WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 25: The Theorem of (Dyadic) MRA

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1 Introduction

In the previous lecture, we discussed that translation and scaling parameter should be discretized. Discretization of the scaling parameter has been described but the translation parameter is still continuous. The scaling parameter had been discretized logarithmically. More specifically, our aim is discretizing the translation parameter with the consideration that wavelet transform is discretized with scaling parameter in powers of two.
Before proceeding to the discretization of translation parameter in the powers of two (i.e. in dyadic scale manner) let us see in short, what we had done earlier.

2 Biorthogonal Filter Bank

Filter banks with different analysis and synthesis wavelets and scaling function are called as 'Biorthogonal Filter Banks'. When we talk about filter bank, we refer to perfect reconstruction filter bank. In general the k^{th} analysis branch takes the input $x(t)$ and subjects it to the filter $\overline{\widehat{\psi}(a_0^k\Omega)}$, where $a_0 > 1$ and k runs over all integer. Output of the k^{th} analysis branch is given as input to the k^{th} synthesis branch whose frequency response is $\widehat{\widetilde{\psi}}(a_0^k\Omega)$. All synthesis branches

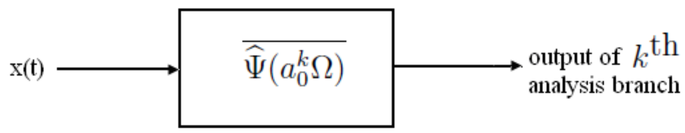

Figure 1: Analysis Branch

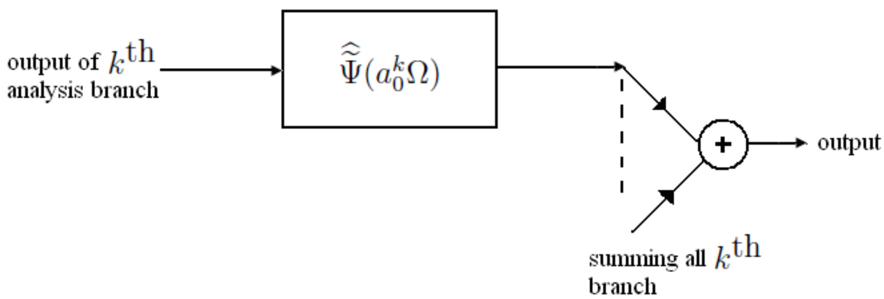

Figure 2: Synthesis Branch

are added together to get output. We have $\widehat{\widetilde{\psi}}(a_0^k\Omega)$ in frequency domain as

$$\widehat{\widetilde{\psi}}(\Omega) = \frac{\widehat{\psi}(\Omega)}{\mathrm{SDS}(\psi, a_0)(\Omega)}$$

where SDS is the sum of dilated spectra.

$$\mathrm{SDS}(\psi, a_0)(\Omega) = \sum_{k=-\infty}^{+\infty} |\widehat{\psi}(a_0^k \Omega)|^2$$

This SDS is bounded by $C1$ and $C2$ so that

$$0 < C1 \leq \mathrm{SDS}(\psi, a_0)(\Omega) \leq C2 < \infty$$

Wavelet ψ is admissible because of $C1$. $\widetilde{\psi}$ was meaningful because of upper and lower bound. $\widetilde{\psi}$ is also admissible and bound on SDS is $\frac{1}{C1}$, $\frac{1}{C2}$. When the $\mathrm{SDS}(\psi, a_0)(\Omega)$ is constant for all Ω, such filter banks are called as 'Orthogonal Filter Banks'.

3 Orthogonal Filter Bank

When filters on analysis side and synthesis side are same (i.e. same wavelet function with same scaling parameter) then these filter banks are called as 'Orthogonal Filter Banks'.

3.1 Construction of orthogonal filter bank

Let us define $\widetilde{\widetilde{\widetilde{\psi}}}(\Omega)$ as

$$\widetilde{\widetilde{\widetilde{\psi}}}(\Omega) = \frac{\widehat{\psi}(\Omega)}{+\sqrt{\mathrm{SDS}(\psi, a_0)(\Omega)}}$$

Because of upper and lower bound on the SDS of ψ, $\mathrm{SDS}(\psi, a_0)(\Omega)$ can be taken in the denominator.

$$0 < \sqrt{C1} \leq \sqrt{\mathrm{SDS}(\psi, a_0)(\Omega)} \leq \sqrt{C2} < \infty$$

With above observation let us prove that the $\widetilde{\widetilde{\psi}}$ is an admissible wavelet function. Consider $\mathrm{SDS}(\widetilde{\widetilde{\psi}}, a_0)(\Omega)$

$$\mathrm{SDS}(\widetilde{\widetilde{\psi}}, a_0)(\Omega) = \frac{\sum_{k=-\infty}^{+\infty} |\widehat{\psi}(a_0^k \Omega)|^2}{\mathrm{SDS}(\psi, a_0)(\Omega)}$$

Sum of dilated spectra is independent of the scaling parameter a_0^k. Replacing Ω by $a_0^k \Omega$ we get

$$\mathrm{SDS}(\psi, a_0)(\Omega) = \mathrm{SDS}(\psi, a_0)(a_0^m \Omega)$$

Proof in general:

$$\begin{aligned} \mathrm{SDS}(\psi, a_0)(a_0^m \Omega) &= \sum_{k=-\infty}^{+\infty} |\widehat{\psi}(a_0^k a_0^m \Omega)|^2 \\ &= \sum_{k=-\infty}^{+\infty} |\widehat{\psi}(a_0^{k+m} \Omega)|^2 \end{aligned}$$

Here m is a constant integer. Therefore, as k runs over all integers, $k + m$ will also run over all integers i.e.

$$\begin{aligned} \mathrm{SDS}(\widetilde{\widetilde{\psi}}, a_0)(\Omega) &= \frac{\sum_{k=-\infty}^{+\infty} |\widehat{\psi}(a_0^k \Omega)|^2}{\mathrm{SDS}(\psi, a_0)(\Omega)} \\ &= 1 \quad \forall \Omega \end{aligned}$$

$\tilde{\tilde{\psi}}$ has the same upper and lower bound and hence it is admissible and orthogonal wavelet. So, it can be used as a wavelet on both the analysis side and the synthesis side. Here we have constructed orthogonal wavelet function from bi-orthogonal wavelet. But still the transition parameter is continuous.
In case of Haar wavelet, it does not satisfy the condition of upper bound equal to lower bound. This is because, in case of Haar, the orthogonality is with respect to discrete shifts in time, and not the continuous shifts. This is a weaker requirement. However, the discrete shifts in time is what we are looking for, as we do not want to retain the whole continuous translation parameter.
In the next section we take a wavelet ψ which has the property of admissibility and reconstructibility and we will study discretization of the translation parameter for the dyadic case i.e. $a_0 = 2$ to construct a dyadic multiresolution analysis.

4 Dyadic Multiresolution Analysis

Examples of dyadic MRA are Haar MRA, Daubechies MRA ($a_0 = 2$). The wavelet obeys the requirement

$$0 < \mathrm{C1} \leq \mathrm{SDS} \leq \mathrm{C2} < \infty, \text{ for all } \Omega$$

The wavelet may not obey this requirement for all a_0, but it obeys this requirement for $a_0 = 2$. So these bounds, in general, depend on a_0. Also, the wavelet admits discretizing the translation parameter. *Should we discretize the translation parameter in the same way in all the branches, or do it differently?*
Let us look at the k^{th} branch. On the k^{th} analysis branch, the output is broadly a band pass function, that is, it is significant in a certain band of frequencies, not around zero. For different values of k, there is a logarithmic variation of the band. We invoke a generalization of the sampling theorem for band pass functions and illustrate it with an example. Consider a bandpass function where the band on Ω lies between π and 2π.

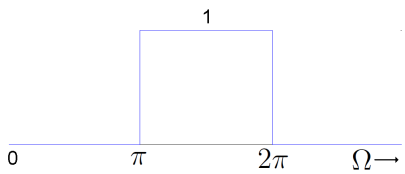

Figure 3: Band Pass Function

When we talk about discretising the translation parameter we are essentially talking about sampling the output of the k^{th} analysis branch and feeding these samples to the k^{th} synthesis branch, instead of the continuous function. So, "*how do we sample the output of the k^{th} branch so that we do not lose anything*", is equivalent to the question "*how do we discretize the translation parameter?*"
We have two options:

1. To sample the signal following the Nyquist criteria, and considering 2π as the highest frequency.

2. To sample it remembering that the band of frequencies occupied by the is only between π and 2π.

In the second case, we can use a sampling rate twice the band occupancy. Here bandwidth is π. Therefore, we could use a sampling rate as 2π. If we simply use the Nyquist criteria, we should have sampling frequency f_S such that

$$2\pi f_s = 4\pi \Rightarrow f_s = 2$$

But we can also do with a sampling frequency f_s such that

$$\begin{aligned} 2\pi f_s &= 2 \text{ times the band occupancy} \\ &= 2\pi \\ \Rightarrow f_s &= 1 \end{aligned}$$

Suppose we do use a sampling rate of 1. Then we are adding all the aliases, which are shifts of the original spectrum by $2\pi k$, for all integer k. The figure 4 shows the original spectrum along with the aliases due to shifting the spectrum by 2π and -2π.

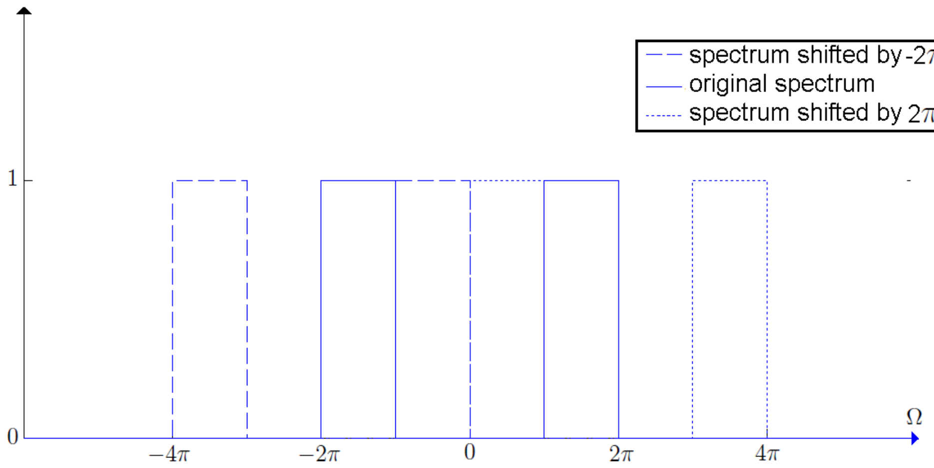

Figure 4: Frequency spectrum of Band Pass Function

The translations by 2π and -2π do not affect the original spectrum. Similarly, translations by 4π and -4π leave the original spectrum unpolluted. For higher translations of the original spectrum, the aliases move further away from the original spectrum, and hence we need not worry about them. The original part of the signal is therefore unaffected. The original signal can be retrieved by putting a bandpass filter between π and 2π. This is the bandpass sampling. This cannot, however, be generalized for any position of the frequency band, i.e. wherever the band of π is put, a sampling rate of 2π may be used, this is not true in general. It is true depending on the position of the band. Hence, the bandpass sampling theorem is a little more complicated than the conventional low pass sampling theorem. It certainly is more economical. In fact, in the dyadic MRA we are essentially invoking the Bandpass Sampling theorem.
The same principle is applicable for bands between 2π and 4π, 4π and 8π and so on. So for different branches on the analysis side, we would need to use different sampling frequencies, and these frequencies will also be related logarithmically. That is exactly what happens in Dyadic MRA. When we go from V_0 to V_1, or from V_1 to V_2, number of points are doubled. Going from V_0 to V_{-1}, number of points is halved. All these are essentially manifestations of the bandpass sampling theorem.
Let us now focus on $a_0 = 2$. We need to use a logarithmic change of the form $2k$ of sampling. On the k^{th} branch, the sampling rate relates to $2k$. This is automatically ensured by the Dyadic MRA axioms.

4.1 Axioms of a Dyadic MRA and Theorem of MultiResolution Analysis

1. **Ladder axiom**

$$\dots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \dots$$

   V_0 is the subspace where the functions are bandpass in a certain band, V_1 is the subspace where functions are bandpass in the next higher band, V_2 the next higher band and so on. Each time the frequency occupancy is doubled. As we go downwards, the frequency occupancy is halved. So, as we go upwards the sampling frequency is doubled, as we go downwards the sampling frequency is halved.

2. **Axiom of perfect reconstruction**

$$\overline{\bigcup_{m\in\mathbb{Z}} V_m} = L_2(\mathbb{R})$$

   When all the incremental subspaces are collected together, we go back to the original input signal.

3. We will always remain in $L_2(\mathbb{R})$ so as we go downwards we are going towards smaller and smaller bands and finally we are going to reach a band with zero power.

$$\bigcap_{m\in\mathbb{Z}} V_m = \{0\}$$

4. If

$$x(t) \in V_0$$

   then

$$x(2^m t) \in V_m$$

   Implicitly, this implies logarithmic sampling.

5. **Axiom of translation**

   If

$$x(t) \in V_0$$

   then

$$x(t-n) \in V_0,\ for\ all\ n \in Z$$

   It essentially says that we have an uniform sampling.

6. **Axiom of orthogonal basis**

   There exists a $\phi(t)$ such that $\{\phi(t-n)\}_{n\in\mathbb{Z}}$ is a basis for V_0. Given axioms 4 and 5, we have a corresponding basis for each of the V_m. This axiom gives us a way to reconstruct the function from samples. The coefficients in the expansion of the function with respect to $\phi(t)$ are like the generalized samples of the function after filtering.

   Now, V_0 is a collective subspace and we are sampling a collective subspace, not an incremental subspace. The theorem of multi-resolution analysis is going to give us an incremental subspace.

Given axioms 1 to 6, there exists a function $\psi(t)$ ($\psi(t) \in L_2(\mathbb{R})$ and $\psi(t) \in V_1$) such that $\{\psi(2^m t - n)\}_{m\in\mathbb{Z}, n\in\mathbb{Z}}$ forms an orthogonal basis for $L_2(\mathbb{R})$.

5 Some Discussions on Biorthogonal Filter Bank

We have seen that the bi-orthogonal wavelets are the wavelets that have different filters $\phi(\Omega)$ and $\tilde{\phi}(\Omega)$ on the analysis and the synthesis side of the filter bank. We have also gone on to see that orthogonality constraint is relaxed on the bi-orthogonal wavelets. That is bi-orthogonal wavelets are not orthogonal to their own discrete dilates and translates. The orthogonality constraint is replaced by removing the equality

$$\mathrm{SDS}(\psi, a_0)(\Omega) = \text{constant}$$

with the inequality

$$0 < C_1 \leq \mathrm{SDS}(\psi, a_0)(\Omega) \leq C_2 < \infty$$

Thus in bi-orthogonal wavelets, the sum of dilated spectra is bound in the positive finite limit $[C_1, C_2]$ for all Ω. The admissibility condition, which assures perfect construction of the original signal from its transformed components, is satisfied due to the lower limit C_1.
A very important question at this stage would be, why biorthogonal wavelets? And how do they differ in properties from orthogonal wavelets?
Biorthogonal wavelets possess some properties which make them more usable than orthogonal wavelets. Most important of these properties is symmetric property of wavelet filter coefficients. Symmetric functions help us to achieve linear phase system which are very important in terms of preserving the signal integrity. By relaxing the condition of orthogonality, we have also achieved a level of freedom in design of our wavelet function. This freedom permits us to design symmetric wavelets functions which possess linear phase. Another important of property of biorthogonal wavelets is that the filter coefficients must posses odd length. Following is a discussion on this property.
Let the scaling function on the analysis side be

$$\phi(t) = \sum_n h(n)\phi(2t - n)$$

In a bi-orthogonal system the scaling function on the synthesis side will be different:

$$\tilde{\phi}(t) = \sum_n \tilde{h}(n)\tilde{\phi}(2t - n)$$

We know that in a bi-orthogonal system, $\phi(t)$ is not orthogonal to its translates. It is orthogonal to the translates of $\tilde{\phi}(t)$

$$\int \phi(t-n)\tilde{\phi}(t-n)dt = 1 \tag{1}$$

$$\int \phi(t-n)\tilde{\phi}(t-m)dt = 0 \qquad n \neq m \tag{2}$$

$$\int \phi(t)\tilde{\phi}(t-n)dt = 0 \qquad n \neq 0$$

For a dyadic system, this condition on the scaling functions can be represented in terms of its filter coefficients using the dilation equation:

$$\phi(t) = \sum_n h(n)\phi(2t - n)$$

$$\int \phi(t)\tilde{\phi}(t-n)dt = 0$$

$$\implies \int \left[\sum_k h(k)\phi(2t-k)\right]\left[\sum_l \tilde{h}(l)\tilde{\phi}(2(t-n)-l)\right] dt = 0$$

Let $l = m - 2n \implies m = l + 2n$

$$\implies \int \left[\sum_m h(m)\phi(2t-m)\right]\left[\sum_m \tilde{h}(m-2n)\tilde{\phi}(2t-m)\right] dt = 0$$

$$\implies \sum_m h(m)\tilde{h}(m-2n) = 0 \tag{3}$$

Similarly for equation (1)

$$\sum_m h(m)\tilde{h}(m) = 1 \tag{4}$$

Hence we observe that analysis and synthesis side responses are orthogonal with shifts of 2. Let us design our $\tilde{h}(n)$ such that it is non-zero in the range $M_1 < n < M_2$ and $h(n)$ is non-zero in the range $N_1 < n < N_2$.

By equations (3) and (4)

$$N_2 - M_1 = 2n + 1$$
$$M_2 - N_1 = 2m + 1$$

where $n, m \in Z$

So,

$$(N_2 - N_1) - (M_1 - M_2) = 2(n + m + 1)$$