

1 Introduction

In the previous lecture we have seen that the Time bandwidth product is the product of time variance(σ_t^2) and frequency variance(σ_Ω^2) of the given function. Time bandwidth product $\sigma_t^2 \sigma_\Omega^2$ for any function $x(t) \in L_2(\mathbb{R}) \geq 0.25$. Gaussian function $x(t) = e^{-t^2/2}$ is an example of optimal function in sense of time bandwidth product. A more general optimal function is of the form $e^{\gamma_0 t^2/2}$, where $Re(\gamma_0)$ is negative and γ_0 can be complex. Gaussian function is optimal but it is unrealizable in practice.

Why do we say that the Gaussian is physically unrealizable? Take for example the exponential time waveform or the exponential time waveform modulated by a sinusoid. These are easily realizable. Circuits which comprise of resistors, inductors, capacitors when excited say with a step or a sinusoid give us either exponentially decaying sinusoids or exponentially decaying transients and therefore those are easy to generate with physical system. So it is difficult to realize Gaussian waveform in physical system and it can only be approximated.

However, we will see that a cascade of two simple systems realizes a function which is close to an optimal.

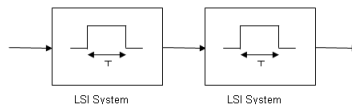


Figure 1: Cascade of two LSI system

In case of Haar scaling function time bandwidth product is infinite. Suppose we took a cascade of two systems each of whose impulse response is essentially a pulse of width T , *i.e.*, instead of taking one pulse, take a cascade of them as shown in Figure 1. This together forms a composite LSI system. The impulse response of this composite LSI system is the convolution of two pulses which will result in a triangular pulse as shown in Figure 2. Now we will calculate the time

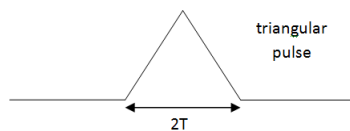


Figure 2: Triangular pulse obtained by convolution of two pulses

bandwidth product of this triangular pulse and compare it with time bandwidth product of Gaussian. Triangular pulse is given by

$$x(t) = 1 - |t|, \quad \text{where} \quad |t| \leq 1$$

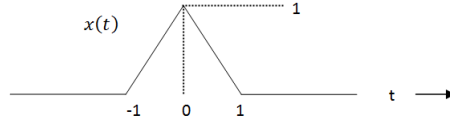


Figure 3: Triangular pulse centered at 0

Recalling that time bandwidth product is invariant to scaling of the dependent variable, independent variable and translation in time and frequency domain, consider a pulse to be centered at origin. Time variance is given by

$$\frac{\|tx(t)\|_2^2}{\|x(t)\|_2^2}$$

As $x(t)$ is symmetric around $t = 0$, squared norm $\|x(t)\|_2^2$ is given by

$$\|x(t)\|_2^2 = 2 \int_0^1 (1-t)^2 dt$$

Let $\lambda = 1 - t$, hence we get

$$\begin{aligned} \|x(t)\|_2^2 &= (2)(-) \int_1^0 (\lambda)^2 d\lambda \\ &= 2 \int_0^1 \lambda^2 d\lambda \\ &= \frac{2}{3} \end{aligned}$$

Now, by symmetry again we have

$$\begin{aligned} \|tx(t)\|_2^2 &= 2 \int_0^1 t^2(1-t)^2 dt \\ &= 2 \int_0^1 t^2(1-2t+t^2) dt \\ &= 2 \int_0^1 (t^2-2t^3+t^4) dt \\ &= 2 \left\{ \frac{t^3}{3} - \frac{2t^4}{4} + \frac{t^5}{5} \right\}_0^1 \\ &= 2 \left\{ \frac{1}{3} - \frac{2}{4} + \frac{1}{5} \right\} \\ &= 2 \frac{10-15+6}{2 \times 15} \\ &= \frac{1}{15} \end{aligned}$$

$$\text{Time variance} = \frac{\|tx(t)\|_2^2}{\|x(t)\|_2^2} = \frac{1/15}{2/3} = \frac{1}{15} \times \frac{3}{2} = 0.1$$

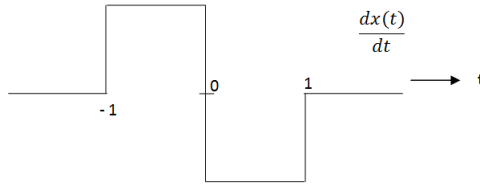


Figure 4: $\frac{dx(t)}{dt}$

Frequency variance is given by

$$\frac{\left\| \frac{dx(t)}{dt} \right\|_2^2}{\|x(t)\|_2^2}$$

and $\frac{dx(t)}{dt}$ is as shown in Figure 4.

From Figure 4, it can be seen that $\frac{dx(t)}{dt}$ has the appearance of a Haar wavelet. So we have

$$\left\| \frac{dx(t)}{dt} \right\|_2^2 = 1^2 \times 1 + 1^2 \times 1 = 2$$

Therefore the frequency variance is

$$\frac{2}{2/3} = 3$$

Now time bandwidth product = time variance \times frequency variance = $0.1 \times 3 = 0.3$.

We note that by cascading the system with itself time bandwidth product is reduced from infinity (in case of Haar) to 0.3 which is close to optimal value of 0.25. In order to go more close to 0.25, it is required to repeatedly convolve the pulse with itself. So now we take cascade of 3 such LSI systems each of whose impulse response is essentially a pulse as shown in Figure 5.

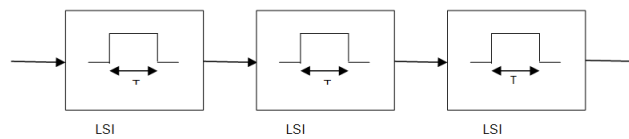


Figure 5: Cascade of 3 LSI system

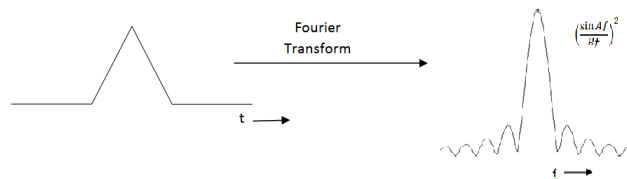


Figure 6: Fourier transform of the triangular pulse

It is important to note that we get a time bandwidth product of 0.3 from not only the compactly

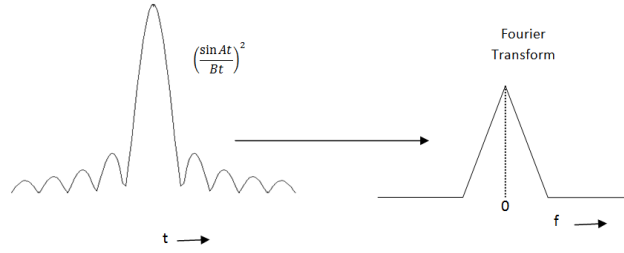


Figure 7: Fourier duality

supported function (used above) but also from non compactly supported function which is evident from the duality principle of Fourier transform. So the time variance for triangular function(in time domain) becomes the frequency variance of the function shown in Figure 7 which is the dual of the triangular function and vice versa.

The time variance of the function $\left(\frac{\sin Af}{Bf}\right)^2 =$ frequency variance of $x(t) = 1 - |t|$ and

frequency variance of $\left(\frac{\sin Af}{Bf}\right)^2 =$ time variance of $x(t) = 1 - |t|$. Therefore time bandwidth product is 0.3.

Hence, it is possible to have the same time bandwidth product for two functions with different shapes. Therefore, two important conclusions can be drawn

- The time bandwidth product is invariant to Fourier transformation.
- We can have two functions one compactly supported and another NOT compactly supported to have the same time bandwidth product $\sigma_t^2 \sigma_\Omega^2$.

2 Time-frequency plane

Putting time and frequency domains together bring out a new idea which is a two variable domain also called a 'Time Frequency Plane'. 'Time-frequency plane' is shown in Figure 8.

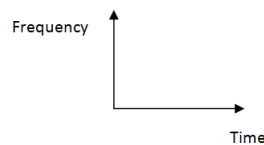


Figure 8: Time frequency plane

It is a plane in which one axis, say horizontal axis represents time and other axis say vertical represents frequency. Occupancy of $x(t) \in L_2(\mathbb{R})$ in time-frequency plane can be thought as being around t_0 , the center in time, from $t_0 + \sigma_t$ to $t_0 - \sigma_t$ on the horizontal axis. On the vertical axis we would like to center it at Ω_0 , the frequency center, and we would spread it from $\Omega_0 - \sigma_\Omega$ to $\Omega_0 + \sigma_\Omega$ as shown in Figure 9. So we could think of the function $x(t)$ as being located in a rectangle which is centered at (t_0, Ω_0) which has a horizontal width of $2\sigma_t$ and vertical spread of $2\sigma_\Omega$ as shown in Figure 9.

A function in $L_2(\mathbb{R})$ occupies a certain area in the time-frequency plane and according to the uncertainty principle this area cannot be smaller than 0.25 units.

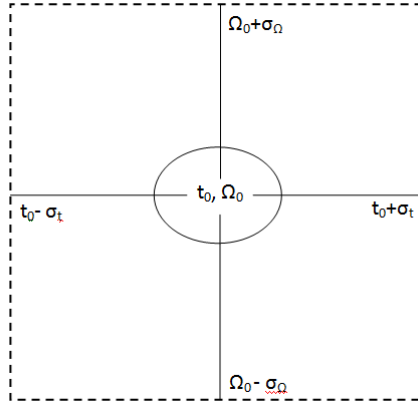


Figure 9: Time frequency plane centered at t_0 and Ω_0

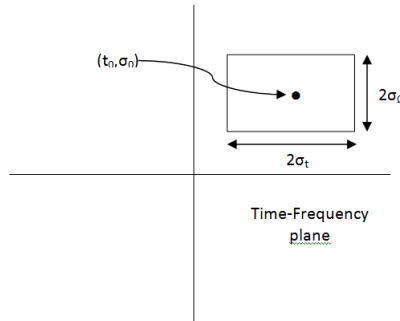


Figure 10: Tiling the time-frequency plane

The area of rectangle area is

$$2\sigma_t \times 2\sigma_\Omega = 4\sigma_t\sigma_\Omega \geq 4\sqrt{0.25} \geq 2 \text{ units}$$

So, the area of the rectangle cannot be smaller than 2 units. Within limitations we can change the width and height of rectangular tile.

3 Tiling the time-frequency plane

Consider a function $y(t)$ to be analyzed and $x(t)$ as a 'tool function'. From Parseval's theorem,

$$\int_{-\infty}^{+\infty} y(t)\overline{x(t)}dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Y(\Omega)\overline{X(\Omega)}d\Omega$$

Physical interpretation:

If we take the projection of the function $y(t)$ on such 'tool function' $x(t)$ in time, we are essentially extracting information about $y(t)$ in the time region between $t_0 + \sigma_t$ and $t_0 - \sigma_t$. Parseval's theorem says that simultaneously we are also extracting information of the Fourier transform of $y(t)$ in a region captured between $\Omega_0 - \sigma_\Omega$ and $\Omega_0 + \sigma_\Omega$. So this is the minimum rectangular area over which we can view $y(t)$. There is a minimum resolution and we cannot achieve finer resolution than this when we look at the two domains together. There are many

different ways in which we can look at the small domain when we are within that uncertainty limit. But tiling has a different interpretation. If we wish to analyze a function then we should do it in time-frequency domain together.

Consider an example of ‘chirp function’ (named after the sound of the birds). When birds chirp crudely the chirp waveform has a pattern which is continuously changing instantaneous frequency in time. It is of the form $\sin(\Omega(t).t)$ where Ω is an instantaneous frequency which is a function of time.

In the time frequency plane $\Omega t = a$ (constant function of time) can be graphically represented as shown in Figure 11. Suppose Ωt is a linear function of time, then graphically we try to trace

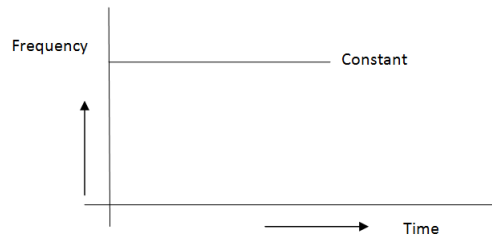


Figure 11: Chirp function

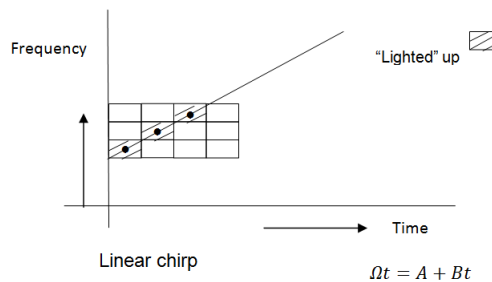


Figure 12: $\Omega t = A + B t$

this pattern using tool function as shown in Figure 12. We only put rectangles which look like as they are shown in Figure 12 and we can never really trace what is happening within the rectangle.

If we think of putting many rectangles in this time frequency plane, then these shaded rectangles are “lighted” up *i.e.* magnitude of dot product of function $y(t)$ (which has the linear chirp nature) with this set of tiles (in which the function essentially is prominent) will be significant. Example: If we look into the time frequency plane each of these rectangles would corresponds to a single point. So it would show points that lie on the line as lighted up. But we cannot go closer than that. We can never know what has happened between these points. Thus, uncertainty principle says that we cannot get instantaneous frequency as a function of time exactly. But we could do as closely as we desire by taking smaller rectangles.