WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 30: Building of Piecewise Linear Scaling Function, Wavelets

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1 Introduction

In previous lecture we begin building piecewise linear MRA. We saw that piecewise linear function obtained by convolving the Haar Scaling function with itself.

$$\phi_1(t) = \phi_0(t) * \phi_0(t) \tag{1}$$

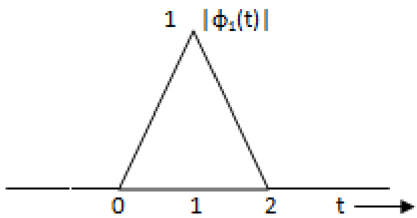
This $\phi_1(t)$ is not orthogonal to its integral translates. This lead to Sum of Translated Spectra (STS) not being a constant.
Since it was only 1, -1 which were the trouble maker translates, but as expected the STS was kind of constant within two positive bound so as a consequence one could extract from $\phi_1(t)$ another function which was orthogonal to its own translates. This new function can be used to build MRA based on piecewise linear scaling function and wavelets.

2 Building of Piecewise linear Scaling Function/Wavelet

We saw that:

$$STS(\phi_1, 2\pi)(\Omega) = \sum_{K=-\infty}^{\infty} |\phi_1(\Omega + 2\pi K)|^2 \tag{2}$$

This is not a constant as required but it lies between two positive bounds.



$$0 < A \leq STS(\phi_1, 2\pi)(\Omega) \leq B < \infty \tag{3}$$

$$\frac{A}{B} = 3 \tag{4}$$

If we scale a function by a constant, the STS is scaled by the Square Magnitude of that constant.
Now, we could define a constant function $\tilde{\phi_1}(\Omega)$ as:

$$\hat{\tilde{\phi_1}}(\Omega) = \frac{\tilde{\phi_1}(\Omega)}{+\sqrt{STS(\phi_1, 2\pi)(\Omega)}} \tag{5}$$

$$STS(\tilde{\phi_1}, 2\pi)(\Omega) = \frac{STS(\phi_1, 2\pi)(\Omega)}{STS(\phi_1, 2\pi)(\Omega)} = 1(Constant) \tag{6}$$

Both numerator and denominator are greater than 0 and less than ∞ and hence cancelation is possible.
Therefore, $\tilde{\phi}_1(t)$ is orthogonal to all its integer translates *i.e.*,

$$\left\langle \tilde{\phi}_1(t), \tilde{\phi}_1(t-m) \right\rangle = 0 \quad \forall\ m \in Z \tag{7}$$

Now we saw that it is of the form

$$\hat{\tilde{\phi}}_1(\Omega) = \frac{\tilde{\phi}_1(\Omega)}{+\sqrt{\frac{2}{3}(1+\frac{1}{2}\cos\Omega)}} \tag{8}$$

Using binomial expansion or Taylor series expansion, *i.e.*, $(1+\gamma)^R$, $R \in \mathbb{R}$, $|\gamma| < 1$

$$(1+\gamma)^R = 1 + R\gamma + \frac{R(R-1)}{2!}\gamma^2 + \frac{R(R-1)(R-2)}{3!}\gamma^3 + \dots \tag{9}$$

Therefore we get,

$$\hat{\tilde{\phi}}_1(\Omega) = \frac{\tilde{\phi}_1(\Omega)}{\sqrt{2/3}}(1+\frac{1}{2}\cos\Omega)^{-\frac{1}{2}} \tag{10}$$

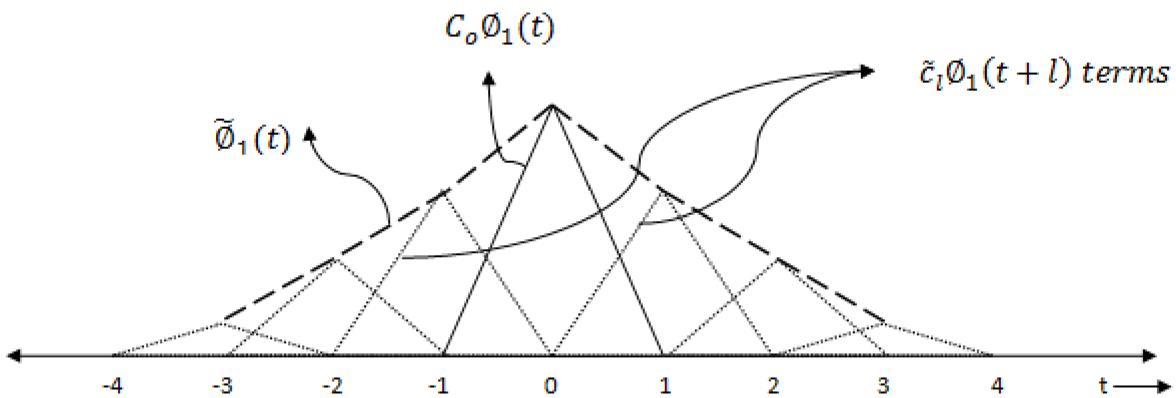
Now, $\cos^N \Omega = (\frac{e^{j\Omega}+e^{-j\Omega}}{2})^N$.
Therefore,

$$\hat{\tilde{\phi}}_1(\Omega) = \sum_{l=-\infty}^{\infty} \tilde{C}_l\, e^{j\Omega l} \hat{\phi}_1(\Omega) \tag{11}$$

The calculation of $\tilde{C}_l$ is cumbersome as for each $\tilde{C}_l$ we have to write a series.
Now, $\tilde{C}_l = \tilde{C_{-l}}$ from symmetry in expanding $\cos\Omega^l$. Therefore if we take Inverse Fourier Transform of equation 11, we get.

$$\tilde{\phi}_1(t) = \sum_{l=-\infty}^{\infty} \tilde{C}_l\, \phi_1(t+l) \tag{12}$$

Therefore, $\tilde{\phi}_1(t)$ is a linear combination of $\phi_1(t)$ and its integer translates.
This shows that $\tilde{\phi}_1(t)$ is piecewise linear. One can calculate few $\tilde{C}_l$, may be for $l = 0$, ± 1, ± 2 and we will see that $\tilde{C}_l$ decays as $l \to +\infty$, $\tilde{C}_l$ and $\tilde{C_{-l}}$ decays as $l \to 0$



Properties of $\tilde{\phi}_1(t)$:

1. Piecewise linear: A sum of piecewise linear function.
2. $\phi_1(t)$ should be orthogonal to all its integral translates *i.e.*,

$$\langle \phi_1(t-m), \phi_1(t-n) \rangle = 0; \quad \forall m, n \in Z; \;\; m \neq n \tag{13}$$

3. $\tilde{\phi}_1(\cdot)$ obeys a dilation equation.

This is critical for MRA because it is this dyadic dilation equation which ensures that $\tilde{\phi}_1(t)$, when dilated by factor of 2 and then translated by all integers, construct basis for next subspace V_0 which is spanned by $\phi_1(t)$ and integer translates and then we have a subspace which is span by $\tilde{\phi}_1(t)$ contracted by a factor of 2 and its integer translates.
We shall use the frequency domain to prove this. Time domain dyadic equation for a general scaling function $\phi(\cdot)$.

$$\phi(t) = \sum_{k=-\infty}^{\infty} h[k] \, \phi(2t-k) \tag{14}$$

Taking Fourier Transform:

$$\hat{\phi}(\Omega) = \frac{1}{2} H\left(\frac{\Omega}{2}\right) \hat{\phi}\left(\frac{\Omega}{2}\right) \tag{15}$$

where $H(\cdot)$ is DTFT of $h[k]$.
To establish a dyadic dilation equation on $\hat{\tilde{\phi}}_1(\cdot)$ we essentially need to consider $\frac{\hat{\tilde{\phi}}_1(\Omega)}{\hat{\tilde{\phi}}_1(\frac{\Omega}{2})}$ and show that this is of the form $\frac{1}{2}H(\frac{\Omega}{2})$ where:

$$H(\Omega) = \sum_{k=-\infty}^{\infty} h[k] \, e^{-jk\Omega} \tag{16}$$

We need to establish essentially that $\frac{\hat{\tilde{\phi}}_1(\Omega)}{\hat{\tilde{\phi}}_1(\frac{\Omega}{2})}$ is a DTFT. That is, it is
(i) Periodic with period 2π
(ii) It is bounded on any interval of 2π so that its IDTFT is calculable.
Boundedness is needed, therefore:

$$\frac{1}{2\pi} \int f(\Omega) \, e^{j\Omega n} d\Omega \;\; must \;\; converge, \;\; where \;\; f(\Omega) = \frac{\hat{\tilde{\phi}}_1(\Omega)}{\hat{\tilde{\phi}}_1(\frac{\Omega}{2})} \tag{17}$$

$$\frac{\hat{\tilde{\phi}}_1(\Omega)}{\hat{\tilde{\phi}}_1(\frac{\Omega}{2})} = \frac{\hat{\phi}_1(\Omega) \left(\frac{2}{3}(1+\frac{1}{2}\cos\frac{\Omega}{2})\right)^{1/2}}{\left(\frac{2}{3}(1+\frac{1}{2}\cos\Omega)\right)^{1/2} \hat{\phi}_1(\frac{\Omega}{2})} \tag{18}$$

Now focus on the term inside the roots, *i.e.*, $\left[\frac{\frac{2}{3}(1+\frac{1}{2}\cos\frac{\Omega}{2})}{\frac{2}{3}(1+\frac{1}{2}\cos\Omega)}\right]^{1/2}$. This is a function of Ω, therefore, its periodic. This is definitely of the form $\frac{1}{2}H(\frac{\Omega}{2})$ with properties of $H(\cdot)$ desired.
The desired properties are:-
If we replace $\Omega \to \frac{\Omega}{2}$ in the above expression(term inside the root), we get,

$$\left[\frac{\frac{2}{3}(1+\frac{1}{2}\cos\Omega)}{\frac{2}{3}(1+\frac{1}{2}\cos 2\Omega)}\right]^{1/2} \tag{19}$$

In this expression, Numerator is periodic with period π and Denominator is periodic with period 2π and hence the overall expression is periodic with period 2π. **Therefore the periodicity is established**.
Now we look at the boundedness. Let us say that in the expression(inside the root):

$$0 < A \leq Numerator \leq B < \infty \tag{20}$$

And

$$0 < A \leq Denominator \leq B < \infty \tag{21}$$

We have already proved this. Therefore we can say that the fraction is also between two positive bounds as:

$$0 < \frac{A}{B} \leq \frac{Numerator}{Denominator} \leq \frac{B}{A} < \infty \tag{22}$$

So, lets write equation 18 as:

$$\frac{\hat{\tilde{\phi}}_1(\Omega)}{\hat{\tilde{\phi}}_1(\frac{\Omega}{2})} = \frac{\hat{\phi}_1(\Omega)}{\hat{\phi}_1(\frac{\Omega}{2})}(Q) = \frac{P}{Q} \tag{23}$$

where Q obeys periodicity and boundedness as required.
Now P as we know obeys the requirement. Therefore, the ratio $\frac{\hat{\tilde{\phi}}_1(\Omega)}{\hat{\tilde{\phi}}_1(\frac{\Omega}{2})}$ obeys the requirement.
Hence $\tilde{\phi}_1(\cdot)$ must obey a dyadic dilation equation:

$$\tilde{\phi}_1(t) = \sum_{k=-\infty}^{\infty} \tilde{h}[k]\, \tilde{\phi}_1(2t-k) \tag{24}$$

$$\tilde{h}[k] \overset{DTFT}{\rightarrow} \tilde{H}(\Omega) \tag{25}$$

$$i.e., \quad \tilde{H}[\Omega] = \sum_{k=-\infty}^{\infty} h[k]\, e^{-jk\Omega} \tag{26}$$

Therefore,

$$\frac{1}{2}\tilde{H}(\frac{\Omega}{2}) = \frac{\hat{\tilde{\phi}}_1(\Omega)}{\hat{\tilde{\phi}}_1(\frac{\Omega}{2})} = \frac{\hat{\phi}_1(\Omega)}{\hat{\phi}_1(\frac{\Omega}{2})}Q = \frac{1}{2}C_o(1 + 2e^{-j\frac{\Omega}{2}} + e^{-j2\frac{\Omega}{2}}) \left[\frac{\frac{2}{3}(1+\frac{1}{2}\cos\frac{\Omega}{2})}{\frac{2}{3}(1+\frac{1}{2}\cos\Omega)}\right]^{1/2} \tag{27}$$

3 Summary

We have now established the existence as well as method to calculate $\tilde{H}[k]$, though the calculation is highly cumbersome. If we have $\tilde{H}[k]$, we have the impulse response of the low pass filter in orthogonal MRA. Once we have dyadic dilation equation, the coefficient of equation give low pass filter response. Once we have the analysis low pass filter, we can construct all the other filter.
We can observe that IDTFT of $\tilde{H}[k]$ will be of ∞ length, will be infinitely non casual and also irrational. This makes this a non-realizable filter. To get an orthogonal piecewise linear MRA one require unrealizable filter.
Once we know impulse response of the low pass filter of analysis side, we also know how to construct wavelet because analysis high pass filter impulse response coefficient will construct the wavelet from scaling function.