

## 1 Introduction

In this lecture we will explore some more avenues of the concept of MRA. First we will complete few details of the proof of the theorem of MRA attempted in previous lectures and then move to different variants of MRA.

## 2 Inner product of Wavelet function $\psi(t)$ and Scaling function $\phi(t - m)$

We had already shown that,  $\psi(t) \in V_1$  and  $\psi(t)$  is expanded as

$$\psi(t) = \sum_{n \in \mathbb{Z}} g[n] \phi(2t - n) \quad (1)$$

where  $g[n]$  is impulse response of the analysis high pass filter *i.e.* corresponding to the inverse  $Z$ -transform of  $z^{-(L-1)}H(-z^{-1})$ , where  $H(z)$  is analysis low pass filter. Expressing  $\phi(t)$  in terms of the basis of the  $V_1$  involves coefficients of low pass analysis filter. Expressing wavelet  $\psi(t)$  in terms of the basis of the  $V_1$  involves coefficients of high pass analysis filter. For example, in the Haar case, High pass filter impulse response is given as

$$\begin{bmatrix} 1 & -1 \end{bmatrix}$$

↑

And accordingly Haar wavelet is given as

$$\psi_{Haar}(t) = 1 \cdot \phi(2t) - 1 \cdot \phi(2t - 1) \quad (2)$$

The wavelet and scaling functions in general are given as

$$\begin{aligned} \psi(t) &= \sum_{n \in \mathbb{Z}} g[n] \phi(2t - n) \\ \phi(t - m) &= \sum_{n_1 \in \mathbb{Z}} h[n_1] \phi(2t - 2m - n_1) \end{aligned}$$

The inner product between wavelet function  $\psi(t)$  and scaling function  $\phi(t)$  shifted by  $m$  is given as

$$\langle \psi(t), \phi(t - m) \rangle = \sum_n \sum_{n_1} g[n] \overline{h[n_1]} \langle \phi(2t - n), \phi(2t - 2m - n_1) \rangle \quad (3)$$

The inner product term on the right hand side can be evaluated as

$$\langle \phi(2t - n), \phi(2t - 2m - n_1) \rangle = \int_{-\infty}^{+\infty} \phi(2t - n) \overline{\phi(2t - 2m - n_1)} dt \quad (4)$$

Let  $2t = \lambda$ . It gives  $2dt = d\lambda$  and  $t : -\infty \rightarrow +\infty \Rightarrow \lambda = -\infty \rightarrow +\infty$  With this substitution the above integral turns out to be,

$$\begin{aligned}\langle \phi(2t - n), \phi(2t - 2m - n_1) \rangle &= \frac{1}{2} \int_{-\infty}^{+\infty} \phi(\lambda - n) \overline{\phi(\lambda - 2m - n_1)} d\lambda \\ \langle \phi(2t - n), \phi(2t - 2m - n_1) \rangle &= \frac{1}{2} \delta[n - (2m + n_1)]\end{aligned}$$

Putting value in equation 3 we get,

$$\langle \psi(t), \phi(t - m) \rangle = \frac{1}{2} \sum_n \sum_{n_1} g[n] \overline{h[n_1]} \delta[n - (2m + n_1)] \quad (5)$$

Dropping  $\sum_n$ , we get

$$\langle \psi(t), \phi(t - m) \rangle = \frac{1}{2} \sum_{n_1} g[2m + n_1] \overline{h[n_1]} \quad (6)$$

The above expression is the cross-correlation of  $g[\cdot]$  and  $h[\cdot]$  evaluated at  $2m$ ,  $\forall m \in \mathbb{Z}$ . So the  $Z$ -transform of the cross-correlation  $g[\cdot]$  of  $h[\cdot]$  is,

$$G(z)H(z^{-1}) = z^{-(L-1)}H(-z^{-1})H(z^{-1}) \quad (7)$$

as  $G(z) = z^{-(L-1)}H(-z^{-1})$ . For simplicity let us assume that the impulse response to be real. Consider that

$$\begin{aligned}z^{-(L-1)}H(-z^{-1})H(z^{-1}) + z^{-(L-1)}H(-z^{-1})H(z^{-1})|_{z \leftarrow -z} &= 0 \\ z^{-(L-1)}H(-z^{-1})H(z^{-1}) + (-1)^{L-1}z^{-(L-1)}H(z^{-1})H(-z^{-1}) &= 0\end{aligned}$$

Here  $L$  is even. The cross-correlation of  $g[\cdot]$  and  $h[\cdot]$  is zero for all value of  $2m$ . So

$$\langle \psi(t), \phi(t - m) \rangle = 0 \quad \forall m \in \mathbb{Z} \quad (8)$$

Similarly the inner product between scaling function and its translate by  $m$  is given as

$$\langle \phi(t), \phi(t - m) \rangle = \frac{1}{2} \sum_n h[2m + n] \overline{h[n]} \quad (9)$$

The above expression shows the auto-correlation of the impulse response of the low pass filter in the analysis side. Now, the design equations for an orthogonal filter bank ensures that

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = \text{constant} \quad (10)$$

Autocorrelation sequence is '0' at all even locations  $2m$  except  $m = 0$  where  $m$  is an integer. Power complementarity property manifests as the orthogonality of impulse response to its own even translates and that also becomes a part of establishing orthogonality of  $\phi(t)$  to its own translates.

### 3 Variants of MRA

Now we explore the possibility of some different MRAs by asking following questions.

1. Must we have essentially the same analysis and synthesis filters?

The answer is ‘No’, because JPEG-2000 standard for data compression employs bi-orthogonal filter bank in which the analysis and synthesis filter banks are different.

When we move from Continuous Wavelet Transform to Discrete Wavelet Transform going through a step where we discretize the scale parameter logarithmically, either have different wavelets say  $\phi(t)$  and  $\tilde{\phi}(t)$  on analysis and synthesis sides and build different filter banks and get perfect reconstruction or use the same wavelet on analysis and synthesis sides but the same wavelet will be different from the wavelet that we started. In case, the discretization of scale parameter was such that sum of dilated spectra is not constant for all frequencies that lie between two positive bounds, we could bring in new wavelet which gave us an orthogonal analysis and synthesis filter bank, wavelet would be same  $\tilde{\phi}(t)$  but different from original wavelet where we started.

2. Do the filter in filter bank has to be finite impulse response filters?

The answer is ‘No, not necessarily’.

3. Must we always iterate on the low-pass branch?

The answer is again ‘No, not always. In wave packet analysis we also iterate on the high pass branch’.

By discrete wavelet transform generally we obtain a tree structure by iterating the filter bank on the low pass branch which is shown in Figure 1.

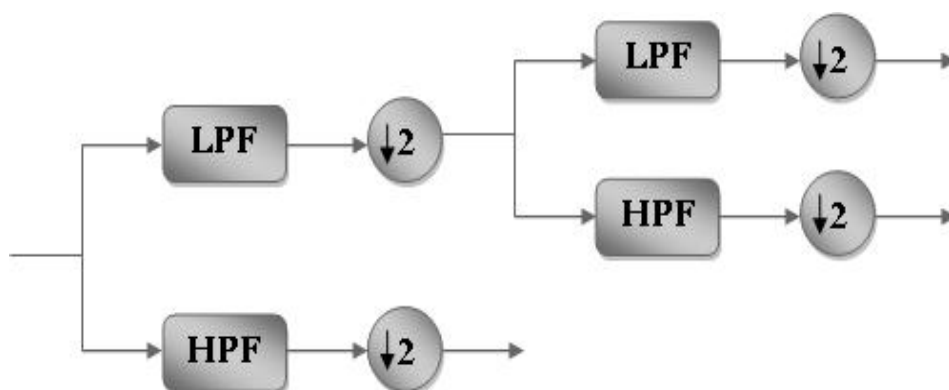


Figure 1: Discrete Wavelet Transform

However we can also iterate on the high pass branch and obtain the wave packet transform. The tree looks like as shown in Figure 2.

## 4 Introduction of Bi-orthogonal Filter Banks

Now let us try to look for an elaborate answer for the first question using the Haar MRA. In the case of Haar, the dilation equation for  $\phi_0(t)$ ,

$$\phi_0(t) = \phi_0(2t) + \phi_0(2t - 1) \tag{11}$$

This can be seen in Figure 3. Now we calculate  $\phi_0(t) * \phi_0(t)$ . Given  $h(t) * g(t) = r(t)$  then

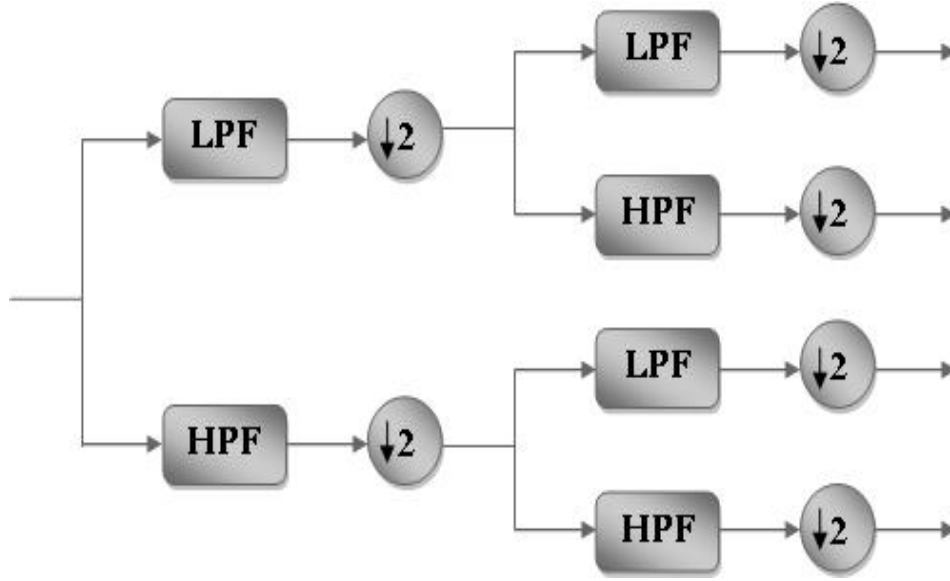


Figure 2: Wavelet Packet Transform

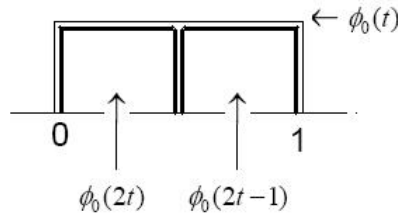


Figure 3: Dilation equation for  $\phi_0(t)$

what is  $h(at + b) * g(at + c)$ ? For same scaling  $a \in \mathbb{R}$ ,

$$h(at + b) * g(at + c) = \int_{-\infty}^{+\infty} h(a\lambda + b)g[a(t - \lambda) + c]d\lambda \quad (12)$$

Let  $a\lambda + b = \gamma$ ,  $a \neq 0$  and  $a \in \mathbb{R}$ . If  $a > 0$ ,  
 $d\gamma = ad\lambda$ , and  $\lambda : -\infty \rightarrow +\infty \Rightarrow \gamma = -\infty \rightarrow +\infty$   
 If  $a < 0$ ,  
 $d\gamma = ad\lambda$ , and  $\lambda : -\infty \rightarrow +\infty \Rightarrow \gamma = +\infty \rightarrow -\infty$   
 In general,

$$\begin{aligned} \int_{-\infty}^{+\infty} h(a\lambda + b)g[a(t - \lambda) + c]d\lambda &= \frac{1}{|a|} \int_{-\infty}^{+\infty} h(\gamma)g(-\gamma + at + b + c)d\gamma \\ \int_{-\infty}^{+\infty} h(a\lambda + b)g[a(t - \lambda) + c]d\lambda &= \frac{1}{|a|} \int_{-\infty}^{+\infty} h(\gamma)g(at + b + c - \gamma)d\gamma \\ \int_{-\infty}^{+\infty} h(a\lambda + b)g[a(t - \lambda) + c]d\lambda &= \frac{1}{|a|} h * g|_{(at+b+c)} \end{aligned}$$

Using this above equation, denoting  $\phi_0(t) * \phi_0(t) = \phi_1(t)$ . The dilation equation of  $\phi_1(t)$  is,

$$\phi_1(t) = \frac{1}{2}[\phi_1(2t) + 2\phi_1(2t - 1) + \phi_1(2t - 2)] \quad (13)$$

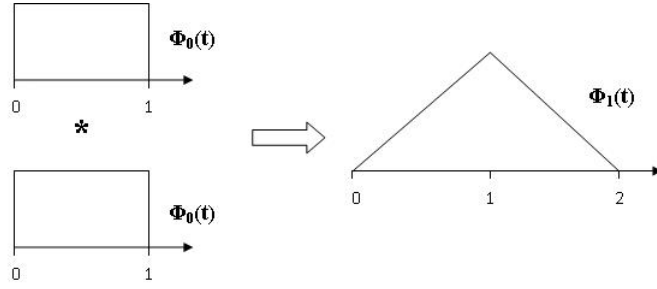


Figure 4: Convolution of  $\phi_0(t)$  with itself gives  $\phi_1(t)$

where  $\phi_1(t)$  is shown in Figure 4

Also,

$$\phi_0(2t) * \phi_0(2t - 1) = \phi_0(2t - 1) * \phi_0(2t) = \frac{1}{2}\phi_1(2t - 1) \quad (14)$$

This dilation equation is shown in Figure 5.

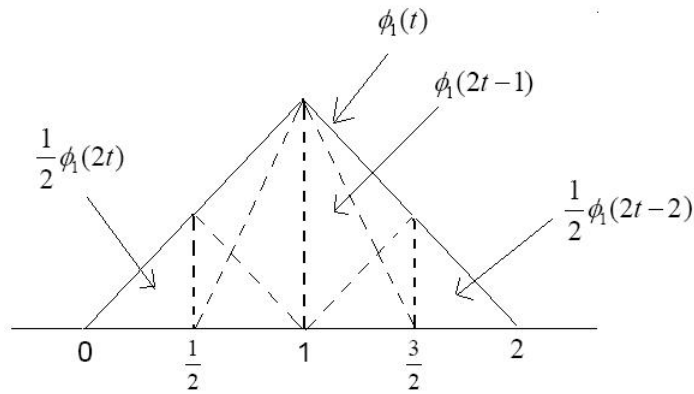


Figure 5: Dilation equation of  $\phi_1(t)$

Coefficient in dilation equation  $\phi_1(t) = \frac{1}{2}\phi_1(2t) + \phi_1(2t - 1) + \frac{1}{2}\phi_1(2t - 2)$  are

$$\begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \uparrow & & \end{bmatrix}$$

And the corresponding filter is,

$$\begin{aligned} H(z) &= \frac{1}{2} + 1 \cdot z^{-1} + \frac{1}{2} \cdot z^{-2} \\ H(z) &= \frac{1}{2}(1 + z^{-1})^2 \end{aligned}$$

But this scaling function is not orthogonal to all its integer translates.

The Figure 6 shows that the  $\phi_1(t)$  is not orthogonal to its translate by 1, *i.e* the axiom of orthogonality is not obeyed.

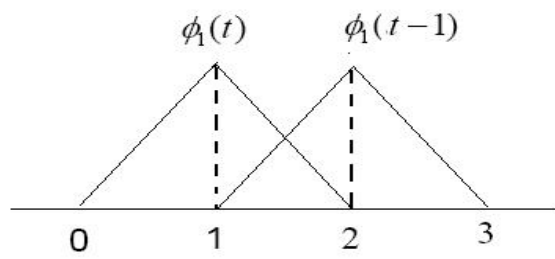


Figure 6:  $\phi_1(t)$  is not orthogonal to its translate by 1