WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 32: Noble Identities and Haar Wavepacket Transform

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1 Introduction

In the previous lecture, we have introduced the idea of Wavepacket transform. In Wavepacket transform, along with low pass branch of filter bank, high pass branch is also further decomposed. This decomposition of high pass branch has some counter intuitive observations in frequency domain. Being observed Wavepacket transform in ideal filter bank, in this lecture we will see Haar Wavepacket transform. Along with this, a concept of Noble Identities is also discussed in great detail.

2 Noble Identity

Noble Identities occur frequently when we want to iterate the filter bank in which case we are often required to combine down and upsamplers and different filters. Noble Identities are useful in dealing with cascade of sampling rate changes and cascade of filters.

Noble Identity for Downampler:

Consider a downsampler by 2 followed by a filter as shown in Figure 1. Noble identity il-

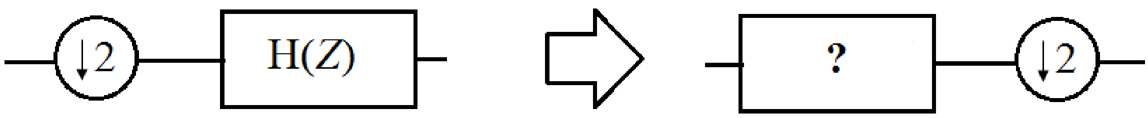


Figure 1: Interchanging the positions of downsampler and filter to get Noble Identity

lustrates how to interchange the positions of downsampler and filter. When the positions of filter and downsampler are interchanged, what will be the nature of new filter? In order to get the answer, consider Figure 2, in which output $y[n]$ is convolution of impulse response $h[n]$ of filter and sequence $x_1[n]$. Sequence $x_1[n]$ is related to input sequence $x[n]$ as $x_1[n] = x[2n]$ for $\forall n \in \mathbb{Z}$.

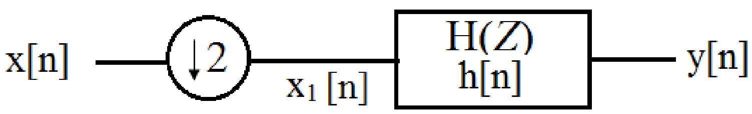


Figure 2: Output sequence as convolution of input sequence and impulse response

We can write,

$$\begin{aligned} y[n] &= x_1[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x_1[k]h[n-k] \quad (1) \\ &= \sum_{l=-\infty}^{\infty} h[l]x_1[n-l] \quad (2) \end{aligned}$$

Equation 1 can be rewritten as

$$y[n] = \sum_{k=-\infty}^{\infty} x[2k]h[n-k] \quad (3)$$

Here, we are trying to get an equivalent system in which, a filter is followed by a downsampler and which will give the expression given by equation 3. Also, we can write equation 2 as

$$y[n] = \sum_{l=-\infty}^{\infty} h[l]x[2n-2l] \quad (4)$$

In equation 4, the index '$2n$' is due to downsampling operation. If '$2n$' is replaced by 'n', we arrive at a point after a filter in an equivalent system. So, we have expression

$$\sum_{l=-\infty}^{\infty} h[l]x[n-2l] \quad (5)$$

which is a convolution in which $h[l]$ is located at '$2l'^{th}$ points and at other places it is zero.

$$\begin{aligned} h_1[n] &= 0 \qquad n \ is \ odd \\ &= h\left[\frac{n}{2}\right] \qquad Otherwise \end{aligned}$$

Hence,

$$y[n] = \sum_{l=-\infty}^{\infty} h_1[l]x[2n-l] \quad (6)$$

This indicates that impulse response of an equivalent filter before downsampler is the impulse response of original filter but upsampled by two. This is shown in Figure 3. This is called as

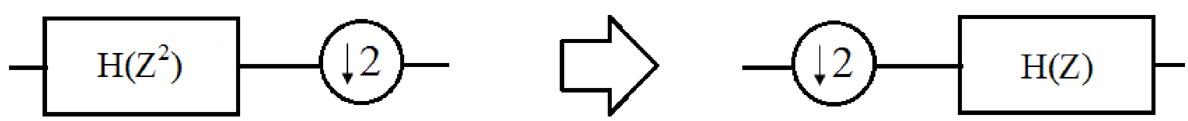


Figure 3: Noble Identity for downsampler by 2

a Noble identity for downsampler.

This equivalent structure however has disadvantage over the previous structure in terms of number of computations. In new structure, the samples obtained after convolution are discarded by downsampler which results in wastage of computations carried out during convolution process. However, in original structure, as downsampling is done first, there is no question of wastage in number of computations in convolution.

Noble Identity for Upsampler:

The Noble Identity for Upsampler can be derived through the concept of 'transposition'. In case of signal flow graph (with no up and downsampler), its transpose is obtained by reversing the direction of each arrow and keeping constant multiplier same. The summing point and branching points in previous graph become branching points and summing points respectively. The similar operation can be done if signal flow graph contains up and downsampler with only change, that downsampler in original graph becomes upsampler in transposed graph with same factor and vice versa.

So applying the rules of transposition to Noble identity of downsampler we get Noble identity for upsampler as shown in Figure 4. Noble identity of upsampler tells that the operation of

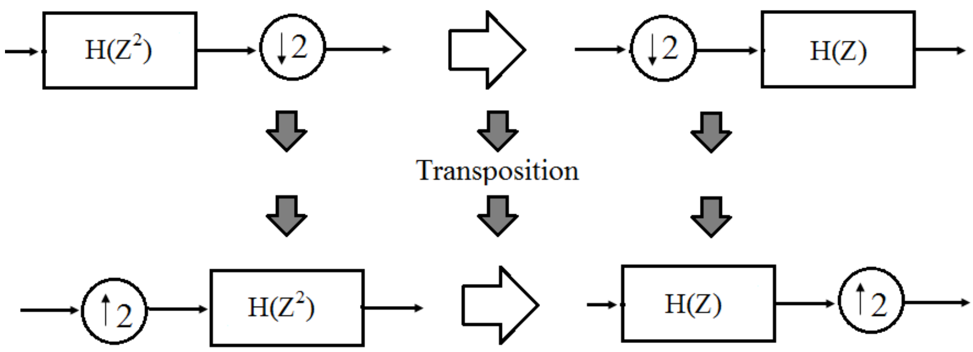


Figure 4: Noble Identity for upsampler

filtering followed by upsampling is same as the operation of upsampling followed by filtering with impulse response of this filter being upsampled version of impulse response of original filter.

3 Haar Wavepacket transform

With the introduction of Noble identities, we will employ them in the Haar Wavepacket transform. Figure 5 shows filter bank corresponding to Haar Wavepacket transform. Now, considering one branch at a time e.g. upper branch and applying Noble Identity we get an equivalent branch as shown in Figure 6. In Figure 6, we note that downsampler by 4 is the result of two cascaded downsamplers by two.

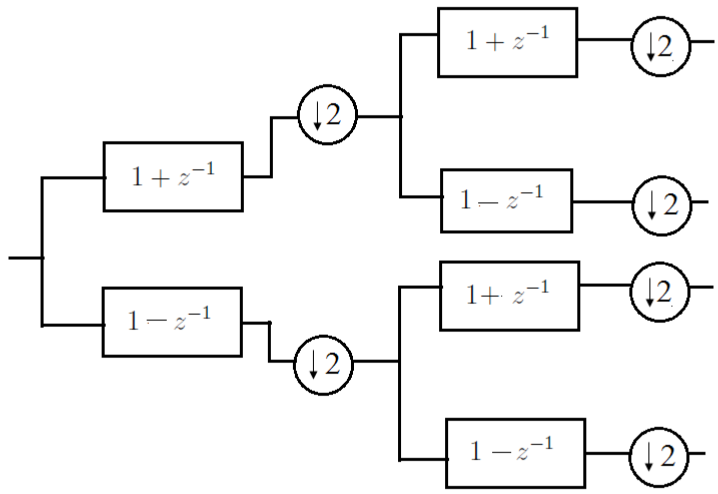


Figure 5: Filter bank employing Haar Wavepacket transform

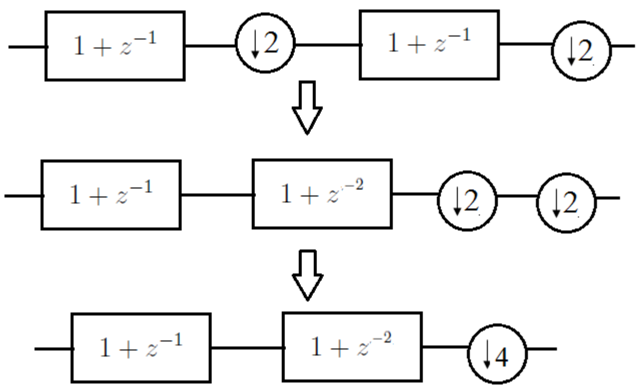


Figure 6: Application of Noble Identity for downsampler on the branch of Haar filter bank

So, there are four filters highlighted in the process of Wavepacket transform. Figure 7 shows what is happening during the entire process. Subspace V_2 is decomposed into subspaces V_1 and W_1. V_1 is decomposed into V_0 and W_0. The important feature of Wavepacket transform is that W_1 is also further decomposed into subspaces say W_{10} and W_{11}.

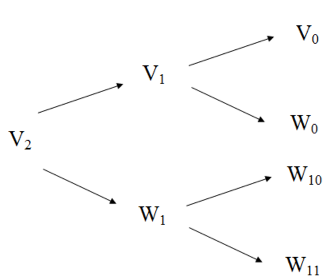


Figure 7: Decomposition of subspaces in Wavepacket transform

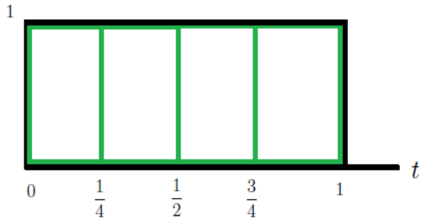
Let us look at the filter on the upper branch. Use of Noble Identity results in a filter as

$$(1 + z^{-1})(1 + z^{-2}) = 1 + z^{-1} + z^{-2} + z^{-3}$$

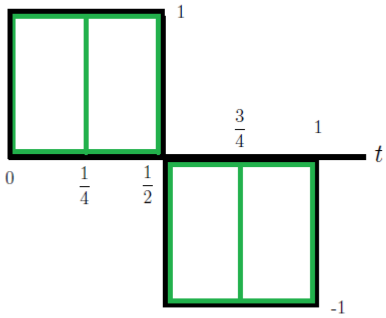
The sequence corresponding to above expression i.e. [1,1,1,1] tells us how to express the basis of V_0 in terms of bases of V_2. In other words, $\phi(t)$ (basis of V_0)can be expressed as linear combination of $\phi(4t)$ (basis of V_2). Note that, downsampling by 4 results in dilation of $\phi(t)$ by factor of 4, which effectively results in going from V_0 to V_2. So, we can write this as,

$$\phi(t) = \phi(4t) + \phi(4t - 1) + \phi(4t - 2) + \phi(4t - 3)$$

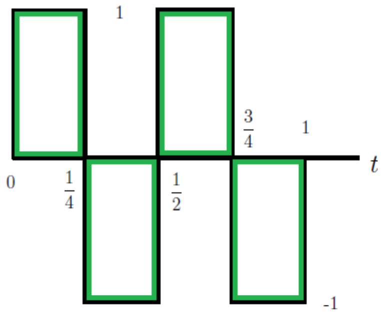
Similarly, we can write expressions for other filters such that corresponding sequences represent bases of W_0,W_{10} and W_{11} in terms of bases of V_2. This is explained in Figure 8.



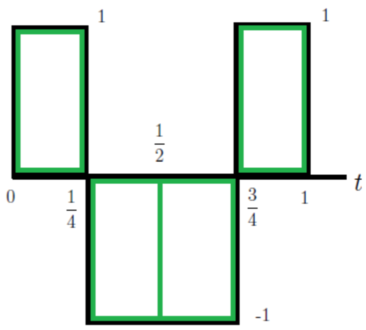
$$V_2 \rightarrow V_1 \rightarrow V_0$$
$$(1+z^{-1})(1+z^{-2}) = 1 + z^{-1} + z^{-2} + z^{-3}$$
$$The\ sequence = [1, 1, 1, 1]$$
$$\uparrow$$
$$\text{Basis for } V_0 = \phi(4t) + \phi(4t-1) + \phi(4t-2) + \phi(4t-3)$$



$$V_2 \rightarrow V_1 \rightarrow W_0$$
$$(1+z^{-1})(1-z^{-2}) = 1 + z^{-1} - z^{-2} - z^{-3}$$
$$The\ sequence = [1, 1, -1, -1]$$
$$\uparrow$$
$$\text{Basis for } W_0 = \phi(4t) + \phi(4t-1) - \phi(4t-2) - \phi(4t-3)$$
(i.e. Haar Wavelet)



$$V_2 \rightarrow W_1 \rightarrow W_{10}$$
$$(1-z^{-1})(1+z^{-2}) = 1 - z^{-1} + z^{-2} - z^{-3}$$
$$The\ sequence = [1, -1, 1, -1]$$
$$\uparrow$$
$$\text{Basis for } W_{10} = \phi(4t) - \phi(4t-1) + \phi(4t-2) - \phi(4t-3)$$



$$V_2 \rightarrow W_1 \rightarrow W_{11}$$
$$(1-z^{-1})(1-z^{-2}) = 1 - z^{-1} - z^{-2} + z^{-3}$$
$$The\ sequence = [1, -1, -1, 1]$$
$$\uparrow$$
$$\text{Basis for } W_{11} = \phi(4t) - \phi(4t-1) - \phi(4t-2) + \phi(4t-3)$$

Figure 8: Bases for V_0,W_0, W_{10} and W_{11}. Basis for V_2 and its translations are shown in green.