

The word Lattice is used for a homogeneous(modular) repetitive pattern or symbol.

1 Haar Analysis Filter Bank

The lowpass filter is given as,

$$H_{low}(Z) = 1 + Z^{-1}$$

and the highpass filter is given as,

$$Z^{-1}H_{low}(-Z^{-1}) = Z^{-1}(1 - Z) = -1 + Z^{-1}$$

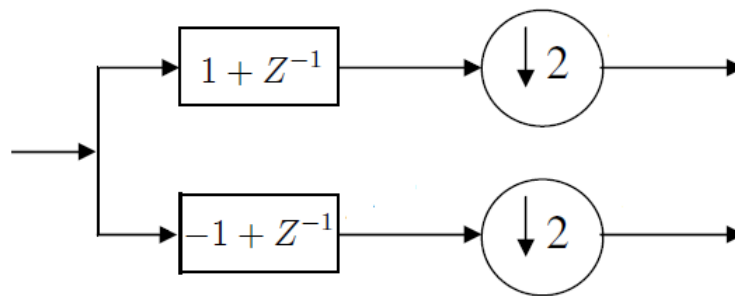


Figure 1: Haar Analysis Filter Bank

Here, we perform convolution first and then pass through downsampler. Hence, actually, we are “wasting” half computation. Here, we are unnecessary calculating odd samples which are finally washed off.

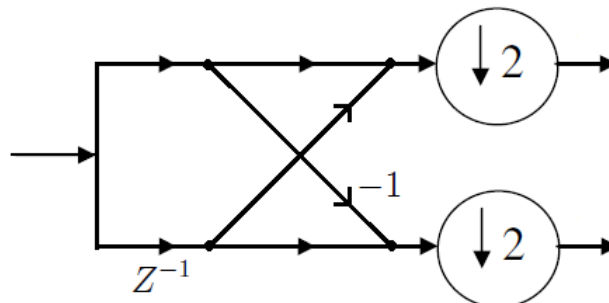


Figure 2: Lattice structure for the Haar Analysis Filter Bank

Hence, we must look for a more efficient structure, which downsamples first and then compute the convolution. We must invoke the “noble” identities. We need to recast the analysis filters.

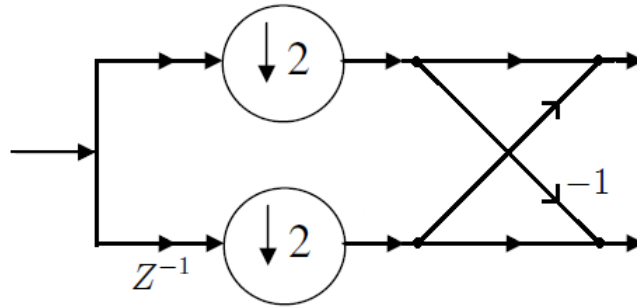


Figure 3: Efficient structure for the Haar Analysis Filter Bank

We can redraw the Haar analysis filter bank as shown in fig. 2.

The downsamplers $\downarrow 2$ can “jump” across adders, constant multipliers and branch points. Hence, we can obtain structure as shown in fig. 3. This is computationally efficient structure. Let us generalize this structure (fig 4). This is the one stage or one module of lattice.

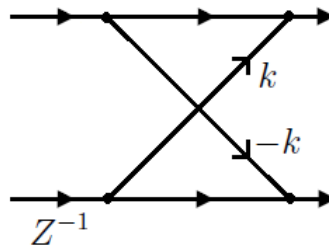


Figure 4: One stage of lattice in generalized form

Inductive Assumption:

A conjugate quadrature pair has been created at the input of this module. It's first stage is as shown in fig. 5,

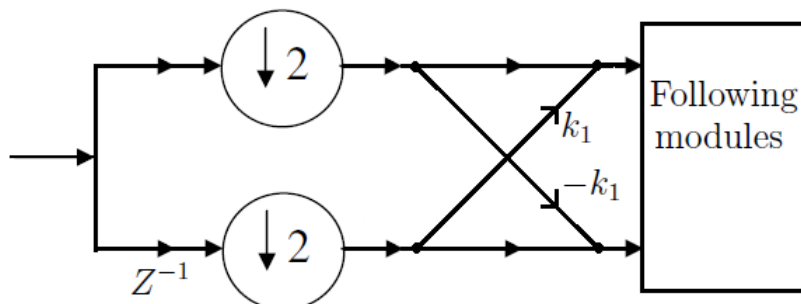


Figure 5: First stage

and it's l^{th} stage is as shown in fig. 6.

Inductive Step:

The inductive step is essentially to prove that,

$$B(Z) = Z^{-(L+2-1)} A(-Z^{-1})$$

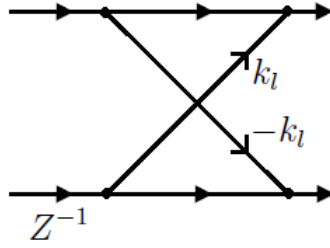


Figure 6: l^{th} stage

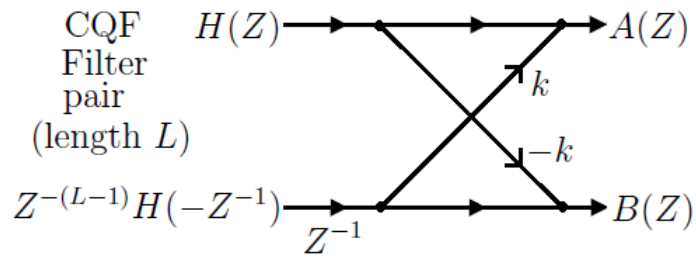
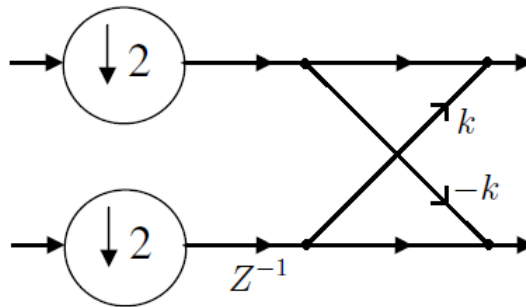
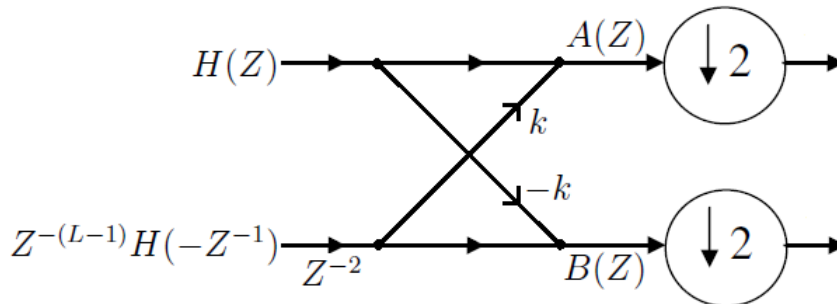


Figure 7: Inductive step

Here, $Z^{-(L+2-1)}$ indicates length of filter is increased by 2. Here, we will use noble identities to prove this.



By, applying noble identity of downsamplers, we get,



Now,

$$\begin{aligned} A(Z) &= H(Z) + kZ^{-2}Z^{-(L-1)}H(-Z^{-1}) \\ &= H(Z) + kZ^{-(L+2-1)}H(-Z^{-1}) \end{aligned}$$

$$\begin{aligned} B(Z) &= -kH(Z) + Z^{-2}Z^{-(L-1)}H(-Z^{-1}) \\ &= -kH(Z) + Z^{-(L+2-1)}H(-Z^{-1}) \end{aligned}$$

We essentially need to consider $A(Z)$ is lowpass filter and find $Z^{-(L+2-1)}A(-Z^{-1})$. Hence, it can given as,

$$\begin{aligned} Z^{-(L+2-1)}A(-Z^{-1}) &= Z^{-(L+2-1)}\{H(-Z^{-1}) + k(-Z)^{L+2-1}H(Z)\} \\ &= Z^{-(L+2-1)}H(-Z^{-1}) + k(-1)^{L+2-1}H(Z) \end{aligned}$$

by inductive assumption, L is even. Hence,

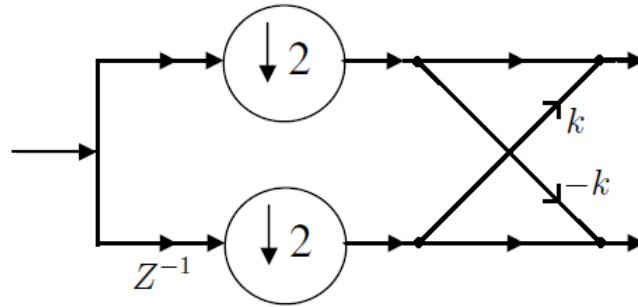
$$\begin{aligned} Z^{-(L+2-1)}A(-Z^{-1}) &= Z^{-(L+2-1)}H(-Z^{-1}) - kH(Z) \\ &= B(Z) \end{aligned}$$

The inductive step is complete. Hence,

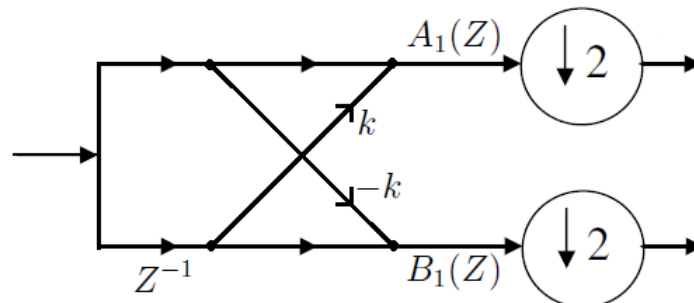
$$B(Z) = Z^{-(L+2-1)}A(-Z^{-1})$$

Basis Step:

We need to study the relation between the two outputs in the following figure.



By applying noble identity, we can obtain,



Here,

$$\begin{aligned} A_1(Z) &= 1 + kZ^{-1} \\ B_1(Z) &= -k + Z^{-1} \end{aligned}$$

And, indeed,

$$B_1(Z) = Z^{-1}A_1(-Z^{-1})$$

Basis step is also complete. Therefore, it is proved that the given structure generate the CQF on analysis side, by mathematical induction.

Now, $(m + 1)^{th}$ stage is given as, Here,

$$\begin{aligned} H_m(Z) \\ \tilde{H}_m(Z) \\ = Z^{-(2m-1)}H_m(-Z^{-1}) \end{aligned} \Rightarrow \begin{cases} H_{m+1}(Z) \\ \tilde{H}_{m+1}(Z) \end{cases}$$

$$\begin{aligned} H_{m+1}(Z) &= H_m(Z) + Z^{-2}k\tilde{H}_m(Z) \\ &= H_m(Z) + kZ^{-2}Z^{-(2m-1)}H_m(-Z^{-1}) \\ &= H_m(Z) + kZ^{-(2m+2-1)}H_m(-Z^{-1}) \end{aligned}$$

and

$$\begin{aligned} \tilde{H}_{m+1}(Z) &= -kH_m(Z) + Z^{-2}\tilde{H}_m(Z) \\ &= -kH_m(Z) + Z^{-2}Z^{-(2m-1)}H_m(-Z^{-1}) \\ &= -kH_m(Z) + Z^{-(2m+2-1)}H_m(-Z^{-1}) \end{aligned}$$

Our objective in synthesis or construction means is, Now,

$$\begin{aligned} H_{m+1}(Z) \\ \tilde{H}_{m+1}(Z) \end{aligned} \Rightarrow \begin{cases} H_m(Z) \\ \tilde{H}_m(Z) \end{cases}$$

$$H_{m+1}(Z) = H_m(Z) + kZ^{-2}Z^{-(2m-1)}H_m(-Z^{-1})$$

Here, $H_m(Z)$ is of length $2m$. Hence,

$$H_m(Z) = h_0 + h_1Z^{-1} + \dots + h_{2m-1}Z^{-(2m-1)}$$

Hence, $\tilde{H}_m(Z)$ is essentially,

$$\tilde{H}_m(Z) = h_{2m-1} + h_{2m-2}Z^{-1} + \dots + h_0Z^{-(2m-1)}$$

Therefore, h_0k is the coefficient of the highest power of Z^{-1} in $H_{m+1}(Z)$.

Now, we have mechanism to obtain k . Once we have k , we should have a mechanism to pill off last stage. We will complete the construction of lattice stage in next lecture.