

Lecture 26: Proof of the Theorem of Dyadic MRA

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## 1 Introduction

One way to interpret discretization of translation parameter is to raise issue of sampling of bandpass signal instead of bandlimited signal. In the discretization of translation parameter in the space  $V$ , we are talking about band limited function with band doubling each time around frequency zero. So sampling frequency is needed to be doubled.

Ideal Bandpass reconstruction filters is having unrealizable impulse response. Wavelets is a way of bandpass sampling and reconstruction practically. Under axioms of MRA can we extract a function  $\psi(t)$  which will allow band-pass sampling?

## 2 Proof of the Theorem of Dyadic MRA in Time Domain

Consider the function  $f(t)$  in the incremental subspace  $W_0$ . The characteristics of the function  $f(t)$  are:

- $f(t)$  is orthogonal to every translate of  $\phi(t)$  i.e.  $\phi(t - m)$ , where  $m \in \mathbb{Z}$ .
- $f(t - m) \in V_1$  and  $f(t)$  can be expressed in terms of  $\phi(2t - n)$ , where  $n \in \mathbb{Z}$ .

Let  $f(t) = \sum_{n=-\infty}^{+\infty} f[n]\phi(2t - n)$  where  $f[n]$  are the coefficients of expansion. We know that  $\phi(t) \in V_0 \subset V_1$ .  $\phi(t)$  can be expanded in terms of  $\phi(2t - n)$

$$\phi(t) = \sum_{n=-\infty}^{+\infty} h[n]\phi(2t - n)$$

where  $h[n]$  is the low pass impulse response coefficient.

$$\phi(t - m) = \sum_{n=-\infty}^{+\infty} h[n]\phi(2t - 2m - n)$$

Using the orthogonality property,

$$\langle f(t), \phi(t - m) \rangle = 0$$

$$\left\langle \sum_n f[n]\phi(2t - n), \sum_l h[l]\phi(2t - 2m - l) \right\rangle = 0 \quad \forall m$$

We now invoke the orthogonality of  $\phi(\cdot)$  with its own translates as

$$\langle \phi(2t - k_1), \phi(2t - k_2) \rangle = \int_{-\infty}^{+\infty} \phi(2t - k_1) \overline{\phi(2t - k_2)} dt$$

Putting  $2t = \lambda$  we get

$$\langle \phi(\lambda - k_1), \phi(\lambda - k_2) \rangle = \frac{1}{2} \int_{-\infty}^{+\infty} \phi(\lambda - k_1) \overline{\phi(\lambda - k_2)} d\lambda = \frac{1}{2} \delta[k_1 - k_2]$$

Thus,

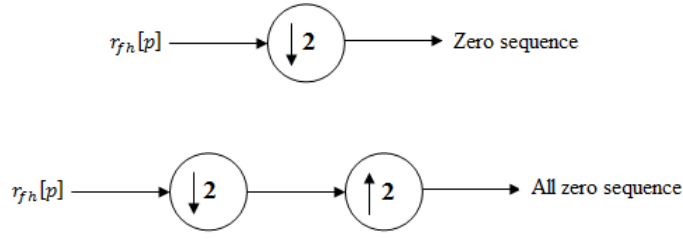
$$\langle f(t), \phi(t - m) \rangle = \sum_n \sum_l f[n] \overline{h[l]} \langle \phi(2t - n), \phi(2t - 2m - l) \rangle = 0$$

The above term is nonzero only for  $n = 2m + l$ .

$$\langle f(t), \phi(t - m) \rangle = \frac{1}{2} \sum_l f[2m + l] \overline{h[l]}$$

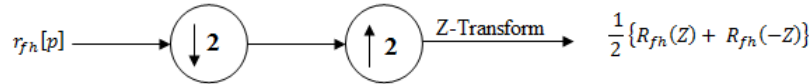
We are essentially looking at the cross correlation of the sequences  $f[\cdot]$  and  $h[\cdot]$ . Cross correlation is often denoted by:

$$r_{fh}[p] = \sum_l f[p + l] \overline{h[l]}$$



Therefore,  $r_{fh}[p]_{p=2m} = 0 \quad \forall \quad m \in \mathbb{Z}$  i.e. the cross correlation of  $f[\cdot]$  and  $h[\cdot]$  evaluated at all even shifts is zero. In  $Z$ -domain

$$r_{fh}[p] \longrightarrow R_{fh}(z)$$



$$\begin{aligned} R_{fh}(z) + R_{fh}(-z) &= 0 \\ F(z)H(z^{-1}) + F(-z)H(-z^{-1}) &= 0 \\ \frac{F(z)}{F(-z)} &= -\frac{H(-z^{-1})}{H(z^{-1})} \end{aligned}$$

$$F(z) = -\Lambda(z)H(-z^{-1}) \tag{1}$$

$$F(-z) = \Lambda(z)H(z^{-1}) \tag{2}$$

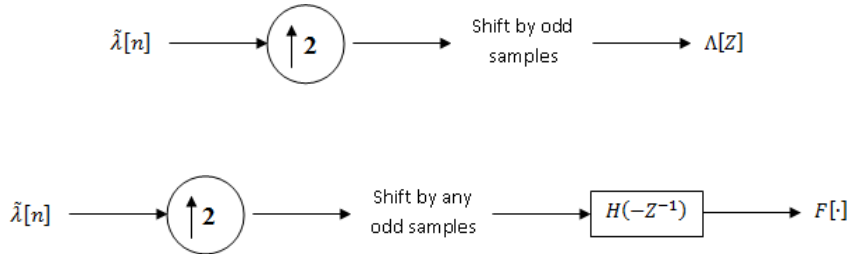
Putting  $z = -z$  in equation(1) we get,

$$F(-z) = -\Lambda(-z)H(z^{-1}) \tag{3}$$

Comparing equation(2) and (3) we have:

$$\begin{aligned} \Lambda(z) &= -\Lambda(-z) \\ \Lambda(z) + \Lambda(-z) &= 0 \end{aligned}$$

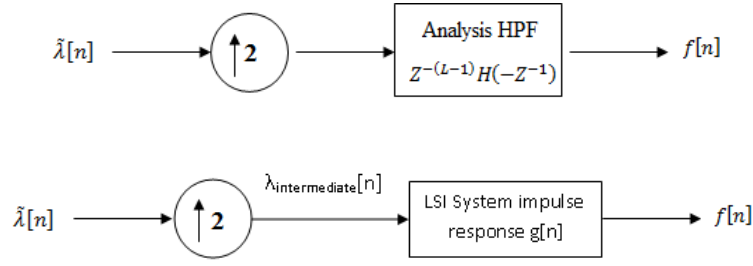
In terms of sequences, if  $\Lambda(z)$  is a  $Z$ -transform of a sequence, then sequence should be zero at all even locations. Sequence could have been obtained by up sampling another sequence and shifting by one place. By upsampling by 2 we introduce zero at odd position. For making zero sequence it should be shifted by odd number of samples.



We could in particular choose odd number of samples:  $L - 1$ , where  $L$  is low pass analysis filter length. Also we know that  $z^{-(L-1)}H(-z^{-1})$  is essentially the analysis HPF.

$$f(t) = \sum_{n=-\infty}^{+\infty} f[n]\phi(2t - n)$$

Let  $g[n]$  be inverse  $Z$ -transform of  $z^{-(L-1)}H(-z^{-1})$  where  $g[n]$  is the impulse response of the analysis HPF.



$$\begin{aligned}
 f[n] &= \lambda_{intermediate}[n] * g[n] \\
 &= \sum_{k=-\infty}^{+\infty} \lambda_{intermediate}[k]g[n-k]
 \end{aligned}$$

$\lambda_{intermediate}[k]$  is non zero only at  $2k$  (even) interval.

$$\begin{aligned}
 f[n] &= \sum_{k=-\infty}^{+\infty} \lambda_{intermediate}[2k]g[n-2k] \\
 &= \sum_{k=-\infty}^{+\infty} \tilde{\lambda}[k]g[n-2k] \\
 f(t) &= \sum_{n=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \tilde{\lambda}[k]g[n-2k]\phi(2t-n) \\
 &= \sum_{k=-\infty}^{+\infty} \tilde{\lambda}[k] \sum_{n=-\infty}^{+\infty} g[n-2k]\phi(2t-n)
 \end{aligned}$$

Substituting  $n - 2k = q$

$$\begin{aligned}
 f(t) &= \sum_{k=-\infty}^{+\infty} \tilde{\lambda}[k] \sum_{q=-\infty}^{+\infty} g[q]\phi(2t-q-2k) \\
 &= \sum_{k=-\infty}^{+\infty} \tilde{\lambda}[k] \sum_{q=-\infty}^{+\infty} g[q]\phi(2(t-k)-q)
 \end{aligned}$$

Here,  $(t - k)$  denotes shift in continuous variable  $t$  by  $k$ . Now let us define

$$\psi(t) = \sum_{q \in \mathbb{Z}} g[q]\phi(2t - q)$$

It follows that  $\psi(\cdot) \in V_1$ . Thus effectively we have,

$$f(t) = \sum_k \tilde{\lambda}[k]\psi(t - k)$$

This equation proves that the proto-type function  $f(t)$  in orthogonal complement of  $V_0$  in  $V_1$  i.e.  $W_0$  is expressible in terms of an integer translates of the function  $\psi(t)$ . If we could capture the single function  $\psi(t)$  and all of its integer translates, then these form the bases which could span  $W_0$  i.e.  $(\psi(t - k))_{k \in \mathbb{Z}}$  spans  $W_0$ .

The proof for the theorem of dyadic multiresolution analysis is almost complete except to demonstrate

- $(\psi(t - k))_{k \in \mathbb{Z}}$  forms a set of an orthogonal bases.
- $\langle \psi(t - k), \phi(t - m) \rangle = 0 \quad \forall k, m \in \mathbb{Z}$

This will be explained in the next lecture.

The same theorem can be proved in Frequency domain as well, which is given below

### 3 Proof of the Theorem of Dyadic MRA in Frequency Domain

We shall follow the same steps for the proof of MRA in frequency domain as we did in time domain.

$$\phi(t) \in V_0 \subset V_1$$

$\phi(t)$  can be expanded in terms of the low pass filter impulse response  $h[n]$  and  $\phi(2t - n)$  as

$$\phi(t) = \sum_{n=-\infty}^{\infty} h[n]\phi(2t - n)$$

Taking a fourier transform on both sides

$$\hat{\phi}(\Omega) = \frac{1}{2}H\left(\frac{\Omega}{2}\right)\hat{\phi}\left(\frac{\Omega}{2}\right) \quad (4)$$

(The fourier series expansion of equation 4 has been explained in previous lectures). Also  $\phi(t)$  is perpendicular to  $\phi(t - n), \forall n \in \mathbb{Z}, n \neq 0$ .

We can define the autocorrelation function

$$R_{\phi\phi}(\tau) = \int_{-\infty}^{\infty} \phi(t + \tau)\overline{\phi(t)}dt$$

The fourier transform of  $R_{\phi\phi}(\tau)$  is  $|\hat{\phi}(\Omega)|^2$ . Sampling  $R_{\phi\phi}(\tau)$  at  $\tau = n \in \mathbb{Z}$  and taking fourier transform gives

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(\Omega + 2\pi k)|^2$$

Since  $\phi(t)$  is orthogonal to its integer translates, the following relation holds true

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(\Omega + 2\pi k)|^2 = C_0$$

Take a typical function,  $f(t)$ , in orthogonal complement of  $V_0$  in  $V_1$ , then

$$f(t) = \sum_{n=-\infty}^{\infty} f[n]\phi(2t - n)$$

Taking the cross correlation of  $f(t)$  and  $\phi(t)$

$$\int_{-\infty}^{\infty} f(t + \tau)\overline{\phi(t)}dt = R_{f\phi}(\tau)$$

If we sample  $R_{f\phi}(\tau)$  we get a zero sequence. That is  $R_{f\phi}(n) = 0 \forall n \in \mathbb{Z}$ .

$$\implies \sum_{k=-\infty}^{\infty} \hat{F}(\Omega + 2\pi k)\overline{\hat{\phi}(\Omega + 2\pi k)} = 0, \text{ where } \hat{F}(\Omega)\overline{\hat{\phi}(\Omega)} \text{ is the fourier transform of } R_{f\phi}(\tau)$$

It was shown previously that

$$\begin{aligned} \hat{f}(\Omega) &= \frac{1}{2}\hat{F}\left(\frac{\Omega}{2}\right)\hat{\phi}\left(\frac{\Omega}{2}\right) \\ \hat{\phi}(\Omega) &= \frac{1}{2}\hat{H}\left(\frac{\Omega}{2}\right)\hat{\phi}\left(\frac{\Omega}{2}\right) \end{aligned} \quad (5)$$

Substituting in equation 5 gives

$$\sum_{k=-\infty}^{\infty} \frac{1}{2}\hat{F}\left(\frac{\Omega}{2} + \pi k\right)\overline{\hat{H}\left(\frac{\Omega}{2} + \pi k\right)}\left|\hat{\phi}\left(\frac{\Omega}{2} + \pi k\right)\right|^2 = 0$$

The left hand side can be expanded as follows

$$\sum_{k \in \mathbb{Z}} \frac{1}{2} \hat{F} \left( \frac{\Omega}{2} + 2\pi k \right) \overline{\hat{H} \left( \frac{\Omega}{2} + 2\pi k \right)} \left| \hat{\phi} \left( \frac{\Omega}{2} + 2\pi k \right) \right|^2 + \sum_{k \in \mathbb{Z}} \frac{1}{2} \hat{F} \left( \frac{\Omega}{2} + 2\pi k + \pi \right) \overline{\hat{H} \left( \frac{\Omega}{2} + 2\pi k + \pi \right)} \left| \hat{\phi} \left( \frac{\Omega}{2} + 2\pi k + \pi \right) \right|^2 = 0$$

Let  $\frac{\Omega}{2} = \nu$

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \frac{1}{2} \hat{F}(\nu + 2\pi k) \overline{\hat{H}(\nu + 2\pi k)} |\hat{\phi}(\nu + 2\pi k)|^2 + \sum_{k \in \mathbb{Z}} \frac{1}{2} \hat{F}(\nu + 2\pi k + \pi) \overline{\hat{H}(\nu + 2\pi k + \pi)} |\hat{\phi}(\nu + 2\pi k + \pi)|^2 &= 0 \\ \implies \hat{F}(\nu) \overline{\hat{H}(\nu)} \sum_{k \in \mathbb{Z}} |\hat{\phi}(\nu + 2\pi k)|^2 + \hat{F}(\nu + \pi) \overline{\hat{H}(\nu + \pi)} \sum_{k \in \mathbb{Z}} |\hat{\phi}(\nu + 2\pi k + \pi)|^2 &= 0 \\ \implies (\hat{F}(\nu) \overline{\hat{H}(\nu)} + \hat{F}(\nu + \pi) \overline{\hat{H}(\nu + \pi)}) C_0 = 0, C_0 \neq 0 & \\ \implies (\hat{F}(\nu) \overline{\hat{H}(\nu)} + \hat{F}(\nu + \pi) \overline{\hat{H}(\nu + \pi)}) &= 0 \\ \implies \frac{\hat{F}(\nu)}{\hat{F}(\nu + \pi)} = -\frac{\overline{\hat{H}(\nu + \pi)}}{\overline{\hat{H}(\nu)}} & \quad (6) \end{aligned}$$

Thus we can write

$$\begin{aligned} \hat{F}(\nu) &= R(\nu) \overline{\hat{H}(\nu + \pi)} \\ \hat{F}(\nu + \pi) &= R(\nu + \pi) \overline{\hat{H}(\nu + 2\pi)} \\ &= R(\nu + \pi) \overline{\hat{H}(\nu)} \end{aligned} \quad (7)$$

by comparing the denominators in eqn[6]

$$\hat{F}(\nu + \pi) = -R(\nu) \overline{\hat{H}(\nu)} \quad (8)$$

Comparing the above equations

$$\begin{aligned} R(\nu) &= -R(\nu + \pi) \\ R(\nu) + R(\nu + \pi) &= 0 \\ R(\nu) &= \frac{\hat{F}(\nu)}{\overline{\hat{H}(\nu + \pi)}} \end{aligned}$$

We can conceive of  $R(\nu)$  as DTFT in its own right. That is

$$\sum_{n=-\infty}^{\infty} r[n] e^{-j\Omega n} = R(\Omega)$$

$R(\nu + \pi)$  in frequency domain implies the sequence  $r[n]$  is multiplied by  $(-1)^n$  that is

$$r[n] + (-1)^n r[n] = 0$$

Let

$$\begin{aligned} \sum_{n=-\infty}^{\infty} r[n] Z^{-n} &= R(Z) \\ \implies R(Z) &= Z^{-1} R_1(Z^2) \end{aligned}$$

(A shift by one means we get a function of  $Z^2$  which has samples only at even positions) Thus

$$\hat{F}(\Omega) = R(\nu) \overline{\hat{H}(\nu + \pi)}|_{\nu=\Omega}$$

Also  $f(t) = \sum_{n=-\infty}^{\infty} f[n] \phi(2t - n)$  implies

$$\begin{aligned} \hat{f}(\Omega) &= \frac{1}{2} \hat{F} \left( \frac{\Omega}{2} \right) \hat{\phi} \left( \frac{\Omega}{2} \right) \\ &= \frac{1}{2} R \left( \frac{\Omega}{2} \right) \hat{H} \left( \frac{\Omega}{2} + \pi \right) \hat{\phi} \left( \frac{\Omega}{2} \right) \\ &= \frac{1}{2} e^{-j\frac{\Omega}{2}} R_1(e^{j\Omega}) \overline{\hat{H} \left( \frac{\Omega}{2} + \pi \right)} \hat{\phi} \left( \frac{\Omega}{2} \right) \end{aligned} \quad (9)$$

Denote  $\hat{\psi}(\Omega) = \frac{1}{2}e^{-j\frac{\Omega}{2}}\overline{H\left(\frac{\Omega}{2} + \pi\right)}\hat{\phi}\left(\frac{\Omega}{2}\right)$

$$\implies \hat{f}(\Omega) = R_1(e^{j\Omega})\hat{\psi}(\Omega)$$

substituting for the value of  $R_1(e^{j\Omega})$

$$\implies \hat{f}(\Omega) = \sum_{n=-\infty}^{\infty} r_1[n]e^{-j\Omega n}\hat{\psi}(\Omega)$$

Taking inverse Fourier Transform

$$f(t) = \sum_{n=-\infty}^{\infty} r_1[n]\psi(t-n)$$

Now we have proved that any function  $f(t)$ , which belongs to  $W_0$ , can be spanned by a function  $\psi(t)$  and its integer translates. Now our goal is to prove that  $\psi(t)$  is orthogonal to its integer translates.

We have

$$\hat{\psi}(\Omega) = \frac{1}{2}e^{-j\frac{\Omega}{2}}\overline{H\left(\pi - \frac{\Omega}{2}\right)}\hat{\phi}\left(\frac{\Omega}{2}\right)$$

The last step will be to prove that the generic function obtained satisfies the orthogonality with its translates, which will be

$$\sum_{k=-\infty}^{\infty} |\hat{\psi}(\Omega + 2\pi k)|^2 = \text{constant}$$

Note: Kindly refer to the tutorial set of lecture-27 for the proof of the above equation.