WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 21: STFT and Wavelet Transform

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1 Short Time Fourier Transform

The short time Fourier transform, we begin with choosing an appropriate window function: $v(t)$. What we desire from the window function is finite time variance and finite frequency variance. This is required to make the time-frequency(t-f) tile of finite area in the t-f plane as will be seen later.
For finite time variance we know that

$$tv(t) \in L_2(\mathbb{R})$$

given $v(t) \in L_2(\mathbb{R})$.
Also for finite frequency variance

$$\frac{dv(t)}{dt} \in L_2(\mathbb{R})$$

given $v(t) \in L_2(\mathbb{R})$.
The simplest window that can be chosen is rectangular window. Such a window immediately disqualifies as it does not have finite frequency variance. Examples of window function that have finite variance in time and frequency:

- Triangular Window: $v(t) = 1 - |t|$
- Gaussian Window: $v(t) = e^{\frac{-t^2}{2}}$
- Raised Cosine Window: $v(t) = 1 + cos(t)$

The idea in the short time Fourier transform is to create a continuum of dot products of such a window function modulated by a sinusoid with the input function. Let,

$$x(t) \in L_2(\mathbb{R}) \tag{1}$$

Then the STFT (short time Fourier transform) is given as

$$STFT(X, V)(\tau_0, \Omega_0) = \text{ dot product of x(t) with } v(t - \tau_0)e^{j\Omega t} \tag{2}$$

The arguments in the first bracket are the secondary arguments namely X, V and the second bracket holds the primary arguments. Writing the STFT in equation format:

$$STFT(X, V)(\tau_0, \Omega_0) = \int_{-\infty}^{\infty} x(t)\overline{v(t - \tau_0)e^{j\Omega_0 t}}\, dt \tag{3}$$

The bar represents complex conjugation.
Hence the STFT extracts a piece of the function and takes its Fourier transform. Invoking the Parseval's theorem, the above expression has an equivalent in the frequency domain *i.e.*, the product of the Fourier transforms of $x(t)$ and $v(t - \tau_0)e^{j\Omega t}$ respectively, integrated from $-\infty$

to $+\infty$.
The Fourier transform of $v(t-\tau_0)e^{j\Omega_0 t}$:

$$\int_{-\infty}^{\infty} v(t-\tau_0)e^{j\Omega_0 t}e^{-j\Omega t}\,dt \tag{4}$$

Let t-$\tau_0 = k$

$$= \int_{-\infty}^{\infty} v(k)e^{j(\Omega_0-\Omega)(k+\tau_0)}\,dk \tag{5}$$

$$= e^{j(\Omega_0-\Omega)\tau_0}V(\Omega-\Omega_0) \tag{6}$$

$V(\Omega)$ denotes the Fourier transform of the function $v(t)$, putting in the Parseval's theorem expression

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)\overline{V(\Omega-\Omega_0)e^{j(\Omega_0-\Omega)\tau_0}}\,d\Omega \tag{7}$$

$$\frac{e^{-j\Omega_0\tau_0}}{2\pi}\int_{-\infty}^{\infty} X(\Omega)\overline{V(\Omega-\Omega_0)e^{j\Omega\tau_0}}\,d\Omega \tag{8}$$

the above expression looks like the inverse Fourier transform evaluated at τ_0.

Duality Interpretation of Fourier transform

The frequency interpretation term:

$$\frac{e^{-j\Omega_0\tau_0}}{2\pi}\int_{-\infty}^{\infty} X(\Omega)\overline{V(\Omega-\Omega_0)e^{j\Omega\tau_0}}\,d\Omega \tag{9}$$

The time interpretation term:

$$\int_{-\infty}^{\infty} x(t)\overline{v(t-\tau_0)e^{j\Omega_0 t}}\,dt \tag{10}$$

The STFT creates fixed shape tiles, where τ_0 is the movement along time and Ω_0 is the movement along frequency

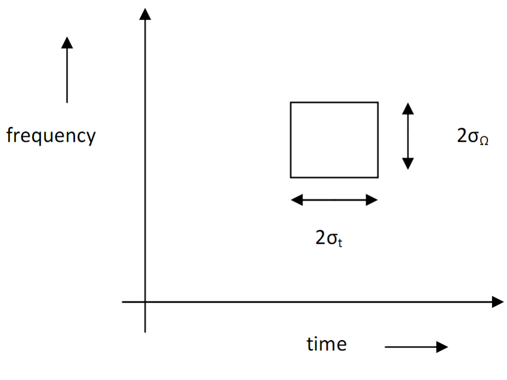


Figure 1: Tiling of time-frequency plane

So far we have seen only dyadic DWT and translations happen in unit steps in V_0. The continuous version of the wavelet transform is essentially an inner product of $x(t)$ with translates

and dilates of the wavelet $\psi(\frac{t-\tau_0}{s_0})$
Continuous wavelet transform(CWT)

$$\left\langle x(t), \frac{1}{\sqrt{s_0}}\psi(\frac{t-\tau_0}{s_0})\right\rangle = \frac{1}{\sqrt{s_0}}\int_{-\infty}^{\infty} x(t)\overline{\psi(\frac{t-\tau_0}{s_0})}dt$$

Problem of normalization caused due to change in the norm of the wavelet function upon dilation, hence the factor of $\frac{1}{\sqrt{s_0}}$ is required to normalize it. Interpreting in the frequency domain using the Parseval's relationship implies that the above expression is equivalent to inner product of $X(\Omega)$ with Fourier transform of $\psi(\frac{t-\tau_0}{s_0})$. Fourier transform of $\psi(\frac{t-\tau_0}{s_0})$, first taking care of the dilation, since

$$\frac{1}{\sqrt{s_0}}\psi(\frac{t}{s_0}) \rightarrow \text{ Fourier Transform } \rightarrow \sqrt{s_0}\Psi(s_0\Omega) \tag{11}$$

Hence the Fourier transform of

$$\frac{1}{\sqrt{s_0}}\psi(\frac{t-\tau_0}{s_0}) \rightarrow \text{Fourier Transform} \rightarrow \sqrt{s_0}\Psi(s_0\Omega)e^{-j\Omega\tau_0}$$

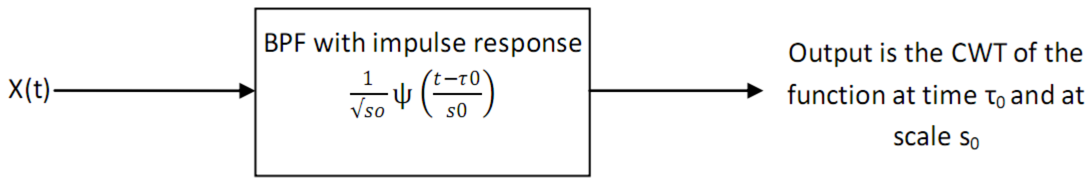
The continuous wavelet transform (CWT) is both a function of τ_0 and s_0 with $\tau_0 \in \mathbb{R}$ and $s_o \in \mathbb{R}^+$

$$CWT(x,\psi)(\tau_0, s_0) = \int_{-\infty}^{\infty} \frac{x(t)}{\sqrt{s_0}}\overline{\psi(\frac{t-\tau_0}{s_0})}\, dt \tag{12}$$

or using the Parseval's relation

$$CWT(x,\psi)(\tau_0, s_0) = \frac{\sqrt{s_0}}{2\pi}\int_{-\infty}^{\infty} X(\Omega)\overline{\Psi(s_0\Omega)}e^{j\Omega\tau_0}\, d\Omega$$

- Provided we recall the nature of the Fourier transform for Ψ, considering the Haar case the magnitude pattern
- We are multiplying X by a bandpass function and taking the inverse Fourier transform, hence we are extracting the frequency component of CWT which corresponds to the dilates of Ψ.
- If we accept that Ψ is a bandpass function then



- There is a continuum of such filters.