WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 23: Admissibility and Discretization of Scale

Prof.V.M.Gadre, EE, IIT Bombay

1 Introduction

In the previous lecture we built up the idea of decomposition and reconstruction from STFT and CWT. The central theme in decomposition and reconstruction was to project a function to be decomposed on the the basis vector and to reconstruct function from its component by multiplying each component by the vector in that direction. STFT is indexed by translation and modulation while CWT is indexed by translation and scale. Translation can be dealt easily but for scale we need an additional weighting factor when we reconstruct.

2 Admissibility

Important steps in reconstructing $x(t)$ from $CWT(x,\psi)(\tau,s)$ are as follows.
The innermost integral involved is called component corresponding to $CWT(x,\psi)(\tau,s)$ and is given by

$$\frac{1}{2\pi}\int \hat{x}(\Omega)s^{\frac{1}{2}}\overline{\hat{\psi}(s\Omega)}e^{j\Omega\tau}d\Omega \tag{1}$$

The two outer integral takes cares of translation and scale parameters. The triple integral is as follows:

$$\int_{s=0}^{\infty}\int_{\tau=-\infty}^{\infty} CWT(x,\psi)(\tau,s)\frac{1}{s^{\frac{1}{2}}}\psi\left(\frac{t-\tau}{s}\right)f(s)ds\,d\tau \tag{2}$$

Here $CWT(x,\psi)(\tau,s)$ is the component , $\frac{1}{s^{\frac{1}{2}}}\psi(\frac{t-\tau}{s})$ is a unit vector and $f(s)$ is the weighting function to deal with the scale. Writing all together, the triple integral becomes

$$\frac{1}{2\pi}\int_{s=0}^{\infty}\int_{\tau=-\infty}^{\infty}\int_{\Omega=-\infty}^{\infty}\hat{x}(\Omega)s^{\frac{1}{2}}\overline{\hat{\psi}(s\Omega)}e^{(j\Omega\tau)}\frac{1}{s^{\frac{1}{2}}}\psi\left(\frac{t-\tau}{s}\right)f(s)ds\,d\Omega\,d\tau \tag{3}$$

The approach used to evaluate this integral is that we take $d\tau$ first, ds second and $d\Omega$ in the last. After taking care of $\int_{-\infty}^{\infty}d\tau$ what was left, by choosing $f(s)=\frac{1}{s^2}$ is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{x}(\Omega)\int_{0}^{\infty}\left|\hat{\psi}(s\Omega)\right|^2\frac{ds}{s}e^{j\Omega t}d\Omega \tag{4}$$

We can observe that if we make $\int_0^\infty|\hat{\psi}(s\Omega)|^2\frac{ds}{s}$ independent of Ω then it becomes a constant and comes out of the integral, hence we get

$$\frac{1}{2\pi}C_\psi\int_{-\infty}^{\infty}\hat{x}(\Omega)e^{j\Omega t}d\Omega \tag{5}$$

This C_ψ tells us the factor by which $x(t)$ is multiplied in the reconstruction. Now consider the integral $\int_0^\infty|\hat{\psi}(s\Omega)|^2\frac{ds}{s}$. Putting $s\Omega=\alpha$ we get,

Case I: $\Omega > 0, \alpha : 0 \to +\infty$ when $s : 0 \to +\infty$
$d\alpha = \Omega ds, \alpha = \Omega s$. Therefore $\frac{d\alpha}{\alpha} = \frac{ds}{s}$

$$\int_0^\infty \left|\hat{\psi}(s\Omega)\right|^2 \frac{ds}{s} = \int_0^\infty \left|\hat{\psi}(\alpha)\right|^2 \frac{d\alpha}{\alpha} \tag{6}$$

We can see that the right hand side of the above equation is based on the function ψ and independent of Ω.
Case II: $\Omega < 0, \alpha : 0 \to -\infty$ when $s : 0 \to +\infty$. So the integral becomes $\int_0^{-\infty} |\hat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha}$. Substituting $\alpha = -\beta$ we get

$$\int_0^\infty \left|\hat{\psi}(s\Omega)\right|^2 \frac{ds}{s} = \int_0^\infty \left|\hat{\psi}(-\beta)\right|^2 \frac{d\beta}{\beta} \tag{7}$$

Now making integral (on s) independent of Ω,

$$\int_0^\infty \left|\hat{\psi}(\alpha)\right|^2 \frac{d\alpha}{\alpha} = \int_0^\infty \left|\hat{\psi}(-\beta)\right|^2 \frac{d\beta}{\beta} < \infty \tag{8}$$

The above integral must be positive as the integration is on non-negative integrands.
If $\psi(t)$ is real then, $\hat{\psi}(-\beta) = \overline{\hat{\psi}(\beta)}$. Therefore,

$$\left|\hat{\psi}(-\beta)\right|^2 = \left|\hat{\psi}(\beta)\right|^2 \tag{9}$$

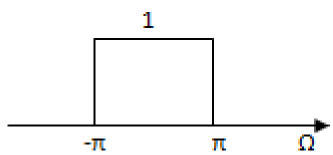
However, if $\psi(t)$ is a complex function, then we need to take care of positive and the negative part of the spectrum separately. If we use complex wavelet and we insist that spectrum is one sided then we must ensure that the signal has no component on other side, therefore in that case the particular condition can be removed. For example, we take complex wavelet where we do not take care of the negative part of the spectrum, so condition which involve $|\hat{\psi}(-\beta)|^2$ is not obeyed, then we may only deal with such 'x' which have non-zero components and therefore 'x' must be complex having non-zero components on the positive part of the spectrum *i.e.*, $\Omega > 0$. Conversely for $\Omega < 0$ original spectrum should have no part on positive side.
So, we conclude that, *admissibility allows the function to be called a wavelet. It is the condition required for reconstruction of a function from its CWT.*

$$\int_0^\infty \left|\hat{\psi}(\alpha)\right|^2 \frac{d\alpha}{\alpha} = \int_0^\infty \left|\hat{\psi}(-\beta)\right|^2 \frac{d\beta}{\beta} < \infty \tag{10}$$

Above integrals are know as admissibility integrals.

3 Admissibility Integral for $\Omega = 0$ and $\Omega = \infty$

Consider the case of $\Omega > 0$ (with real ψ this is enough). Now, according to admissibility condition $\int_0^\infty |\hat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha}$ must be finite. We will now have a look at $\Omega = 0$ and $\Omega = \infty$. Consider $|\hat{\psi}(\alpha)|^2$ to be as shown below.

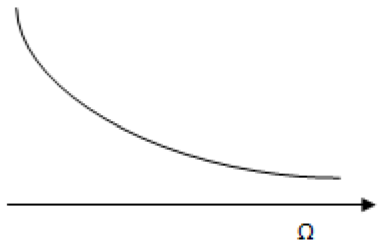


It is obvious that it cannot be a wavelet as:

$$\int_0^\infty |\hat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha} = \int_0^\pi 1.\frac{d\alpha}{\alpha} \tag{11}$$

$$\int_0^\pi \frac{d\alpha}{\alpha} = \ln\alpha|_0^\pi \tag{12}$$

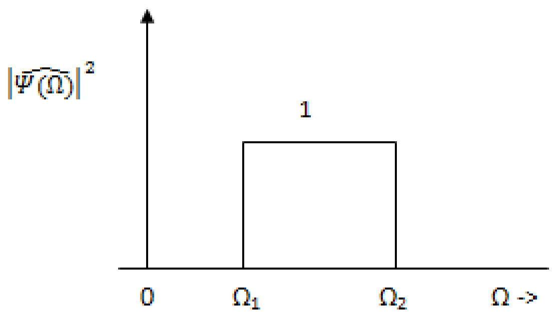
The above integral diverges. This trouble comes from region around $\Omega = 0$. Consider another example as shown below.



Here $|\hat{\psi}(\alpha)|^2$ will be a constant (say C_0) as $\alpha \to \infty$, therefore

$$\int_{Large\,value}^\infty |\hat{\psi}(\alpha)|^2 \frac{d\alpha}{\alpha} \approx \int_{Large\,value}^\infty C_0 \frac{d\alpha}{\alpha} = \ln\alpha|_{Large\,value}^\infty \tag{13}$$

Here also we observe that the integral diverges. This time the trouble comes from the region around $\Omega \approx \infty$. *Hence, it is clear that a fourier spectrum of a function should have value zero around* $\Omega = 0$ *as well as* $\Omega \approx \infty$. Can we allow following $|\hat{\psi}(\Omega)|^2$?



$$\int_0^\infty \left|\hat{\psi}(\alpha)\right|^2 \frac{d\alpha}{\alpha} = \int_{\Omega_1}^{\Omega_2} 1.\frac{d\alpha}{\alpha} = \ln\frac{\Omega_2}{\Omega_1} \tag{14}$$

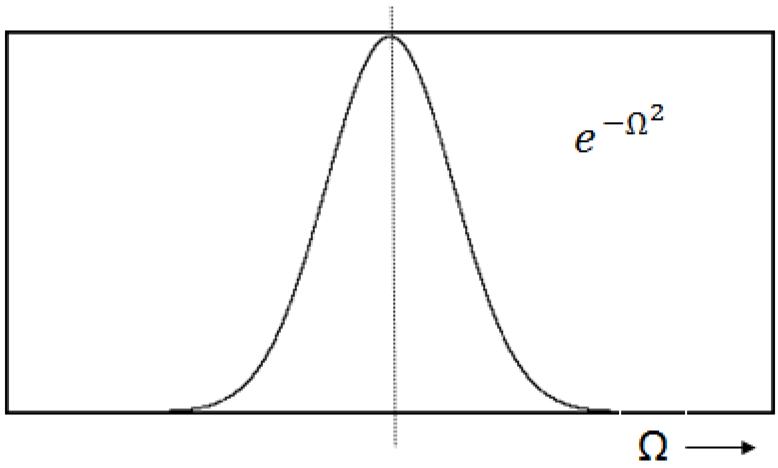
This is finite and therefore acceptable. A bandpass function is what we can accept for a function to become an admissible wavelet.

4 Analysis of gaussian function

An important property of gaussian function is that fourier transform of gaussian function is gaussian function in frequency domain.
Lets us consider the function.

$$\frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} \tag{15}$$

Fourier transform of gaussian function will be of form $e^{-\Omega^2}$ which is also gaussian in nature. The question which arises is whether gaussian function is an admissible wavelet?



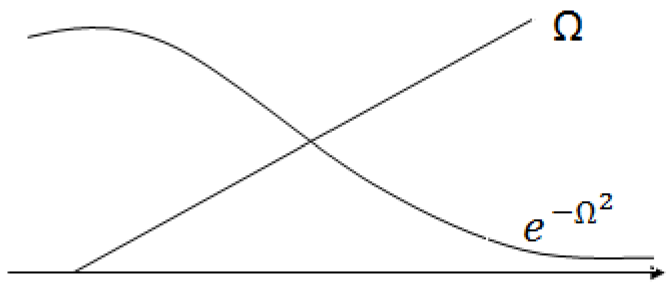
As we can see from the graph, at zero frequency the spectrum does not vanishes therefore even for a small area near zero frequency let say,

$$\int_0^1 |e^{-\Omega^2}|^2 \frac{d\Omega}{\Omega} \tag{16}$$

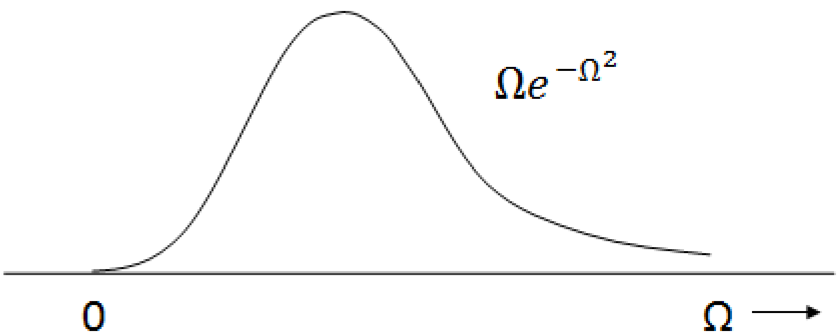
This integral diverges and hence gaussian is not admissible.

Consider a derivative of gaussian function,

$$\frac{d}{d\Omega}(e^{-\Omega^2}) = -2\Omega e^{-\Omega^2} \approx \Omega e^{-\Omega^2} \quad (For\ consideration) \tag{17}$$



Their product will look like



This is an admissible wavelet as admissibility integral converges.

$$\int_0^\infty \left|\Omega e^{-\Omega^2}\right|^2 \frac{d\Omega}{\Omega} \Rightarrow \int_0^\infty \Omega^2 e^{-2\Omega^2} \frac{d\Omega}{\Omega}$$

$$\Rightarrow \int_0^{\infty} \Omega e^{-2\Omega^2} d\Omega$$

Putting $\Omega^2 = \lambda$, therefore $d\lambda = 2\Omega d\Omega$, hence

$$\int_0^{\infty} e^{-2\lambda} \frac{1}{2} d\lambda \tag{18}$$

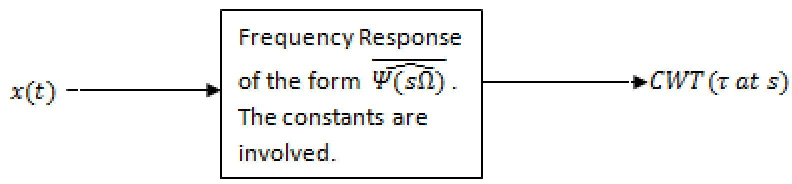
We observe that this integral converges. Hence, derivative of gaussian function is an admissible function. But this function no longer remain optimal in the sense of time frequency.
Now, $\frac{d}{d\Omega}$ is equivalent to multiplication by t in time domain. The inverse Fourier transform of $\Omega e^{-\Omega^2}$ has the same form (Try to show this as an exercise). Try doing the same for the second derivative of gaussian function which is known as Mexican Hat function.

5 Discretization of scale parameter

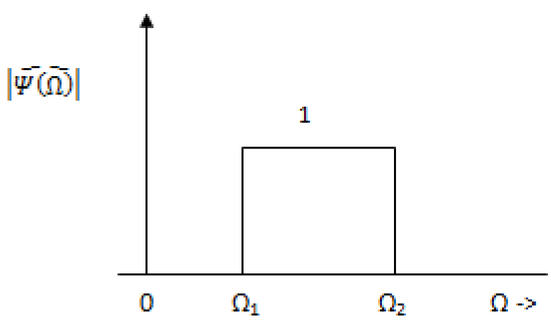
Admissibility is adequate when we talk about reconstructing from CWT. But it is difficult thing, to do numerically, to construct two dimensional continuous parameter τ and s with one dimensional function $x(t)$. Hence we discretize the scale.

5.1 Condition of scale discretization

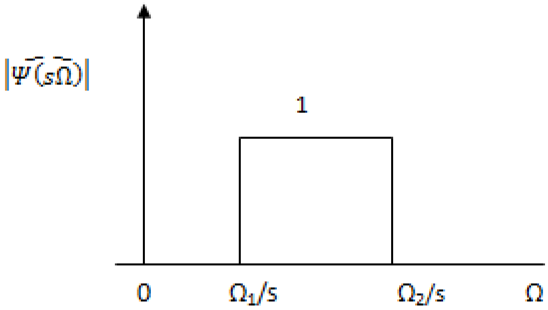
When we build CWT,



Ideal Band Pass function has a nature as shown below.



Therefore the magnitude of $\hat{\psi}(s\Omega)$ will go from $\frac{\Omega_1}{s}$ to $\frac{\Omega_2}{s}$.



We know that $\Omega_1 > 0$ and $\Omega_2 > 0$, therefore $\frac{\Omega_1}{s}$ and $\frac{\Omega_2}{s}$ will also be greater than zero. With change of 's', the band of bandpass filter will move along the positive part of the spectrum. Therefore, the natural condition to discretize the scale parameter is to ensure that we are covering the whole spectrum. When we scale by the factor of 's' we are also scaling the center frequency and the band. So there is logarithmic change. The natural kind of discretization to consider for scale parameter is logarithmic discretization. In general we allow,

$$s = a_0^k \quad for \quad k \in \mathbb{Z}, \; a_0 > 1 \tag{19}$$

We will see the discretization of scale in more detail in next lecture.