

1 Introduction

In previous lecture, variants of multi-resolution analysis (MRA) were briefly introduced. In particular, MRA with different analysis and synthesis filters, also known as biorthogonal filter-banks in perfect reconstruction framework, MRA with infinite impulse response (IIR) and MRA which also iterates on highpass branch along with the lowpass branch were discussed. In this and subsequent lectures we will discuss these variants in detail. We start with biorthogonal filter-banks utilized in Joint Photographic Experts Group 2000 (JPEG 2000) image compression standard in this lecture. We start with piecewise linear function as a scaling function at synthesis side and apply alias cancelation and perfect reconstruction requirements to achieve transfer functions of filters involved. This filter-bank differs from orthogonal MRA in non-orthogonality of scaling function to its integer translates. In this lecture, we will see how JPEG 2000 filter-bank is derived for spline type (piecewise linear with desirable interpolation characteristic) scaling function. In subsequent lectures, extension to orthogonal MRA for piecewise linear functions will be discussed, which explicitly reveals simplicity offered by biorthogonal filter-banks by sacrificing orthogonality requirement of scaling and wavelet functions.

2 Construction of biorthogonal filter bank

Dilation equation for Haar scaling function can be written as

$$\phi_0(t) = \phi_0(2t) + \phi_0(2t - 1)$$

where $\phi_0(t) \in V_0$ of Haar MRA. Convolution of this function with itself yields piecewise linear function which can be represented in the following manner.

$$\phi_1(t) = \frac{1}{2}\phi_1(2t) + \phi_1(2t - 1) + \frac{1}{2}\phi_1(2t - 2)$$

Clearly this is a triangular function as shown in Figure 1. This function comes from the class of piecewise polynomial interpolants, also called as splines. Z -transform of this function may be depicted in the following manner.

$$\Phi_1(z) = (1 + z^{-1})^2$$

Noting that $\phi_1(t)$ is a piecewise linear function. As seen in the last lecture, this function is not orthogonal to its integer translates. However, it is orthogonal to all its translates $\phi_1(t - m)$ for $m \geq 2$. We want to obtain following kind of filter-bank structure for this scaling function, which is similar to orthogonal MRA.

Let us select (without loss of generality) $G_0(z)$ as $(1 + z^{-1})^2$. At this point, we may note that once alias cancelation and perfect reconstruction conditions are met, we may interchange analysis and synthesis filters. Alias cancelation condition may be given in the following manner:

$$H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0$$

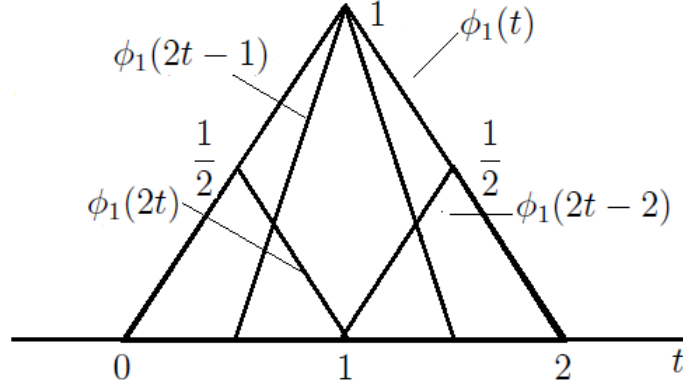


Figure 1: Triangular wave resulting from convolution of Haar scaling function with itself.

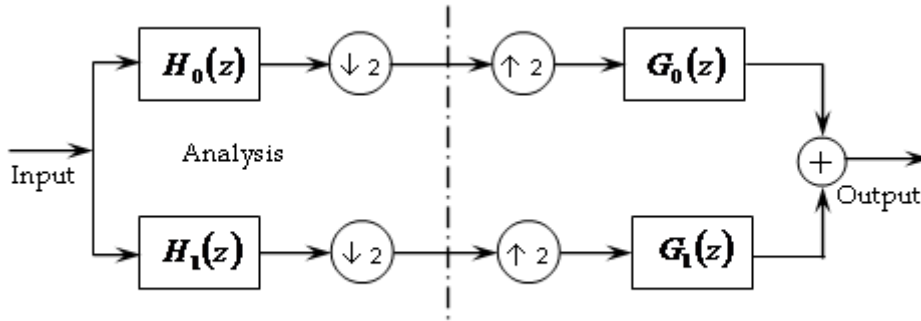


Figure 2: Desired filter-bank structure from piecewise linear function of Figure 1

If we replace z by $-z$ we get

$$H_0(z)G_0(-z) + H_1(z)G_1(-z) = 0$$

which is indeed a requirement of alias cancellation for analysis and synthesis filters interchanged. Also, perfect reconstruction condition may be depicted in the following manner:

$$H_0(z)G_0(z) + H_1(z)G_1(z) = C_0 z^{-D}$$

If we replace G_0 by H_0 and G_1 by H_1 and vice versa, we get the same condition back, which indicates that perfect reconstruction condition is also satisfied for this new filter-bank. Hence, we may say that once filter-bank satisfies alias cancellation and perfect reconstruction conditions, analysis and synthesis filters can be interchanged to get another 2-band perfect reconstruction filter-bank. However, if there are two separate scaling functions at analysis and synthesis side then a ‘smoother’ scaling function is employed at the reconstruction side for more ‘appealing’ reconstruction.

Using alias cancellation condition for this case, relationship between various filter transfer functions can be obtained. Such relationship is shown next.

$$H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0$$

$$\frac{G_0(z)}{G_1(z)} = -\frac{H_1(-z)}{H_0(-z)}$$

Equating numerator and denominator we get the following.

$$G_0(z) = -H_1(-z)$$

$$G_1(z) = H_0(-z)$$

Taking perfect reconstruction condition into consideration, we can get important relationship to obtain analysis lowpass transfer function from synthesis low pass transfer function. Consider the following relationship.

$$H_0(z)G_0(z) + H_1(z)G_1(z) = C_0z^{-D}$$

However, $G_1(z) = H_0(-z)$ and $G_0(z) = -H_1(-z)$, putting these value in above equation we get

$$H_0(z)G_0(z) - G_0(-z)H_0(-z) = C_0z^{-D}$$

Let us denote $\kappa_0(z) = G_0(z)H_0(z)$ then we have the following relationship

$$\kappa_0(z) - \kappa_0(-z) = C_0z^{-D}$$

This relationship indicates that we kill all even samples in inverse Z -transform of $\kappa_0(z)$ and out of the remaining odd samples preserve only one sample (namely, the D^{th} sample) to be nonzero, all other odd samples are set to be zero. If we select same degree of smoothness at analysis and synthesis sides then we may select $H_0(z)$ to have two zeros at $z = -1$. In other words, we select $H_0(z)$ to have $(1 + z^{-1})^2$ factor in its transfer function.

In order to have a linear phase characteristic, we need symmetry in transfer function. Taking note of this and without considering effect of causality, we may extend $H_0(z)$ by introducing a factor $(1 + h_0z^{-1} + z^{-2})$. We may note that by introducing this factor along with the factor $(1 + z^{-1})^2$, we have retained symmetry and only one degree of freedom that we have is in terms of parameter h_0 .

$$H_0(z) = (1 + z^{-1})^2(1 + h_0z^{-1} + z^{-2})$$

In retaining symmetry, we could have used some constant, say h_1 , for coefficients of z^0 and z^{-2} and retaining h_0 as coefficient of z^{-1} . Apparently, we could have had two degrees of freedom in such case. However, the factor introduced here essentially scales the whole filter transfer function by a constant. Scale factor for a transfer function is not really important for magnitude and phase characteristics. Also, we generally scale all the transfer functions in filter-bank to have some fixed norm. Hence, scaling can also be taken care of at the time of normalization.

In other words, we introduce only as many degrees of freedom as required to get something novel in terms nature of frequency response and not in terms of over-all scaling which might always be adjusted while normalizing the impulse response. Therefore, keeping only one degree of freedom, condition can be imposed on $G_0(z)H_0(z)$, which may be depicted in the following manner:

$$\begin{aligned} G_0(z)H_0(z) &= (1 + z^{-1})^2(1 + z^{-1})^2(1 + h_0z^{-1} + z^{-2}) \\ G_0(z)H_0(z) &= (1 + 4z^{-1} + 6z^{-2} + 4z^{-3} + z^{-4})(1 + h_0z^{-1} + z^{-2}) \end{aligned}$$

which is nothing but the convolution as shown below

$$\begin{array}{cccccc} 1 & 4 & 6 & 4 & 1 & * & 1 & h_0 & 1 \\ \uparrow & & & & & & \uparrow & & \\ 0 & & & & & & 0 & & \end{array}$$

which gives the following sequence

$$\begin{array}{cccccc} 1 & (4 + h_0) & (7 + 4h_0) & (8 + 6h_0) & (7 + 4h_0) & (4 + h_0) & 1 \\ \uparrow & & & & & & \\ 0 & & & & & & \end{array}$$

As discussed above, we do not have any relationship for even samples as they were removed by $\kappa_0(z) - \kappa_0(-z) = C_0 z^{-D}$ relationship. However, we have some information about the nature of odd samples; accordingly, we need to retain only one sample out of them. Notice that essentially if we remove two $(4 + h_0)$ samples, we achieve what we wanted. In other words, if we make $(4 + h_0) = 0$, we retain only one sample out of all odd samples. Therefore, we select $h_0 = -4$.

Hence

$$H_0(z) = (1 + z^{-1})^2(1 - 4z^{-1} + z^{-2})$$

Again for expansion, if we denote $H_0(z)$ by an impulse response (by series), we can denote the above relationship in terms of convolution of two series.

$$\begin{array}{cccccc} 1 & 2 & 1 & * & 1 & -4 & 1 \\ \uparrow & & & & \uparrow & & \\ 0 & & & & 0 & & \end{array}$$

Carrying out this convolution gives the following sequence:

$$1 \quad -2 \quad -6 \quad -2 \quad 1$$

Therefore, $H_0(z)$ turns out to be $1 - 2z^{-1} - 6z^{-2} - 2z^{-3} + z^{-4}$, which is a lowpass filter of the analysis side. We note that length of this filter is 5. Hence, we started out with

$$G_0(z) = 1 + 2z^{-1} + z^{-2}$$

and obtained

$$H_0(z) = 1 - 2z^{-1} - 6z^{-2} - 2z^{-3} + z^{-4}$$

Here, $G_0(z)$ is of length 3 and $H_0(z)$ is of length 5. This filter-bank is known as the JPEG 5/3 filter bank, where 5/3 refers to the lengths of impulse responses of low pass filters on the synthesis and the analysis side.

3 JPEG 2000 filter bank

JPEG-2000 compression standard admits two kinds of filter banks: a 5/3 filter-bank and a 9/7 filter-bank. As we have seen, 5/3 refers to lengths of impulse responses in a 5/3 filter-bank. Similarly, 9/7 also refers to lengths of impulse responses of a 9/7 filter-bank. We may consider these lengths to belong to either lowpass filters on analysis and synthesis sides or to belong to their highpass counterparts. We could also think of them as the lengths of the impulse responses of the analysis filters or the lengths of the synthesis filters. Lengths of impulse responses in the JPEG 2000 5/3 filter bank may be represented as shown in Figure 3.

We may use above noted relationships between various filter transfer functions to get transfer functions of all filters involved in JPEG 2000 5/3 filter bank. Now,

$$G_0(z) = -H_1(-z) \Rightarrow H_1(z) = -G_0(-z) = -(1 - z^{-1})^2$$

Further

$$G_1(z) = H_0(-z) \Rightarrow G_1(z) = 1 + 2z^{-1} - 6z^{-2} + 2z^{-3} + z^{-4}$$

Filter transfer functions for JPEG 2000 may be summarized in the following manner:

$$H_0(z) = 1 - 2z^{-1} - 6z^{-2} - 2z^{-3} + z^{-4}$$

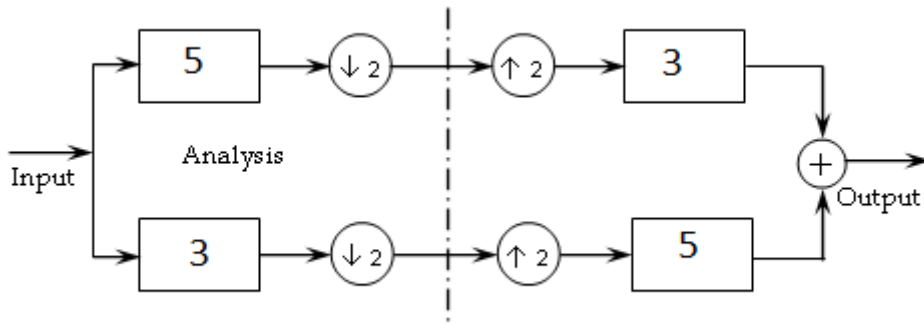


Figure 3: JPEG 2000 5/3 filter-bank

$$G_0(z) = 1 + 2z^{-1} + z^{-2}$$

$$H_1(z) = -G_0(-z) = -(1 - z^{-1})^2$$

$$G_1(z) = 1 + 2z^{-1} - 6z^{-2} + 2z^{-3} + z^{-4}$$

Note that individual sums of coefficients of highpass filters on analysis and synthesis sides are zero. This simply indicates presence of zero (null) at zero frequency. In fact, in both $H_1(z)$ and $G_1(z)$, there is a factor of $(1 - z^{-1})^2$, which indicates presence of two zeros at zero frequency. This substantiates our intuition that both filters should be highpass filters in nature.

Our major goal was to achieve the perfect reconstruction filter-bank from a scaling function, which along with its integer translates do not form an orthogonal basis set. Due to such property orthogonal MRA is not possible with such scaling function. In such case, we could still achieve perfect reconstruction filter-bank using this scaling function, however, we ended up getting different scaling functions at analysis and synthesis sides. In perfect reconstruction framework, such MRA is known as biorthogonal MRA. Idea of biorthogonal basis vectors may be explained more lucidly in a 2-D vector space as explained next.

4 Biorthogonal basis vectors

Consider a set of vectors consisting of \hat{u}_1 and \hat{u}_2 , which are orthogonal to each other, i.e., their inner product is zero. One such representation is shown in Figure 4.

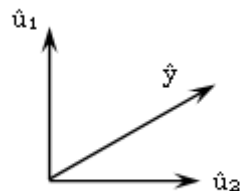


Figure 4: Orthogonal basis vectors (\hat{u}_1 and \hat{u}_2)

In this case, representing any other vector in this vector space involves projecting the vector over \hat{u}_1 and \hat{u}_2 by taking inner products with respect to such basis vectors. If vectors \hat{u}_1 and

\hat{u}_2 are linearly independent but not orthogonal to each other then also they form basis set for corresponding vector space. However, the ease of representation of vector \hat{y} with respect to such basis vectors is lost. This can be seen from Figure 5, which denotes two linearly independent vectors in a 2-D vector space.

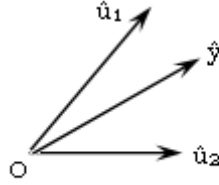


Figure 5: Orthogonal basis vectors (\hat{u}_1 and \hat{u}_2)

Biorthogonal basis vectors can help in such case to restore the ease of representation. We want \hat{y} to be represented in terms of $c_1\hat{u}_1 + c_2\hat{u}_2$. Consider a set of vectors \tilde{u}_1 and \tilde{u}_2 , selected such that \tilde{u}_1 is orthogonal to \hat{u}_1 and \tilde{u}_2 is orthogonal to \hat{u}_2 . Further $\langle \hat{u}_1, \tilde{u}_1 \rangle = 0$ and $\langle \hat{u}_2, \tilde{u}_2 \rangle = 0$, These vectors may be represented as shown in Figure 6.

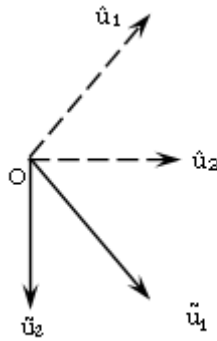


Figure 6: Idea of biorthogonal basis vectors

Now, we can take inner product of \hat{y} with \tilde{u}_1 and obtain its projection over \hat{u}_2 . Similarly, inner product of \hat{y} with \tilde{u}_2 yields projection over \hat{u}_1 . In this way, we can obtain representation of \hat{y} in terms of linearly independent basis vectors \hat{u}_1 and \hat{u}_2 . This idea is extended to the generation of a biorthogonal filter-bank. We can compare similarity by noting that in biorthogonal filter-bank there exist two scaling functions and two corresponding wavelet functions, one each at analysis and synthesis side. However, neither of them (scaling functions) forms an orthogonal set with its integer translates.