WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture19: Evaluating and bounding time-bandwidth product

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1 Introduction

Since the last few lectures we have been working on to evaluate a measure for the joint resolution of time and frequency. We have so far sorted out that given a wavelet function, the product of the time and frequency variances could give us a precise idea as to how well we can focus in both the domains simultaneously. This time bandwidth product is invariant to translation and modulation in time domain. Also the time bandwidth product is invariant to scaling of the dependent and the independent variable and is a direct function of the shape. We have derived the necessary expressions for the time and frequency variances and will try to evaluate it further and find out the constraint induced by nature on this time bandwidth product. We will essentially try to find the lower bound on the time bandwidth product. This will also give us a mathematical proof of the uncertainty that exists in nature when we try to focus in both the domains simultaneously.

2 Evaluation of time-bandwidth product $\sigma_t^2\sigma_\Omega^2$

Let us recall the expressions for the time and frequency variances which are given as:

$$\begin{aligned}\sigma_t^2 &= \frac{\int_{-\infty}^{+\infty} t^2|x(t)|^2dt}{\int_{-\infty}^{+\infty}|x(t)|^2dt}\\ &= \frac{\|tx(t)\|_2^2}{\|x(t)\|_2^2}\end{aligned} \tag{1}$$

$$\begin{aligned}\sigma_\Omega^2 &= \frac{\int_{-\infty}^{+\infty} |\frac{d}{dt}x(t)|^2dt}{\int_{-\infty}^{+\infty}|x(t)|^2dt}\\ &= \frac{\|\frac{d}{dt}x(t)\|_2^2}{\|x(t)\|_2^2}\end{aligned} \tag{2}$$

The time bandwidth product can therefore be obtained from equations (1) and (2) as:

$$Time\ bandwidth\ product = \frac{\|tx(t)\|_2^2}{\|x(t)\|_2^2}\frac{\|\frac{d}{dt}x(t)\|_2^2}{\|x(t)\|_2^2} \tag{3}$$

NOTE:
While stating the above equations we have put $t_0 = 0$ and $\sigma_0 = 0$. We could do this because the time bandwidth product is invariant to time and frequency domain shifts. Thus, without the loss of generality we can always obtain a function centered at origin in both the domains either by shifting or modulation.

Let us now evaluate the numerator first:

$$numerator \quad = \quad \|tx(t)\|_2^2 \; \|\frac{d}{dt}x(t)\|_2^2 \tag{4}$$

To simplify it further we need to interpret the functions $tx(t)$ and $\frac{d}{dt}x(t)$ as vectors $\vec{v_1}$ and $\vec{v_2}$ respectively. Now as per the basic principle of inner product of two vectors:

$$\begin{aligned}
< \vec{v_1}, \vec{v_2} > &= |\vec{v_1}||\vec{v_2}| \cos\theta \qquad (\theta = \; angle\ between\ the\ two\ vectors) \\
| < \vec{v_1}, \vec{v_2} > |^2 &= |\vec{v_1}|^2|\vec{v_2}|^2 \cos^2\theta \\
Therefore, \quad | < \vec{v_1}, \vec{v_2} > |^2 &\leq |\vec{v_1}|^2|\vec{v_2}|^2 \qquad (as \;\; \cos^2\theta \leq 1)
\end{aligned}$$

This principle can be generalized to the functions viewed as vectors and in fact is a very important theorem in functional analysis called **Cauchy Schwarz inequality**. The theorem states that if there are two functions say f_1 and f_2 such that $f_1, f_2 \in L_2(\mathbb{R})$ then,

$$| < f_1, f_2 > |^2 \quad \leq \quad |f_1|^2|f_2|^2 \tag{5}$$

Thus, from equation (5) the numerator in equation (4) can be written as:

$$numerator \geq \left|\left\langle tx(t), \frac{d}{dt}x(t)\right\rangle\right|^2 \tag{6}$$

NOTE:
As per the inequality we are assuming that the functions belong to space $L_2(\mathbb{R})$. Therefore to make this inequality liable in our time bandwidth evaluation it is important to choose $x(t)$ such that $tx(t) \in L_2(\mathbb{R})$ and $\frac{d}{dt}x(t) \in L_2(\mathbb{R})$. If this condition is not satisfied then the integral diverges and there cannot be a lower bound.

To illustrate the above condition let us consider an example as in figure 1:
Consider,

$$x(t) \quad = \quad 1 \qquad\qquad 0 \leq t \leq 1$$

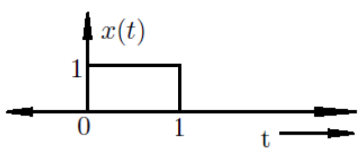

Figure 1: Function belonging to $L_2(\mathbb{R})$

The function $\frac{d}{dt}x(t)$ will look like in figure 2:
Now the function in figure 2 is not square integrable because it contains two impulses which have infinite energy. This can be well understood by zooming the view a bit on the impulse.

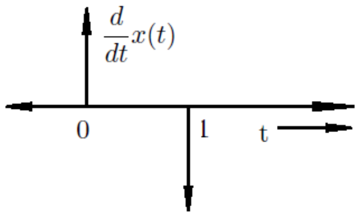

Figure 2: function $\frac{d}{dt}x(t)$

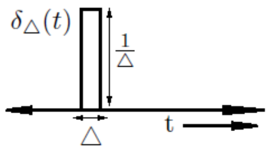

Figure 3: Delta function

The zoomed view of the impulse will be as in figure 3
Now,

$$\lim_{\triangle\to 0}\int_{-\infty}^{+\infty}\delta_\triangle^2(t)dt$$

diverges.
Therefore, for the time bandwidth product to have a lower bound the function and its derivative both should belong to $L_2(\mathbb{R})$.

The RHS of equation (6) can be written as:

$$\left|\left\langle tx(t), \frac{d}{dt}x(t)\right\rangle\right|^2 \;=\; \left|\int_{-\infty}^{+\infty} tx(t)\overline{\frac{d}{dt}x(t)}dt\right|^2 \tag{7}$$

Now for any complex number Z we have the following relation:

$$\mid Z\mid^2 \geq \mid Re\{Z\}\mid^2$$

Therefore from equations (6) and (7)

$$numerator \;\geq\; \left|Re\left\{\int_{-\infty}^{+\infty} tx(t)\overline{\frac{d}{dt}x(t)}dt\right\}\right|^2 \tag{8}$$

REMARK:
t being a real variable $\overline{\frac{d}{dt}x(t)} = \frac{d}{dt}\overline{x(t)}$

Thus,

$$\begin{aligned} numerator &\geq \left|Re\left\{\int_{-\infty}^{+\infty} tx(t)\frac{d}{dt}\overline{x(t)}dt\right\}\right| \\ &\geq \left|\int_{-\infty}^{+\infty} tRe\{x(t)\frac{d}{dt}\overline{x(t)}\}dt\right|^2 \end{aligned} \tag{9}$$

Now,

$$
\begin{aligned}
Re\{x(t)\frac{d}{dt}\overline{x(t)}\} &= \frac{1}{2}\{x(t)\frac{d}{dt}\overline{x(t)} + \overline{x(t)}\frac{d}{dt}x(t)\} \\
&= \frac{1}{2}\frac{d}{dt}\{x(t)\overline{x(t)}\} \\
&= \frac{1}{2}\frac{d}{dt} \mid x(t) \mid^2 \qquad (10)
\end{aligned}
$$

Therefore from equations (9) and (10)

$$
numerator \;\geq \left|\frac{1}{2}\int_{-\infty}^{+\infty} t\frac{d}{dt} \mid x(t) \mid^2 dt\right|^2 \qquad (11)
$$

Solving the integral term by parts we get;

$$
\begin{aligned}
\int_{-\infty}^{+\infty} t\frac{d}{dt} \mid x(t) \mid^2 dt &= [t \mid x(t) \mid^2]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \mid x(t) \mid^2 dt \qquad (12) \\
&= -\int_{-\infty}^{+\infty} \mid x(t) \mid^2 dt
\end{aligned}
$$

NOTE:
We have agreed that $\int_{-\infty}^{+\infty} t^2 \mid x(t) \mid^2 dt$ should be finite for a bound to exist. Therefore for the integral to converge the function $t^2 \mid x(t) \mid^2$ should decay to zero value as $t \to +\infty$ and $t \to -\infty$. $t^2 \mid x(t) \mid^2 \to 0$ grantees $t \mid x(t) \mid^2 \to 0$ as $t \to +\infty$ and $t \to -\infty$. Thus, the term $[t \mid x(t) \mid^2]_{-\infty}^{+\infty}$ in equation (12) becomes zero.
Therefore

$$
\int_{-\infty}^{+\infty} t\frac{d}{dt} \mid x(t) \mid^2 dt \;=\; -\|x\|_2^2 \qquad (13)
$$

Finally form equations (11) and (13) we get;

$$
\begin{aligned}
numerator\ of\ the\ time\ bandwidth\ product &\geq |\frac{1}{2}(-\|x\|_2^2)|^2 \\
&\geq \frac{1}{4}\|x\|_2^2\|x\|_2^2 \qquad (14)
\end{aligned}
$$

Substituting the value of the numerator in equation (3)of the time bandwidth product we get;

$$
\begin{aligned}
time\ bandwidth\ product &\geq \frac{\frac{1}{4}\|x\|_2^4}{\|x\|_2^4} \\
&\geq \frac{1}{4} \qquad (15)
\end{aligned}
$$

CONCLUSION:
The time bandwidth product can never be less than 0.25. The result is fundamental to signal processing and has no relation with the technology and tools available at a particular time. The result tells us that no matter what we do, we can never get a function with finite energy confined beyond a certain range in both time and frequency simultaneously.

Now the optimal function in the sense of time bandwidth product means that the Cauchy Schwarz inequality becomes an equality. Recall that the inequality essentially arises due to the $\cos^2\theta$ term in the vectorial interpretation of the functions. Thus to attain equality $\cos^2\theta = 1$, i.e. we need the vectors $tx(t)$ and $\frac{d}{dt}x(t)$ to be collinear.

NOTE:
Two vectors being collinear means that they should be linearly dependent i.e. one of the two vectors should be a multiple of other.

Thus to get the optimal solution we need to satisfy the given condition:

$$\frac{d}{dt}x(t) = \gamma_0.t.x(t) \qquad (\gamma_0 = \ constant)$$

Therefore solving the above equation;

$$\begin{aligned}
\ln x &= \gamma_0.\frac{t^2}{2} + c_0 && (c_0 = \ constant\ of\ integration) \\
e^{\ln x} &= e^{\gamma_0\frac{t^2}{2}+c_0} \\
x(t) &= e^{c_0}.e^{\gamma_0\frac{t^2}{2}} \\
x(t) &= c.e^{\gamma_0\frac{t^2}{2}} && (c = \ constant) \qquad (16)
\end{aligned}$$

REMARK:
For the function $x(t)$ to be in $L_2(\mathbb{R})$, $|e^{c_0}|^2.|e^{\gamma_0\frac{t^2}{2}}|^2$ should be integrable. This is possible only if γ_0 has a negative real part.

Thus, one optimal function with time bandwidth product 0.25 can be of the form $\mathbf{e^{-\frac{t^2}{2}}}$; the Gaussian. Here $\boldsymbol{\gamma_0 = -1}$ and $\mathbf{c = 1}$ as in equation (16).