

1 Introduction

In this lecture, we shall continue to discuss one more variant of the MRA. In the previous lectures, we have studied the construction of a biorthogonal filter bank along with the example of the 5/3 filter bank in JPEG 2000. In biorthogonal filter bank we essentially have the filters of unequal lengths. With this approach we get the linear phase and symmetry in the impulse response of the filters. Further, we could extend what we did in the Haar case to a piecewise linear function. So, if we take 5/3 filter bank and if we look at the LPF of length 3, the filter has an impulse response of $(1 + z^{-1})^2$. It would give us the triangular function as the scaling function. The disadvantage with the triangular function was that it was not orthogonal to all its integer translates. It was orthogonal once you translate it by two units or more.

Now, in this lecture we shall take again the same $(1 + z^{-1})^2$, the length 3 LPF that we have seen in the 5/3 filter bank. But we shall deal with it in a slightly different way and that would bring us to the idea of orthogonal MRA with splines where we need to make a compromise in the nature of the scaling and the wavelet function that we construct and also in the nature of the filter bank that we would build, that underlie this MRA. Eventually we will see that this new approach would result in the filters of infinite length.

2 5/3 Filter Bank

If we look at the length-3 Low pass filter in the 5/3 filter bank, it essentially has $\frac{1}{2}(1 + z^{-1})^2$ as the system function. We know that the corresponding scaling function is $\phi_1(t)$ and it obeys the dilation equation

$$\phi_1(t) = \frac{1}{2}\phi_1(2t) + \phi_1(2t - 1) + \frac{1}{2}\phi_1(2t - 2) \quad (1)$$

$\phi_1(t)$ has an appearance as shown in figure 1. Now our main problem and the reason why we need it to go to an orthogonal filter bank as opposed to a bi-orthogonal filter bank was that this scaling function is not orthogonal to its translates by unity. So, if we translate this by unity and we take the dot product essentially between $\phi_1(t)$ and $\phi_1(t - 1)$, i.e, $\langle \phi_1(t), \phi_1(t - 1) \rangle$ and $\langle \phi_1(t), \phi_1(t + 1) \rangle$ turns out to be nonzero. Hence we are not able to construct an orthogonal MRA out of the function $\phi_1(t)$.

Now we wish to construct orthogonal MRA from a function that look similar to $\phi_1(t)$, or in other word, out of a function that is piecewise linear. Can we build a multi-resolution analysis with a piecewise linear function for ϕ and ψ ? This is the question that we are trying to answer in this lecture and to answer that question, we must relax the requirement for orthogonality.

3 Derivation of orthogonal MRA

If we look at the notion of orthogonality of $\phi(t)$ to its integer translates, we can express this requirement in terms of the autocorrelation of ϕ i.e., the autocorrelation function of $\phi(t)$ at all

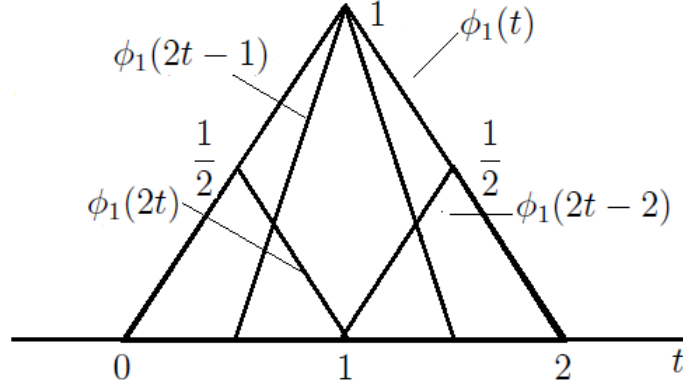


Figure 1: Scaling function $\phi_1(t)$

the integers except at '0' is 0 *i.e.*, $R_{\phi\phi}(\tau) = 0$ for all $\tau \in \mathbb{Z}$ except for $\tau = 0$, where $R_{\phi\phi}(\tau)$ is the autocorrelation function of $\phi(t)$. This is the basic principle of a function being orthogonal to its integer translates.

Let the scaling function be $\phi(t)$ and it is orthogonal to its integer translates $\phi(t - m)$, where $m \in \mathbb{Z}$. Essentially we mean that the autocorrelation function $R_{\phi\phi}(\tau)$ when sampled at $\tau = m, m \in \mathbb{Z}$ (*i.e.*, we sample at all integers, or at a sampling rate of 1) gives an impulse sequence. Mathematically, the sequence is a discrete impulse sequence. When we sample it, the Fourier Transform of the autocorrelation function gets aliased. Fourier Transform of the autocorrelation function is $|\hat{\phi}(\Omega)|^2$ where $\hat{\phi}(\Omega)$ is the Fourier transform of the scaling function $\phi(t)$. In other sense, we can say that Fourier Transform of the autocorrelation function is the power spectral density of the scaling function in the frequency domain. Now sampling $R_{\phi\phi}(\tau)$ at $\tau = m$ for $m \in \mathbb{Z}$, means summing up all the aliases of $|\hat{\phi}(\Omega)|^2$ in the Fourier domain. This can be written as

$$K_0 \sum_{k=-\infty}^{\infty} |\hat{\phi}(\Omega + 2\pi k)|^2, \text{ where } K_0 \text{ is a constant.} \quad (2)$$

Recall that when we sample a continuous function, its Fourier Transform shifts on the frequency axis by every multiple of the sampling frequency and adds up all these translates or aliases. In a way, the sequence $R_{\phi\phi}(\tau) |_{\tau=m}$ is an impulse sequence, then its DTFT, which is shown in equation 2, must be a constant. In order that the $\phi(t)$ is orthogonal to its integer translates, we require that the quantity, sum of aliases of the power spectral density must be a constant. In other words, $K_0 \sum_{k=-\infty}^{\infty} |\hat{\phi}(\Omega + 2\pi k)|^2$ is a constant. We shall call $K_0 \sum_{k=-\infty}^{\infty} |\hat{\phi}(\Omega + 2\pi k)|^2$ as the sum of translated spectra of $\phi(\cdot)$. We shall abbreviate sum of translated spectra by *STS*. $STS(\phi, 2\pi)(\Omega)$ has Ω as primary argument and $\phi, 2\pi$ as secondary arguments. In general,

$$STS(\phi, T)(\Omega) = K_0 \sum_{k=-\infty}^{\infty} |\hat{\phi}(\Omega + Tk)|^2 \quad (3)$$

With this little notational introduction, we take the same strategy as we did when we relaxed the condition for the sum of dilated spectra. We know when we talk about discretizing the scale we need to essentially relax the requirement of sum of dilated spectra to be a constant to where it is between two positive constants. We take similar approach for the *STS*. There exists relationship between relaxation of this requirement in the τ domain (or shift domain) and the frequency domain. The $\phi_1(t)$ is as shown in figure 1. We want the dot product of $\phi_1(t)$

with its integer translates to be zero for all the translates. It is true for translates of 2 and more than 2. But only for translation by 1 and -1, we are asking for relaxation here. The dot product of ϕ with itself would have certain value and it is the energy of the function. And if we take dot product of ϕ with its translates by 1 and -1, they are expected to have a smaller value. So, the auto correlation of this function is not exactly an impulse but close to an impulse. This means it is a non zero for very few values of around $n = 0$ and that manifests in the frequency domain as the STS not being a constant but being between two positive constants. So, for $\phi_1(t)$, by looking the figure 1 we can observe that

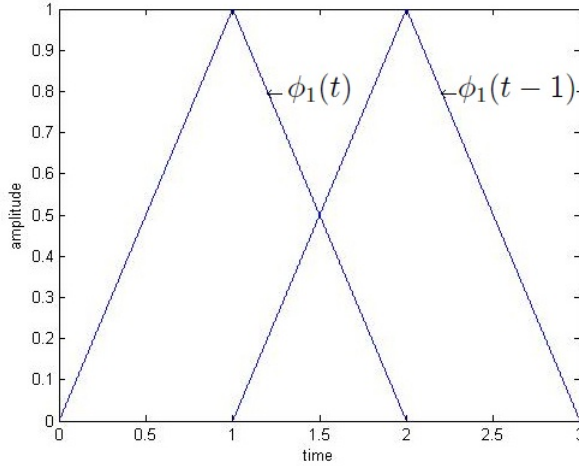


Figure 2: Graphical representation of dot product between $\phi_1(t)$ and $\phi_1(t - 1)$

$$R_{\phi_1\phi_1}(1) = R_{\phi_1\phi_1}(-1)$$

These quantities can be calculated as follows.

We find $\int_{-\infty}^{\infty} \phi_1(t)\phi_1(t - 1)dt$. In fact, this quantity can be easily calculated by shifting both the functions to left by 1 unit as shown in Figure 2. The above equation now becomes

$$R_{\phi_1\phi_1}(1) = \frac{1}{6}$$

Similarly $R_{\phi_1\phi_1}(0)$ can be calculated from the below equation.

$$\begin{aligned} R_{\phi_1\phi_1}(0) &= \int_0^2 \phi_1^2(t)dt \\ &= \frac{2}{3} \end{aligned}$$

Therefore,

$$R_{\phi_1\phi_1}(\tau) \Big|_{\tau=m, m \in \mathbb{Z}} = \left\{ \frac{1}{6}, \frac{2}{3}, \frac{1}{6} \right\}$$

↑

DTFT of $R_{\phi_1\phi_1}(\tau) |_{\tau=m, m \in \mathbb{Z}} = \frac{1}{6}e^{j\Omega} + \frac{2}{3} + \frac{1}{6}e^{-j\Omega}$. Hence, STS becomes

$$\begin{aligned} K_0 \sum_{k=-\infty}^{\infty} |\hat{\phi}(\Omega + 2\pi k)|^2 &= \frac{1}{6}e^{j\Omega} + \frac{2}{3} + \frac{1}{6}e^{-j\Omega} \\ &= \frac{2}{3} \left(1 + \frac{1}{2} \cos \Omega \right) \end{aligned} \quad (4)$$

As expected, this sum is always strictly positive and lies between two positive constants and it can be observed that the constants are $\frac{1}{3}$ and 1 *i.e.*,

$$\frac{1}{3} \leq \frac{2}{3} \left(1 + \frac{1}{2} \cos \Omega \right) \leq 1 \quad (5)$$

So a relaxation of the requirement in time domain has also led to a corresponding relaxation in the frequency domain. Now we can see the sum of translated spectrum lies between two positive bounds. We could say even though $\phi_1(t)$ by itself gives us a multi-resolution analysis, the question arises can we construct an other function $\tilde{\phi}_1(t)$ out of $\phi_1(t)$ by using the sum of translated spectrum in such a way that $\tilde{\phi}_1(t)$ gives us an orthogonal MRA. So, let us define $\tilde{\phi}_1(t)$ in terms of its Fourier Transform as

$$\hat{\tilde{\phi}}_1(\Omega) = \frac{\hat{\phi}_1(\Omega)}{+\sqrt{STS(\phi_1, 2\pi)(\Omega)}} \quad (6)$$

Here denominator is between $\frac{1}{\sqrt{3}}$ and 1. So, this division is a nonzero finite value. The denominator $STS(\phi_1, 2\pi)(\Omega)$ exhibits an important property of periodicity with period 2π .

$$\begin{aligned} STS(\phi, 2\pi)(\Omega) &= STS(\phi, 2\pi)(\Omega + 2\pi) \quad \forall \Omega \\ STS(\phi, 2\pi)(\Omega + 2\pi) &= \sum_{k=-\infty}^{\infty} |\hat{\phi}(\Omega + 2\pi + 2\pi k)|^2 \\ &= \sum_{k=-\infty}^{\infty} |\hat{\phi}(\Omega + 2\pi(k+1))|^2 \end{aligned} \quad (7)$$

As k runs from $-\infty$ to ∞ , $k+1$ also have the same limits. Hence, the summation in equation 7 becomes $STS(\phi, 2\pi)(\Omega)$. Therefore $STS(\tilde{\phi}_1, 2\pi)(\Omega)$ becomes

$$\begin{aligned} STS(\tilde{\phi}_1, 2\pi)(\Omega) &= \sum_{k=-\infty}^{\infty} \frac{|\hat{\phi}_1(\Omega + 2\pi k)|^2}{STS(\phi_1, 2\pi)(\Omega + 2\pi k)} \\ &= \frac{1}{STS(\phi_1, 2\pi)(\Omega)} \sum_{k=-\infty}^{\infty} |\hat{\phi}_1(\Omega + 2\pi k)|^2 \\ &= \frac{STS(\phi_1, 2\pi)(\Omega)}{STS(\phi_1, 2\pi)(\Omega)} \\ &= 1 \end{aligned}$$

From the above result it is clear that $STS(\tilde{\phi}_1, 2\pi)(\Omega)$ is constant and is equal to one. Therefore, we can say that the underlying continuous function $\tilde{\phi}_1(t)$ is orthogonal to its integer translates. To characterize $\tilde{\phi}_1(t)$ lets consider its Fourier domain. We have,

$$\hat{\tilde{\phi}}_1(\Omega) = \frac{\hat{\phi}_1(\Omega)}{+\sqrt{STS(\phi_1, 2\pi)(\Omega)}}$$

Substituting the value of $STS(\phi_1, 2\pi)(\Omega)$ from equation 4, we get

$$\hat{\phi}_1(\Omega) = \frac{\hat{\phi}_1(\Omega)}{\sqrt{\frac{2}{3}(1 + \frac{1}{2} \cos \Omega)}}$$

Expanding the above equation in the form of binomial expansion,

$$\hat{\phi}_1(\Omega) = \hat{\phi}_1(\Omega) \left(\frac{2}{3}\right)^{-\frac{1}{2}} \left(1 + \frac{1}{2} \cos \Omega\right)^{-\frac{1}{2}} \quad (8)$$

This equation is of the form $(1 + \lambda)^R$, $R \in \mathbb{R}$, we know that

$$(1 + \lambda)^R = 1 + R.\lambda + \frac{R(R-1)}{2!}\lambda^2 + \frac{R(R-1)(R-2)}{3!}\lambda^3 + \dots \quad (9)$$

comparing the equations 8 and 9 and expanding equation 8, a typical p^{th} term can be represented as follows

$$K_p \lambda^p = K_p (\cos \Omega)^p \left(\frac{1}{2}\right)^p \quad (10)$$

We know that, $(\cos \Omega)^p = \left(\frac{e^{j\Omega} + e^{-j\Omega}}{2}\right)^p$, using this result in equation 10 and expanding it we get the final expression which looks like

$$\hat{\phi}_1(\Omega) = \sum_{k=-\infty}^{\infty} \tilde{C}_k e^{j\Omega k} \hat{\phi}_1(\Omega) \quad (11)$$

Now, from equation 11 we can find the Inverse DTFT of $\tilde{\phi}_1(\Omega)$ easily. We know that multiplication by the term $e^{j\Omega k}$ in Fourier domain shifts the signal by k in time domain. Therefore, $\tilde{\phi}_1(t)$ turns out to be

$$\tilde{\phi}_1(t) = \sum_{k=-\infty}^{\infty} \tilde{C}_k \phi_1(t + k) \quad (12)$$

From equation 12, $\tilde{\phi}_1(t)$ turns out to be a linear combination of $\phi_1(t)$ shifted by integer translates. When we shift a piecewise linear function by integer translates and add them we still get a piecewise linear function. Hence, $\tilde{\phi}_1(t)$ is a piecewise linear function.

In the next lectures we shall study the nature of \tilde{C}_k and know how to construct an MRA out of $\tilde{\phi}_1(t)$.