WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 22: Reconstruction and Admissibility

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1 Introduction

We have devised Short Time Fourier Transform (STFT) and Continuous Wavelet Transform(CWT) to analyze a signal simultaneously in the time and the frequency domain. STFT is taking Fourier transform of the signal multiplied with a tempered weightage function (the window function) shifted in time and modulated in frequency. CWT is taking dot product of of the signal with normalized wavelet. In this lecture, we will study reconstruction methods of the signal from its STFT and CWT.

2 Reconstruction from STFT

STFT of a signal $x(t)$ with a window function $v(t)$ at time τ and frequency Ω is defined by

$$STFT(x,v)(\tau,\Omega) = \int_{-\infty}^{\infty} x(t)\overline{v(t-\tau)e^{j\Omega t}}dt \tag{1}$$

But this is equivalent to a dot product of the signal $x(t)$ with the translated and modulated window. In other words, this operation is to find out the component of the signal along translated and modulated window function $v(t)$. Here, we try to reconstruct the original signal $x(t)$ by summing the components of STFT along the unit vector in its direction.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} STFT(x,v)(\tau,\Omega)v(t-\tau)e^{j\Omega t}d\tau d\Omega \tag{2}$$

Expanding

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(y)\overline{v(y-\tau)e^{j\Omega y}}dy\; v(t-\tau)e^{j\Omega t}\; d\tau d\Omega \tag{3}$$

Now consider the integral on Ω first

$$\int_{-\infty}^{\infty} e^{-j\Omega y}e^{j\Omega t}d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi e^{j\Omega(t-y)}\; d\Omega \tag{4}$$

which is inverse Fourier transform of a function which is constant ($= 2\pi$) for all Ω

$$= 2\pi\delta(t-y) \tag{5}$$

$\delta(\cdot)$ being continuous impulse function. So the original triple integral turns into

$$2\pi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(y)\overline{v(y-\tau)}\; v(t-\tau)\delta(t-y)\; d\tau dy \tag{6}$$

which gives (using the property of delta function $\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$)

$$= 2\pi\int_{-\infty}^{\infty} x(t)\overline{v(t-\tau)}v(t-\tau)d\tau \tag{7}$$

$$= 2\pi x(t) \int_{-\infty}^{\infty} |v(t-\tau)|^2 d\tau \tag{8}$$

$$= 2\pi\, x(t) \int_{-\infty}^{\infty} |v(z)|^2 dz \tag{9}$$

$$= 2\pi\, x(t)\, ||v||^2 = constant * x(t) \tag{10}$$

where $||v||^2 =< v, v >= L_2$ norm of window function $v(t)$.
In this way, $x(t)$ can be reconstructed from its STFT. But STFT or CWT are continuous transforms which are performed over all values of time and frequency. This means one chooses among a continuum of time and frequency centers and calculates the integral for each such function which is highly impractical. As computations are done in digital computers, we require discrete nature of above transforms. *So, in turn it is needed that τ and ω be discretized.*

3 Time Frequency Tiling: Comparison of STFT and CWT

Now let us compare properties of STFT and CWT from the perspective of time-frequency plane. The STFT moves a tile of constant shape in the time-frequency plane. The minimum possible area of this STFT tile is governed by the time-bandwidth product which is limited by a bound of 0.25 as per the Uncertainty Principle.

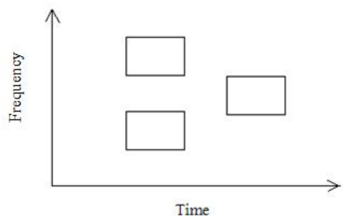

Figure 1: STFT Tiling of Time Frequency Plane

In CWT, tiles have variable shapes but constant area which is also governed by time-bandwidth product. The parameters defining the tiling are time centre τ and scaling parameter s_0. When we increase s_0 we are effectively expanding in time or compressing in frequency and when we decrease s_0 we are effectively compressing in time or expanding in frequency. Movement along the frequency happens because the centre frequency is non zero. The shape of tile remains rectangular but it keeps changing continuously with change in scaling parameter s_0. This is shown in the figure 1.

We can clearly see that τ - time center determines the location of tile on time frequency plane in a direct manner but this is not true for scaling parameter s_0. So we can not reconstruct similar to what we did for STFT, but after allowing a scale factor (which depnds on s_0) reconstruction is achievable as presented in next section.

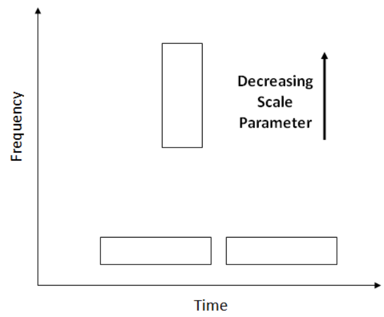

Figure 2: CWT Tiling of Time Frequency Plane

4 Reconstruction from CWT in Frequncy domain approach

Reconstruction of $x(t)$ from CWT$(x, \psi)(\tau, s_0)$ can be written as

$$\int_{s_0=0}^{\infty} \int_{-\infty}^{\infty} CWT(x,\psi)(\tau, s_0)\, \frac{1}{\sqrt{s_0}}\, \psi(\frac{t-\tau}{s_0})\, f(s_0)\, d\tau ds_0 \tag{11}$$

where $f(s_0)$ is a weight factor which is dependent only on s_0 as discussed earlier. In Lecture-21, we used Parsevel's Theorem to arrive at the following expression of CWT:

$$CWT(x,\psi)(\tau, s_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{s_0}\, \hat{X}(\Omega)\overline{\hat{\psi}(s_0\Omega)}\, e^{j\Omega\tau}\, d\Omega \tag{12}$$

where $\hat{X}(\Omega)$ is Fourier Transform of $x(t)$ and $\hat{\psi}(\Omega)$ is Fourier Transform of $\psi(t)$. Substituting this in the reconstruction formulae we get a triple integral,

$$\frac{1}{2\pi} \int_{s_0=0}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{s_0}\, \hat{X}(\Omega)\overline{\hat{\psi}(s_0\Omega)} e^{j\Omega\tau} \frac{1}{\sqrt{s_0}} \psi\left(\frac{t-\tau}{s_0}\right) f(s_0) d\Omega d\tau ds_0 \tag{13}$$

Solving the integral in τ first,

$$I = \int_{-\infty}^{\infty} \psi\left(\frac{t-\tau}{s_0}\right)\, e^{j\Omega\tau}\, d\tau \tag{14}$$

substituting

$$\frac{t-\tau}{s_0} = \lambda$$

$$d\tau = -s_0\, d\lambda$$

$$I = \int_{-\infty}^{\infty} \psi(\lambda)\, e^{j\Omega(t-s_0\lambda)}\, s_0\, d\lambda \tag{15}$$

$$I = e^{j\Omega t} s_0 \int_{-\infty}^{\infty} \psi(\lambda)\, e^{j\Omega(-s_0\lambda)}\, d\lambda \tag{16}$$

which on observation (that integral is a Fourier integral) gives:

$$I = e^{j\Omega t} s_0 \hat{\psi}(s_0\Omega) \tag{17}$$

Substituting this back in the triple integral in equation (13) we get,

$$\frac{1}{2\pi}\int_{s_0=0}^{\infty}\int_{-\infty}^{\infty}\hat{X}(\Omega)\overline{\hat{\psi}(s_0\Omega)}\ f(s_0)\,e^{j\Omega t}\ s_0\hat{\psi}(s_0\Omega)d\Omega\,ds_0 \tag{18}$$

Now solving the integral in s_0,

$$I_1 = \int_{s_0=0}^{\infty}\overline{\hat{\psi}(s_0\Omega)}\,f(s_0)\,s_0\,\hat{\psi}(s_0\Omega)\,ds_0 \tag{19}$$

$$= \int_{s_0=0}^{\infty}|\hat{\psi}(s_0\Omega)|^2\,f(s_0)\,s_0\,ds_0 \tag{20}$$

If we can make this integral I_1 independent of Ω then we will be done with the reconstruction since rest of the term is Inverse Fourier Transform giving $x(t)$. So, only objective left is to make the integral independent of Ω and that is where freedom to choose $f(s_0)$ comes in handy. If we could make

$$s_0\,f(s_0)\,ds_0 = \frac{ds_0}{s_0} \quad or \quad f(s_0) = \frac{1}{s_0^2}$$

we get a valid weight function which is non negative for all values of s_0. Let $s_1 = \Omega s_0$ which implies

$$\frac{ds_1}{s_1} = \frac{ds_0}{s_0} \qquad (\Omega \neq 0)$$

When Ω is positive, limits of the integral I_1 are 0 to ∞ and when Ω is negative, limits of the integral I_1 are 0 to -∞. In both cases, integral is independent of Ω and hence can be treated as a constant further. Remaining integral becomes:

$$constant \times \int_{-\infty}^{\infty}\frac{1}{2\pi}\,\hat{X}(\Omega)e^{j\Omega t}\,d\Omega \tag{21}$$

where the constant is

$$\int_0^{\infty}|\hat{\psi}(s_1)|^2\,\frac{ds_1}{s_1} \qquad \forall\Omega > 0$$

Or

$$\int_0^{-\infty}|\hat{\psi}(s_1)|^2\frac{ds_1}{s_1} \qquad \forall\Omega < 0$$

These two integrals must be finite for perfect reconstruction. This condition is known as 'admissibility condition' of a wavelet function which will be discussed in the next lecture.

5 Reconstruction from CWT in Time domain approach

Reconstruction of $x(t)$ from CWT$(x,\psi)(\tau,s_0)$ can be written as

$$\int_{s_0=0}^{\infty}\int_{-\infty}^{\infty}CWT(x,\psi)(\tau,s_0)\,\frac{1}{\sqrt{s_0}}\,\psi(\frac{t-\tau}{s_0})\,f(s_0)\,d\tau ds_0 \tag{22}$$

where $f(s_0)$ is a weight factor which is dependent only on s_0 as discussed earlier.

$$\int_{s_0=0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x(\lambda)\,\frac{1}{\sqrt{s_0}}\,\overline{\psi(\frac{t-\lambda}{s_0})}\,\frac{1}{\sqrt{s_0}}\,\psi(\frac{t-\tau}{s_0})\,f(s_0)\,d\lambda d\tau ds_0 \tag{23}$$

calculating the integral with respect to the variable τ

$$\int_{-\infty}^{\infty} \overline{\psi(\frac{t-\lambda}{s_0})}\, \psi(\frac{t-\tau}{s_0})\, d\tau \tag{24}$$

Let $\frac{\lambda-\tau}{s_0}$=α then $d\tau$=-$s_0 d\alpha$ Now the integral on τ becomes,

$$\int_{-\infty}^{\infty} s_0\overline{\psi(\alpha)}\, \psi(\frac{t-\lambda}{s_0}+\alpha)\, d\alpha \tag{25}$$

This integral is the auto-correlation function of ψ. Writing the auto-correlation function

$$R_{\psi\ \psi}(\frac{t-\lambda}{s_0}) = \int_{-\infty}^{\infty} \overline{\psi(\alpha)}\, \psi(\frac{t-\lambda}{s_0}+\alpha)\, d\alpha \tag{26}$$

Computing the inverse of the auto-correlation function in the frequency domain

$$R_{\psi\ \psi}(\frac{t-\lambda}{s_0}) = \int_{-\infty}^{\infty} |\hat{\psi}(\Omega)|^2\, e^{j(\frac{t-\lambda}{s_0})\Omega}\, d\Omega \tag{27}$$

Substituting the above equation in the equation 25 ,in the integral α

$$\int_{-\infty}^{\infty} s_0\, |\hat{\psi}(\Omega)|^2\, e^{j(\frac{t-\lambda}{s_0})\Omega}\, d\Omega \tag{28}$$

Let $\frac{\Omega}{s_0} = \beta$ then $d\Omega = s_0 d\beta$, The equation becomes,

$$\int_{-\infty}^{\infty} s_0^2\, |\hat{\psi}(s_0\beta)|^2\, e^{j(t-\lambda)\beta}\, d\beta \tag{29}$$

Substituting this in our original integral,

$$\int_{s_0=0}^{\infty} \int_{\lambda=-\infty}^{\infty} x(\lambda)\, \frac{1}{s_0} \int_{\beta=-\infty}^{\infty} s_0^2\, |\hat{\psi}(s_0\beta)|^2\, e^{j(t-\lambda)\beta}\, d\beta\, f(s_0)\, d\lambda ds_0 \tag{30}$$

Now solving the integral in s_0,

$$I_1 = \int_{s_0=0}^{\infty} |\psi(s_0\beta)|^2\, f(s_0)\, s_0\, ds_0 \tag{31}$$

If we can make this integral I_1 independent of β then we will be done with the reconstruction. So, only objective left is to make the integral independent of β and that is where freedom to choose $f(s_0)$ comes in handy.
If we could make

$$s_0\, f(s_0)\, ds_0 = \frac{ds_0}{s_0} \quad or \quad f(s_0) = \frac{1}{s_0^2}$$

we get a valid weight function which is non negative for all values of s_0. Let $s_1 = \beta s_0$ which implies

$$\frac{ds_1}{s_1} = \frac{ds_0}{s_0} \qquad (\beta \neq 0)$$

When β is positive, limits of the integral I_1 are 0 to ∞ and when β is negative, limits of the integral I_1 are 0 to -∞. In both cases, integral is independent of β and hence can be treated as a constant C_ψ further. Remaining integral becomes:

$$= \int_{\lambda=-\infty}^{\infty} x(\lambda) \int_{\beta=-\infty}^{\infty} C_\psi e^{j(t-\beta)} \, d\beta d\lambda \tag{32}$$

$$= \int_{\lambda=-\infty}^{\infty} 2\pi C_\psi x(\lambda) \, \delta(t-\lambda) \, d\lambda \tag{33}$$

$$= 2\pi C_\psi x(t) \tag{34}$$

where the constant C_ψ is

$$\int_0^{\infty} |\hat{\psi}(s_1)|^2 \, \frac{ds_1}{s_1} \qquad \forall \beta > 0$$

Or

$$\int_0^{-\infty} |\hat{\psi}(s_1)|^2 \frac{ds_1}{s_1} \qquad \forall \beta < 0$$

These two integrals must be finite for perfect reconstruction. This condition is known as 'admissibility condition' of a wavelet function which will be discussed in the next lecture.