1 Introduction

In the last lecture, we saw that we can construct a modular lattice structure for implementing a filter bank. Repetition of the modules will lead to an increase of order by 2 for every module. In this lecture, the idea is to go the opposite way, *i.e.* if we know the final filter response, is it possible to peel off the modules to know earlier system functions in order to construct the lattice structure.

2 The lattice module

A single lattice stage is shown in figure (1). We have shown that the conjugate quadrature
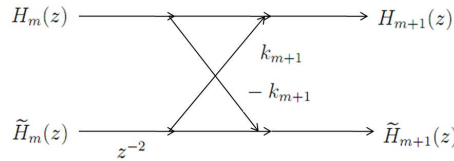

Figure 1: A stage in lattice structure

relationship is preserved, *i.e.* given that

$$\widetilde{H}_m(z) = z^{-(2m-1)} H_m(-z^{-1})$$

we have

$$\widetilde{H}_{m+1}(z) = z^{-(2(m+1)-1)} H_{m+1}(-z^{-1})$$

$$(1)$$

In constructing a lattice structure, we have to go the other way, *i.e.*

$$
\begin{array}{ccc}
H_{m+1}(z) & & H_m(z) \\
& \longrightarrow & \\
\widetilde{H}_{m+1}(z) & & \widetilde{H}_m(z)
\end{array}
\tag{2}
$$

3 Inductive(recursive) lattice relation

The inductive lattice relation is given by

$$H_{m+1}(z) = H_m(z) + k_{m+1} z^{-2} \widetilde{H}_m(z) \tag{3}$$

$$\widetilde{H}_{m+1}(z) = z^{-2} \widetilde{H}_m(z) - k_{m+1} H_m(z) \tag{4}$$

We will determine $H_m(z)$ in terms of $H_{m+1}(z)$ and $\widetilde{H}_{m+1}(z)$ by solving the above two equations for $H_m(z)$. Solving, we get $H_m(z)$ as given in equation (5)

$$H_m(z) = \frac{H_{m+1}(z) - k_{m+1}\widetilde{H}_{m+1}(z)}{1 + k_{m+1}^2} \tag{5}$$

3.1   Obtaining k_{m+1}

In equation (3), $H_{m+1}(z)$ is of length $2(m+1)$ and $H_m(z)$ is of length $2m$. The z^{-2} term increases length by 2. Now we shall inductively show that the coefficient of z^0 in $H_m(z)$ is 1.

3.1.1   Basis step

The system function of first module of lattice is given as

$$H_1(z) = 1 + k_1 z^{-1} \tag{6}$$
$$\widetilde{H}_1(z) = -k_1 + z^{-1} \tag{7}$$

Thus, the coefficient of z^0 is 1 in $H_1(z)$. Now let it also be true for $H_m(z)$. According to (3), the z^0 can come only from $H_m(z)$ as the lowest power of z in the second term would be z^{-2}. The the coefficient of z^0 is "carried forward" from $H_m(z)$ to $H_{m+1}(z)$. Thus, it is proved by induction that coefficient of z^0 in $H_{m+1}(z)$ $\forall m \in \mathbb{N}$ is 1. From equation (1), if

$$H_m(z) = 1 + h_1^m z^{-1} + h_2^m z^{-2} + \cdots + h_{2m+1}^m z^{-(2m-1)} \tag{8}$$

then $\widetilde{H}_m(z)$ is given by

$$\widetilde{H}_m(z) = -h_{2m+1}^m + \cdots - h_1^m z^{-2m} + z^{-(2m-1)} \tag{9}$$

Thus, coefficient of highest power of z, *i.e* $z^{-(2m-1)}$ is 1. The highest negative power of z will come from the second term on the RHS of equation (3). Since k_{m+1} is the multiplier in this term, it is obvious that the coefficient of the highest negative power of z in the filter Z-transform is k_{m+1}. Thus the last coefficient directly gives the value of k_{m+1}. Once we know k_{m+1}, we can peel of one module. Since H_{m+1} is known, we can construct $\widetilde{H}_{m+1}(z)$. Thus we can determine $H_m(z)$ from (5).

3.2   An example

Consider the example of length 4 Daubechies filter with coefficients 1, $h1$, $h2$, $h3$. We have a 2-stage lattice structure to implement this filter. The analysis structure is shown in figure (2).
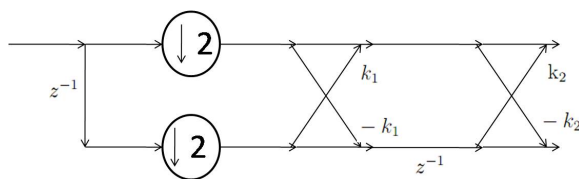

Figure 2: lattice structure for Daubechies length 4 filter

Given the length 4 filter, we have equation (10)

$$k_2 = h_3$$
$$H_2(z) = 1 + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} \tag{10}$$
$$\widetilde{H}_2(z) = -h_3 + h_2 z^{-1} - h_1 z^{-2} + z^{-3}$$

We can calculate $H_1(z)$ from equation (5) as

$$H_1(z) = \frac{H_2(z) - k_2 \widetilde{H}_2(z)}{1 + k_2^2}$$
$$= \frac{H_2(z) - h_3 \widetilde{H}_2(z)}{1 + h_3^2} \tag{11}$$

Consider the numerator only

$$= 1 + h_3^2 + (h_1 - h_2 h_3)z^{-1} + (h_2 + h_1 h_3)z^{-2}$$

The Daubechies filter bank impulse response is orthogonal to its even translates, $i.e.$, the product of

$$
\begin{array}{cccc}
1 & h_1 & h_2 & h_3 \\
 & 1 & h_1 & h_2 & h_3
\end{array}
$$

would be zero, $i.e$

$$h_2 + h_1 h_3 = 0$$

Hence,

$$H_1(z) = \frac{1 + (h_1 - h_2 h_3)z^{-1} + (h_2 + h_1 h_3)z^{-2}}{1 + h_3^2}$$
$$= \frac{(1 + h_3^2) + (h_1 - h_2 h_3)z^{-1}}{1 + h_3^2} \tag{12}$$
$$= 1 + \frac{h_1 - h_2 h_3}{1 + h_3^2} z^{-1}$$

This also gives us the value of k_1.

$$k_1 = \frac{h_1 - h_2 h_3}{1 + h_3^2}$$

The backward recursion in equation (5) is effected by

- k_{m+1} is the coefficient of highest power of z^{-1}.

- As long as this coefficient is real, the denominator $1 + k_{m+1}^2$ poses no problem.

- The length of $H_m(z)$ is reduced by 2, one due to cancelation of highest order coefficient, and one due to the orthogonality of the filter response to its even translates.

4   The synthesis variant

The synthesis side of the lattice structure is essentially the transpose of analysis side. Hence for the first stage, shown in figure (3).
The corresponding synthesis stage is shown in figure (4).
The inductive analysis stage is shown in figure (5).
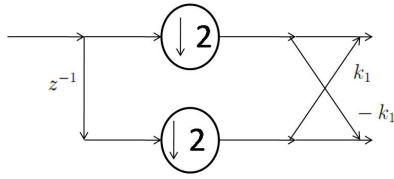The corresponding inductive synthesis stage is shown in figure (6).

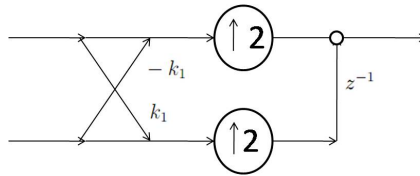Figure 3: Analysis lattice structure first stage

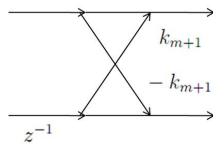Figure 4: Synthesis lattice structure first stage
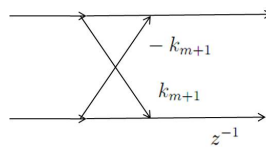

Figure 5: Inductive analysis lattice structure stage

Figure 6: Inductive synthesis lattice structure stage