WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 18: The Time-Bandwidth Product

Prof.Prof.V.M.Gadre, EE, IIT Bombay

1 Introduction

In this lecture, our aim is to define the time Bandwidth Product, that is the product of time variance σ_t^2 and frequency variance σ_Ω^2 and study its properties.

2 Revision

We will revise some basic formulae we introduced in the last lecture.

2.1 The time center t_0

The time center of any waveform $x(t)$ is given by

$$t_0 = \frac{\int_{-\infty}^{\infty} t|x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt} \tag{1}$$

The quantity $\int_{-\infty}^{\infty} |x(t)|^2 dt$ is often represented as $||x||_2^2$ to indicate that it is the squared $\mathbb{L}_2$ norm of $x(t)$.

2.2 The time variance σ_t^2

The time variance of a function $x(t)$ is defined as

$$\sigma_t^2 = \frac{\int_{-\infty}^{\infty} (t-t_0)^2 |x(t)|^2 dt}{||x||_2^2} \tag{2}$$

2.3 The frequency center Ω_0

The frequency center of $\hat{x}(\Omega)$, where $\hat{x}(\Omega)$ is the Fourier transform of $x(t)$, is given by

$$\Omega_0 = \frac{\int_{-\infty}^{\infty} \Omega|\hat{x}(\Omega)|^2 d\Omega}{\int_{-\infty}^{\infty} |\hat{x}(\Omega)|^2 d\Omega} \tag{3}$$

As before, the quantity $\int_{-\infty}^{\infty} |\hat{x}(\Omega)|^2 d\Omega$ is expressed as $||\hat{x}||_2^2$. For real signals, $|\hat{x}(\Omega)|^2$ is an even function of Ω and hence $\Omega_0 = 0$ due to symmetry.

2.4 The frequency variance σ_Ω^2

The frequency variance of a signal spectrum $\hat{x}(\Omega)$ given by

$$\sigma_\Omega^2 = \frac{\int_{-\infty}^{\infty} (\Omega-\Omega_0)^2 |\hat{x}(\Omega)|^2 d\Omega}{||\hat{x}||_2^2} \tag{4}$$

For interpretation of this time and frequency centers, treat $x(t)$, $|x(t)|^2$, $\hat{x}(\Omega)$ and $|\hat{x}(\Omega)|^2$ as masses individually and looking for their Center of Mass. Time and Frequency Variance is variance of density which we constructed out of that mass $x(t)$ and $\hat{x}(\Omega)$, respectively.

3 Time-Bandwidth Product

In this section, we formulate the Time-Bandwidth Product, which is, product of time variance and frequency variance

$$\sigma_t^2\sigma_\Omega^2 = \frac{\int_{-\infty}^{\infty}(t-t_0)^2|x(t)|^2dt}{||x||_2^2}\frac{\int_{-\infty}^{\infty}(\Omega-\Omega_0)^2|\hat{x}(\Omega)|^2d\Omega}{||\hat{x}||_2^2} \tag{5}$$

It is the measure of localization in time and frequency domains, simultaneously and our aim is to minimize this value.

4 Signal transformations

In this section, we shall study the effect of some common signal transformations on the five quantities mentioned above.

4.1 Shifting in time domain

Let the signal $x(t)$, with time center t_0 be shifted in time by amount t_1, i.e.

$$y(t) = x(t-t_1) \tag{6}$$

4.1.1 Effect on time center

Since we are only shifting in time and not changing the magnitude, the $\mathbb{L}_2$ norm square in the denominator will remain unchanged. The new time center will be given as

$$t_{0(new)} = \frac{\int_{-\infty}^{\infty} t|x(t-t_1)|^2dt}{||x||_2^2} \tag{7}$$

Now, substitute $t - t_1 = u$. Thus

$$dt = du$$

The limits of integration remain unchanged. Thus, the new integral will be

$$\begin{aligned} t_{0(new)} &= \frac{\int_{-\infty}^{\infty}(u+t_1)|x(u)|^2du}{||x||_2^2} \\ &= \frac{\int_{-\infty}^{\infty}u|x(u)|^2du}{||x||_2^2} + \frac{\int_{-\infty}^{\infty}t_1|x(u)|^2du}{||x||_2^2} \\ &= t_0 + t_1\frac{\int_{-\infty}^{\infty}|x(u)|^2du}{||x||_2^2} \\ t_{0(new)} &= t_0 + t_1 \end{aligned} \tag{8}$$

because,

$$\int_{-\infty}^{\infty} |x(u)|^2 du = ||x||_2^2$$

4.1.2 Effect on time variance

From the new time center, we can write the expression for the time variance as

$$\sigma^2_{t(new)} = \frac{\int_{-\infty}^{\infty} (t - t_0 - t_1)^2 |x(t - t_1)|^2 dt}{||x||_2^2}$$

substitute

$$t - t_1 = u$$

Thus,

$$\begin{aligned} \sigma^2_{t(new)} &= \frac{\int_{-\infty}^{\infty} (u - t_0)^2 |x(u)|^2 du}{||x||_2^2} \\ \sigma^2_{t(new)} &= \sigma_t^2 \end{aligned} \tag{9}$$

Thus, the time variance is unaffected by a shift in time.

4.1.3 Effect on frequency domain

$$\begin{aligned} & y(t) = x(t - t_1) \\ \Rightarrow\ & \hat{y}(\Omega) = e^{-\jmath\Omega t_1}\hat{x}(\Omega) \\ \Rightarrow\ & |\hat{y}(\Omega)| = |\hat{x}(\Omega)| \\ \Rightarrow\ & ||\hat{y}||_2^2 = ||\hat{x}||_2^2 \end{aligned} \tag{10}$$

Since the magnitude of $\hat{y}(\Omega)$ is same as $\hat{x}(\Omega)$, the frequency center and frequency variance will remain same. The only change is in phase, which is of no consequence in calculating Ω_0 and σ_Ω^2.

4.1.4 Effect on Time Bandwidth Product

$$\sigma^2_{t(new)} = \sigma_t^2$$

$$\sigma^2_{\Omega(new)} = \sigma_\Omega^2$$

$$\sigma^2_{t(new)}\sigma^2_{\Omega(new)} = \sigma_t^2\sigma_\Omega^2$$

Hence, the time bandwidth product is time shift invariant.

4.2 Shifting in frequency domain(or modulation in time domain)

Shifting in frequency domain implies multiplication by a complex exponential in time domain. Thus,

$$\begin{aligned} & y(t) = e^{\jmath\Omega_1 t}x(t) \\ \Rightarrow\ & \hat{y}(\Omega) = \hat{x}(\Omega - \Omega_1) \end{aligned} \tag{11}$$

The frequency center is shifted by $+\Omega_1$ and frequency variance is unchanged. It can be proved similar to that of time shifting in section 4.1. Also note that, since

$$|y(t)| = |x(t)|$$

time center and time variance also remain unchanged. Hence, the time bandwidth product $\sigma_t^2\sigma_\Omega^2$ is also frequency shift invariant.

4.3 Multiplication by a constant

Let

$$\begin{aligned} y(t) &= C_0 x(t) \qquad (C_0 \neq 0) \\ \Rightarrow |y(t)|^2 &= |C_0|^2 |x(t)|^2 \\ \Rightarrow |\hat{y}(\Omega)|^2 &= |C_0|^2 |\hat{x}(\Omega)|^2 \end{aligned} \tag{12}$$

Substituting $y(t)$ in equations (1), (2), (3), (4), (5) we can see that the term $|C_0|^2$ will come outside the integration in both denominator and numerator and will get canceled, thus leaving centers, variances and time bandwidth product unchanged.

4.4 Scaling of independent variable

Let

$$\begin{aligned} y(t) &= x(\alpha t) \qquad (\alpha \in \mathbb{R}, \alpha \neq 0) \\ \Rightarrow \hat{y}(\Omega) &= \frac{1}{|\alpha|}\hat{x}\left(\frac{\Omega}{\alpha}\right) \end{aligned} \tag{13}$$

4.4.1 Effect in time domain

The new time center is given by

$$\begin{aligned} t_{0(new)} &= \frac{\int_{-\infty}^{\infty} t|x(\alpha t)|^2 dt}{\int_{-\infty}^{\infty} |x(\alpha t)|^2 dt} \\ put \ \ \lambda &= \alpha t \\ \Rightarrow d\lambda &= \alpha dt \\ \Rightarrow t_{0(new)} &= \frac{\frac{1}{\alpha^2}\int_{-\infty}^{\infty} \lambda |x(\lambda)|^2 d\lambda}{\frac{1}{\alpha}\int_{-\infty}^{\infty} |x(\lambda)|^2 d\lambda} \\ \Rightarrow t_{0(new)} &= \frac{1}{\alpha} t_0 \end{aligned} \tag{14}$$

Using similar reasoning, it can be proved that

$$\sigma_{t(new)}^2 = \frac{1}{\alpha^2}\sigma_t^2 \tag{15}$$

Note that even if α is negative, the limits of integration would still remain same as there would be reversal of limits twice.

4.4.2 Effect in frequency domain

Starting with $\hat{x}\left(\frac{\Omega}{\alpha}\right)$ it can be proved that

$$\Omega_{0(new)} = \alpha\Omega_0 \tag{16}$$

$$\sigma^2_{\Omega(new)} = \alpha^2\sigma^2_\Omega \tag{17}$$

The multiplier $\left(\frac{1}{\alpha}\right)$ is not considered as it neither affects center nor the variance.(The students should verify the above results as an exercise.)

4.4.3 Effect on time-bandwidth product

The new time bandwidth product will be given by

$$\begin{aligned} \sigma^2_{t(new)}\sigma^2_{\Omega(new)} &= \left(\frac{1}{\alpha^2}\sigma^2_t\right)\left(\alpha^2\sigma^2_\Omega\right) \\ &= \sigma^2_t\sigma^2_\Omega \end{aligned} \tag{18}$$

Thus, the time-bandwidth product is invariant to scaling of the independent variable.

5 Properties of the time-bandwidth product

Thus, to summarize, the time-bandwidth product is invariant to the following operations:

- Shifting waveform in time
- Shifting waveform in frequency(modulation in time)
- Multiplying the function by a constant
- Scaling of independent variable

The time-bandwidth product is thus a robust measure of combined time and frequency spread of a signal. It is essentially a property of the **shape** of the waveform.

Challenge: Can two waveforms with different shapes have the same time bandwidth product?

In the last lecture, we had noted that the time-bandwidth product of the Haar scaling function was ∞. The above results prove that the time-bandwidth product of **any** gate/rectangular function is ∞. The next fundamental question which comes to mind is **what is the minimum value of this product?**

5.1 Simplification of the time-bandwidth formula

Without loss of generality, we can assume that a function has both time and frequency center zero (because that does not affect the time bandwidth product).

$$\sigma^2_t\sigma^2_\Omega = \frac{\int_{-\infty}^{\infty} t^2|x(t)|^2dt}{||x||^2_2}\frac{\int_{-\infty}^{\infty} \Omega^2|\hat{x}(\Omega)|^2d\Omega}{||\hat{x}||^2_2} \tag{19}$$

Now, we simplify the numerator of the frequency variance.

$$\int_{-\infty}^{\infty} \Omega^2 |\hat{x}(\Omega)|^2 d\Omega = \int_{-\infty}^{\infty} |\jmath\Omega\hat{x}(\Omega)|^2 d\Omega \tag{20}$$

Now, we know that if

$$x(t) \xrightarrow{\mathbb{F}} \hat{x}(\Omega)$$
$$\frac{dx(t)}{dt} \xrightarrow{\mathbb{F}} \jmath\Omega\hat{x}(\Omega)$$

By Parseval's theorem,

$$\begin{aligned} ||\hat{x}(\Omega)||_2^2 &= 2\pi||x(t)||_2^2 \\ ||\jmath\Omega\hat{x}(\Omega)||_2^2 &= 2\pi||\frac{dx(t)}{dt}||_2^2 \end{aligned} \tag{21}$$

Using above results in equation (19), we get

$$\begin{aligned} \sigma_t^2\sigma_\Omega^2 &= \frac{||tx(t)||_2^2}{||x||_2^2}\frac{||\frac{dx(t)}{dt}||_2^2}{||x||_2^2} \\ &= \frac{||tx(t)||_2^2.||\frac{dx(t)}{dt}||_2^2}{||x||_2^4} \end{aligned} \tag{22}$$

The next step is to minimize this product that will give us a fundamental bound in nature known as the Uncertainty Bound, which will be discussed in the next lecture.