such rows and each row has B elements. Whereas in analysis polyphase matrix we have B rows and M elements in each row.

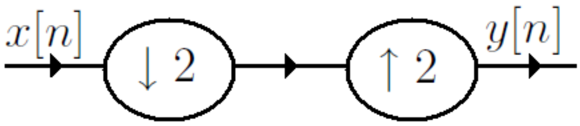
The overall analysis of M-Bank filter bank with B-branches in terms of the polyphase components as be shown as,

$$\underset{M \text{ x } 1}{\begin{bmatrix} \text{Output} \\ \text{Polyphase} \\ \text{Vector} \\ (Z^M) \end{bmatrix}} = \underset{M \text{ x } B}{\begin{bmatrix} \text{Polyphase} \\ \text{Synthesis} \\ \text{Matrix} \\ (Z^M) \end{bmatrix}} \underset{B \text{ x } M}{\begin{bmatrix} \text{Polyphase} \\ \text{Analysis} \\ \text{Matrix} \\ (Z^M) \end{bmatrix}} \underset{M \text{ x } 1}{\begin{bmatrix} \text{Input} \\ \text{Polyphase} \\ \text{Matrix} \\ (Z^M) \end{bmatrix}}$$

Figure 11: M-Bank Filter Bank in terms of Polyphase Components

4 Modulation Approach

In this approach we treat down sampling as a sum of modulations. Consider an example of $M = 2$.



The output $y[n]$ obtained after first down-sampling the input $x[n]$ by 2 and then up-sampling by 2 can be consider as the multiplication of the input $x[n]$ sequence by $\cdots 101010 \cdots$. Here 1 is at every multiple of 2.

In general for any positive integer M, the effect of first down- sampling by M and then up-sampling by M is same as multiplication by a sequence given below:

$$\ldots \quad \underset{\substack{\uparrow \\ Multiples\ of\ M}}{1} \quad \underbrace{0\ 0\ \cdots\ 0}_{(M-1)}\ 1\ 0\ 0\ \cdots 0 \ldots$$

In the modulation approach the idea is instead of decomposing the sequence in time we essentially treat the sequence as a sum of modulations. And we combine the down and up-sampler when we treat it thus as a sum.

In the next lecture we shall go further and learn modulation approach and contrast it with the polyphase approach bringing out the differences and the similarities between the two establish condition for the perfect reconstruction based on both of these approaches.