WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 24: Logarithmic (Dyadic) Discretization

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1 Introduction

In the previous lecture, the admissibility condition for the continuous wavelet transform (CWT) was studied in depth. The admissibility condition was essentially the condition required to reconstruct a signal from its continuous wavelet transform. The continuous wavelet transform was extremely redundant. To use a continuous scale and a continuous translation, meaning a two dimensional representation for a one dimensional entity is extremely redundant. Therefore we exploit the possibilities of discretization of scale as well as the translation parameter. This lectures deals with discretization of the scale parameter. It deals with Logarithmic Discretization (in general) and Dyadic Discretization (in particular). The continuous wavelet transform operates like a filter both on the synthesis side as well as on the analysis side.
The continuous wavelet transform at scale 's' is a filtering operation with a frequency response $\widehat{\psi}(s\Omega)$ as shown in Figure-1 (with some constants which are ignored).

Figure 1: CWT as Filtering Operation

If we take an ideal filter or an ideal wavelet (real), the $\widehat{\psi}(\Omega)$ corresponding to that wavelet was essentially an ideal bandpass filter with cutoff frequencies $\Omega 1$ and $\Omega 2$ as shown in the Figure-2. Only the positive side of the frequency response is shown.

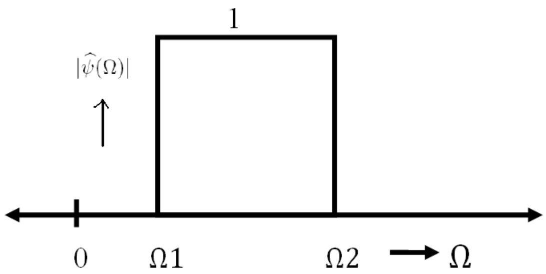

Figure 2: Ideal Wavelet Bandpass Response

Now, if we take the dilation of this band pass function, we again get a band pass function. So, for any $s > 0$, $\widehat{\psi}(\Omega)$ would essentially be as shown in Figure-3.

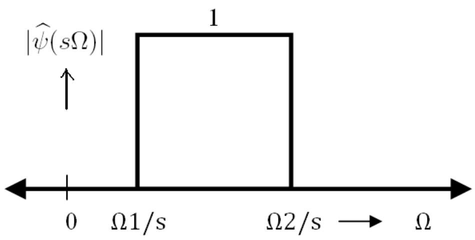

Figure 3: Dilated Wavelet Response

Thus, there is contraction or expansion of the band as well as the center frequency. Now, logarithmic discretization means, the parameter 's' should be discretized as

$$s = a_0^k$$

where 'k' is the set of all integers and $a_0 > 1$.
Since, we are taking 'k' to be the set of all integers (positive as well as negative), it is clear that we need not consider the possibility of a_0 to be less than 1. Now for each such 'k', we have a filter. In the ideal condition (we shall study ideal conditions initially and then degrade to the nonideal conditions eventually), the k^{th} filter would have a frequency response as shown in Figure-4.

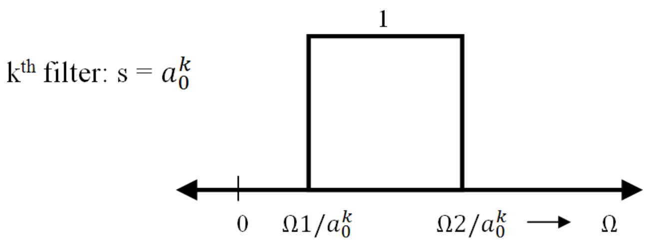

Figure 4: k^{th} ideal Filter Response

Now, if we pass a signal $x(t)$ through this ideal filter and obtain the CWT, the reconstruction should also be done using the same process. In fact, $x(t)$ could be obtained as shown in the Figure-5.

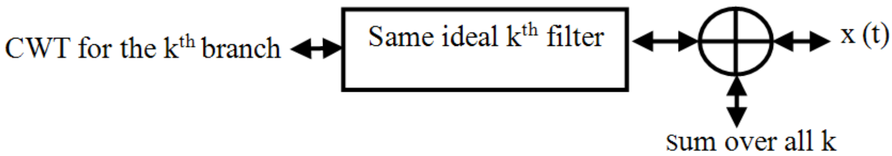

Figure 5: Reconstruction of $x(t)$

Thus, each particular k^{th} branch extracts a particular band on the frequency axis and each branch should have a separate non overlapping band. This could be explained by an example.
Example: Consider a wavelet having ideal frequency response $\widehat{\psi}(\Omega)$ as shown in Figure-6.

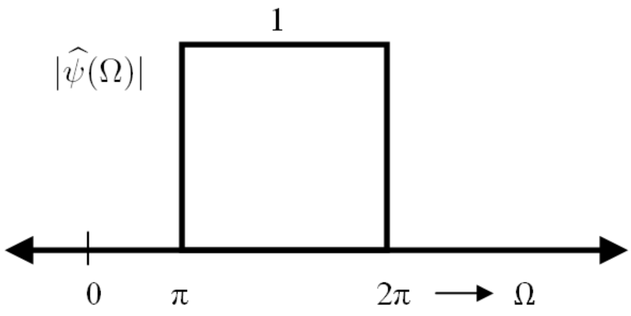

Figure 6: Ideal $\widehat{\psi}(\Omega)$ with positive side of Ω axis

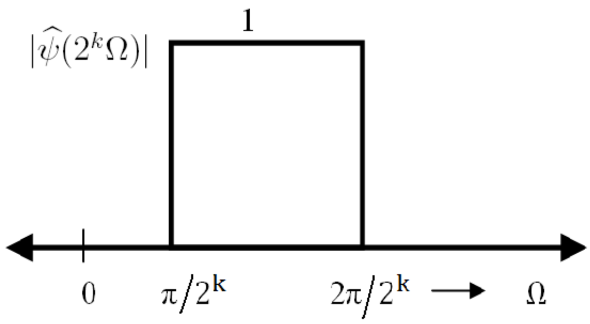

Figure 7: $\widehat{\psi}(2^k\Omega)$ with positive side of Ω

Then, if $a_0 = 2$, then the k^{th} branch would be the following:
Thus for different integers k, these bands would be non overlapping as shown in the figure below.

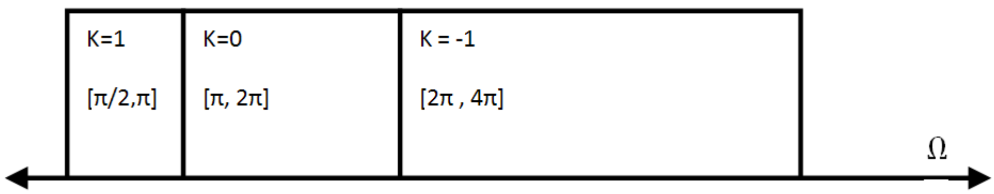

Figure 8: Non-overlapping bands with different 'k'

However we can not obtain such discretization in the frequency domain by time limited functions because of the uncertainty principle.
This discretization of the scale parameter also explains the notion of filter banks. In fact it is equivalent to constructing a filter bank. A filter bank is a collection of filters either with a common input or with all the outputs summed together to get a common output.
Consider the analysis and the synthesis side. It is shown in Figure-9 and Figure-10.
Output of the k^{th} branch of analysis side = $\widehat{X}(\Omega)$(conjugate of $\widehat{\psi}(a_0^k\Omega)$)= $\widehat{X}(\Omega)(\overline{\widehat{\psi}(a_0^k\Omega)})$
Output from the k^{th} branch of synthesis side = $\widehat{X}(\Omega)|\widehat{\psi}(a_0^k\Omega)|^2$
Overall output by summing over all 'k' =$\widehat{X}(\Omega)\sum_{k=-\infty}^{\infty}|\widehat{\psi}(a_0^k\Omega)|^2$

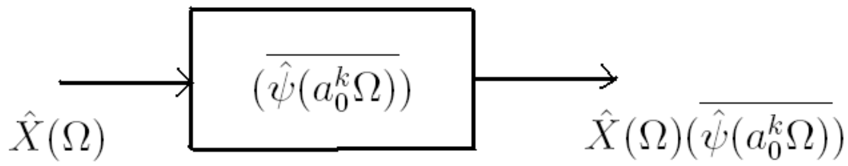

Figure 9: Analysis Filters

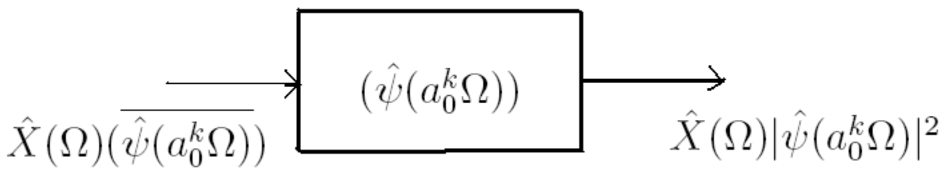

Figure 10: Synthesis Filters

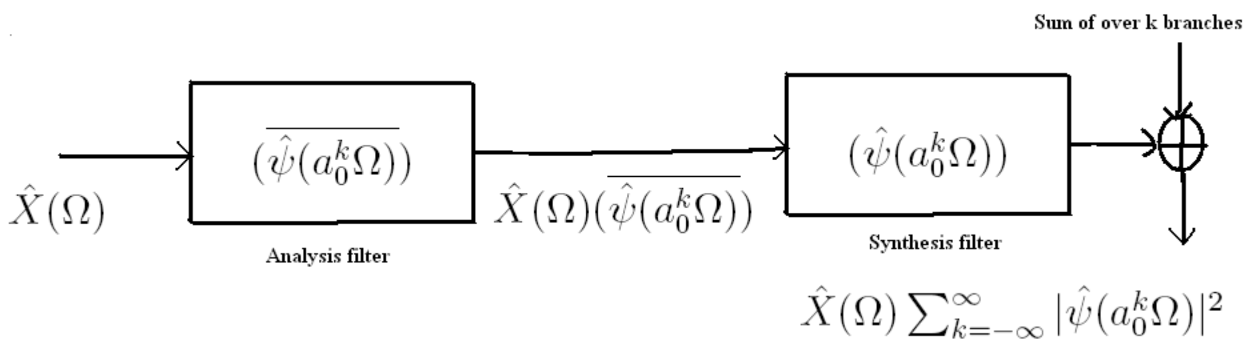

Figure 11: Overall Filter

Ideally, $\sum_{k=-\infty}^{\infty}|\widehat{\psi}(a_0^k\Omega)|^2 = 1$ for all Ω indicating a perfect reconstruction. The challenge here is to obtain $\sum_{k=-\infty}^{\infty}|\widehat{\psi}(a_0^k\Omega)|^2 = 1$ for all Ω using time limited functions.
To meet this challenge, we need to relax in the frequency domain. The first step towards this is not to consider it as a constant. In fact, we can consider it to be lying between two strictly positive constants $c1$ and $c2$. Therefore for designability, we consider

$$0 < c1 \leq \sum_{k=-\infty}^{\infty}|\widehat{\psi}(a_0^k\Omega)|^2 \leq c2 < \infty$$

Thus $0 < c1 \leq c2 < \infty$. Note that $c1$ is strictly greater than 0 and $c2$ strictly less than ∞. By using this condition, we can make a small change on the synthesis filter as follows and obtain perfect reconstruction. We define another function $\widehat{\widehat{\psi}}(\Omega)$ in the frequency domain from $\widehat{\psi}(\Omega)$ as

$$\widehat{\widehat{\psi}}(\Omega) = \frac{\widehat{\psi}(\Omega)}{\sum_{k=-\infty}^{\infty} |\widehat{\psi}(a_0^k\Omega)|^2} \tag{1}$$

Since the condition of $c1$ is imposed on the denominator, it cannot go to 0. Hence we can define such a function. Now, we aim to show that $\widehat{\psi}(\Omega)$ could be used on the analysis side while $\widehat{\widehat{\psi}}(\Omega)$ can be used on the synthesis side. Before proving this, we first show ψ is automatically admissible (ψ is real). As studied in the previous lecture, we consider the following integral for admissibility

$$\Rightarrow \int_0^{\infty} |\widehat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha}$$

Let's break this integral as (2)

$$\int_0^{\infty} |\widehat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha} = \sum_{k=-\infty}^{\infty} \int_{a_0^k}^{a_0^{k+1}} |\widehat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha} \tag{2}$$

Put $\alpha = a_0^k\beta$

$$\Rightarrow \int_{a_0^k}^{a_0^{k+1}} |\widehat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha} = \int_1^{a_0} |\widehat{\psi}(a_0^k\beta)|^2 \frac{d\beta}{\beta} \tag{3}$$

$$\Rightarrow \int_0^{\infty} |\widehat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha} = \int_1^{a_0} \left\{ \sum_{k=-\infty}^{\infty} |\widehat{\psi}(a_0^k\beta)|^2 \right\} \frac{d\beta}{\beta} \tag{4}$$

Since the limits of integration are finite and the argument of integration is also upper bounded by $c2$, we are guaranteed that the integral is convergent and hence $\widehat{\psi}(\Omega)$ is admissible. In fact, the admissibility integral is upper bounded by,

$$\int_1^{a_0} c2 \frac{d\beta}{\beta} = c2 ln(a_0)$$

2 Sum of Dilated Spectra (SDS)

The quantity,

$$\sum_{k=-\infty}^{\infty} |\widehat{\psi}(a_0^k\Omega)|^2$$

is called as Sum of Dilated Spectra (SDS). SDS has primary as well as secondary arguments, where primary arguments a_0, Ω are those which are important and change in a given context and the secondary argument 'ψ' is used for the construction of continuous wavelets transform. Therefore,

$$SDS(\psi, a_0)(\Omega) = \sum_{k=-\infty}^{\infty} |\widehat{\psi}(a_0^k\beta)|^2 \tag{5}$$

where

$$0 < c1 \leq SDS(\psi, a_0)(\Omega) \leq c2 < \infty$$

and $c2$ guarantees admissibility.

Now let us check the admissibility condition for $\widehat{\widehat{\psi}}(\Omega)$. From (1),

$$\widehat{\widehat{\psi}}(\Omega) = \frac{\widehat{\psi}(\Omega)}{SDS(\psi, a_0)(\Omega)} \tag{6}$$

For this purpose, we need to consider $SDS(\widetilde{\psi}, a_0)$

$$|\widehat{\widehat{\psi}}(a_0^k\Omega)|^2 = \frac{|\widehat{\psi}(a_0^k\Omega)|^2}{\{\sum_{k=-\infty}^{\infty} |\widehat{\psi}(a_0^l a_0^k\Omega)|^2\}^2} \tag{7}$$

$$|\widehat{\widehat{\psi}}(a_0^k\Omega)|^2 = \frac{|\widehat{\psi}(a_0^k\Omega)|^2}{\{SDS(\psi, a_0)(\Omega)\}^2} \tag{8}$$

$$SDS(\widetilde{\psi}, a_0)(\Omega) = \frac{\sum_{k=-\infty}^{\infty} |\widehat{\psi}(a_0^k\Omega)|^2}{\{SDS(\psi, a_0)(\Omega)\}^2} \tag{9}$$

$$SDS(\widetilde{\psi}, a_0)(\Omega) = \frac{SDS(\psi, a_0)(\Omega)}{\{SDS(\psi, a_0)(\Omega)\}^2} \tag{10}$$

Cancelation from the numerator and denominator is valid because of the bounds $c1$, $c2$.

$$SDS(\widetilde{\psi}, a_0)(\Omega) = \frac{1}{SDS(\psi, a_0)(\Omega)} \tag{11}$$

We know that, $0 < c1 \leq SDS(\psi, a_0)(\Omega) \leq c2 < \infty$, hence

$$\infty > \frac{1}{c1} \geq \frac{1}{SDS(\psi, a_0)(\Omega)} \geq \frac{1}{c2} > 0$$

i.e.,

$$0 < \frac{1}{c2} \leq SDS(\widetilde{\psi}, a_0)(\Omega) \leq \frac{1}{c1} < \infty$$

So, $\widetilde{\psi}$ is also an admissible wavelet.
k^{th} branch of the synthesis side using $\widetilde{\psi}$ is shown in Figure-12.

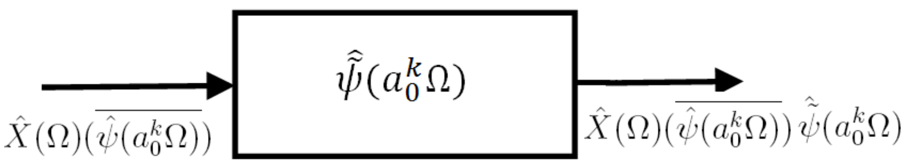

Figure 12: Synthesis Filter with $\widehat{\widehat{\psi}}(t)$

Hence, output of the analysis and synthesis filters together can be written as

$$= \widehat{x}(\Omega) \times \sum_k \overline{\widehat{\psi}(a_0^k\Omega)} \times \widehat{\widehat{\psi}}(a_0^k\Omega)$$

$$= \widehat{x}(\Omega) \times \sum_k \overline{\widehat{\psi}(a_0^k\Omega)} \times \frac{\widehat{\psi}(a_0^k\Omega)}{SDS(\psi, a_0)(\Omega}$$

$$= \widehat{x}(\Omega) \times \frac{SDS(\psi, a_0)(\Omega)}{SDS(\psi, a_0)(\Omega)}$$

$$= \widehat{x}(\Omega)$$

This gives perfect reconstruction.
Hence, if we allow $SDS(\psi, a_0)(\Omega)$ to lie between two constants $c1$ and $c2$, we can allow two different wavelets on the analysis side and the synthesis side. Thus we have 'ψ' on the analysis side and '$\widetilde{\psi}$' on the synthesis side. This leads to a different class of filter banks.
Note: If $c1 = c2$ then ψ, $\widetilde{\psi}$ are the same, that gives us orthogonal filter banks. However, here, we have not yet discretized the translation parameter. We have discretized only the scale. So it does not follow that Haar wavelet has SDS as a constant because the translation parameter is discrete in the Haar case. In particular, if $a_0 = 2$, we get a dyadic orthogonal filter bank.