Lecture 29: Orthogonal Multiresolution Analysis With Splines

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Self Evaluation Quizzes

Q 1. Prove that Fourier transform of autocorrelation of a real function $\phi(t)$ is the power spectral density function of $\phi(t)$ in frequency domain.
Ans. We have by definition,

$$
\begin{aligned}
R_{\phi\phi}(\tau) &= \int_{-\infty}^{\infty} \phi(t)\phi(t+\tau)dt, \quad \text{Now substituting } t+\tau = k, \text{ we get} \\
R_{\phi\phi}(\tau) &= \int_{-\infty}^{\infty} \phi(k)\phi(k-\tau)dk \\
&= \int_{-\infty}^{\infty} \phi(k)\phi(-(\tau-k))dk
\end{aligned}
$$

By observing the above equation clearly we see that $R_{\phi\phi}(\tau)$ is the convolution of $\phi(t)$ with $\phi(-t)$. We know that convolution in time domain is multiplication in frequency domain. Let the Fourier transform of $\phi(t)$ be $\hat{\phi}(\Omega)$ then Fourier transform of $\phi(-t)$ is $\hat{\phi}(-\Omega)$. Therefore Fourier transform of $R_{\phi\phi}(\tau)$ is,

$$
\begin{aligned}
\hat{R}_{\phi\phi}(\Omega) &= \hat{\phi}(\Omega)\hat{\phi}(-\Omega) \\
&= |\hat{\phi}(\Omega)|^2
\end{aligned}
$$

$|\hat{\phi}(\Omega)|^2$ is essentially the power spectral density function of $\phi(t)$ in frequency domain. Hence proved.

Q 2. Prove that autocorrelation of any function $\phi(t)$ is symmetric and has a maximum value at $t = 0$.
Ans. In question 1 we have made an important observation that autocorrelation of a function $\phi(t)$ is convolution of $\phi(t)$ with $\phi(-t)$ *i.e.*,

$$
\begin{aligned}
R_{\phi\phi}(\tau) &= \phi(\tau) * \phi(-\tau), \text{ which implies} \\
R_{\phi\phi}(-\tau) &= \phi(-\tau) * \phi(\tau)
\end{aligned}
$$

We know that convolution follows commutative property. Therefore the above two equations have the same values. Therefore,

$$R_{\phi\phi}(\tau) = R_{\phi\phi}(-\tau)$$

Hence symmetry is proved. Now, $R_{\phi\phi}(0)$ is the area under the curve $|\hat{\phi}(\Omega)|^2$ which is always positive. Therefore, $R_{\phi\phi}(0) \geq 0$ for any function ϕ. Now consider a new function $\phi_1(t)$ where,

$$\phi_1(t) = \phi(t) - \phi(t+\tau)$$

Now $R_{\phi_1\phi_1}(0) \geq 0$

$$
\begin{aligned}
&\Rightarrow R_{\phi_1\phi_1}(0) = \int_{-\infty}^{\infty} \phi_1(t)\phi_1(t)dt \geq 0, \\
&\Rightarrow \int_{-\infty}^{\infty} (\phi(t) - \phi(t+\tau))(\phi(t) - \phi(t+\tau))dt \geq 0, \\
&\Rightarrow \int_{-\infty}^{\infty} \phi(t)^2 dt + \int_{-\infty}^{\infty} \phi(t+\tau)^2 dt - 2\int_{-\infty}^{\infty} \phi(t)\phi(t+\tau)dt \geq 0, \\
&\Rightarrow R_{\phi\phi}(0) + R_{\phi\phi}(0) - 2R_{\phi\phi}(\tau) \geq 0, \\
&\Rightarrow R_{\phi\phi}(0) \geq R_{\phi\phi}(\tau)
\end{aligned}
$$

Hence proved.