WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 16: Ideal Time-Frequency Behaviour

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1 Introduction

In the previous lecture we have looked at the Fourier Transform of the scaling function (Father wavelet) $\phi(t)$ and the wavelet function (Mother wavelet) $\psi(t)$ in the Haar Multiresolution Analysis. In this lecture we will see, what is the ideal situation that we are driving towards. We have made some observation about the nature of the magnitude of $\widehat{\phi}(\Omega)$ and $\widehat{\psi}(\Omega)$. We have noted, when we take dot product of $x(t)$ and a translate of $\phi(t)$, the magnitudes of the Fourier Transforms of $x(\cdot)$ and $\phi(\cdot)$ are getting multiplied. We have observed that the nature of the Fourier Transform of the $\phi(\cdot)$ and also that of $\psi(\cdot)$ was such that it emphasizes some bands of frequencies of the underlying function $x(t)$.

2 Frequency localization by $\phi(t)$ and $\psi(t)$

The magnitude of Fourier Transform of $\phi(t)$ is shown in Figure 1.

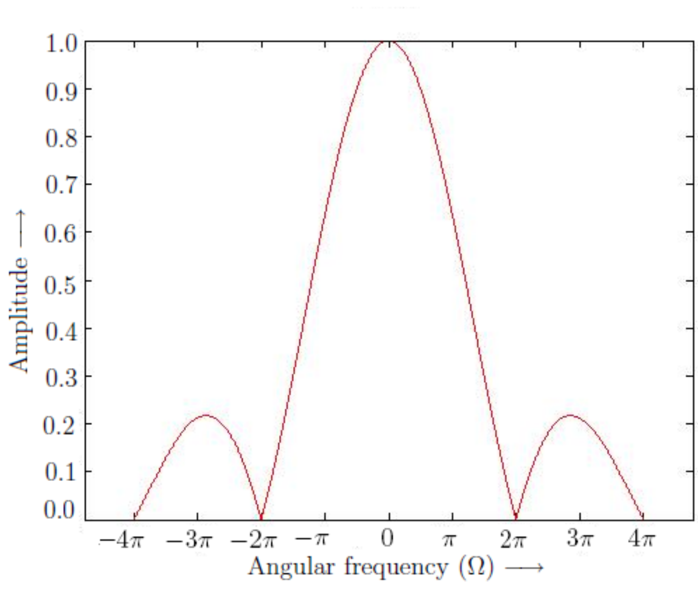

Figure 1: Magnitude of Fourier Transform of $\phi(t)$

Magnitude of Fourier Transform of $\psi(t)$ is

$$|\widehat{\psi}(\Omega)| = \frac{\sin^2\left(\frac{\Omega}{4}\right)}{\left(\frac{\Omega}{4}\right)}$$

To sketch its waveform, we first sketch $\frac{\sin\left(\frac{\Omega}{4}\right)}{(\frac{\Omega}{4})}$ and $\sin\left(\frac{\Omega}{4}\right)$ and then we shall multiply these two waveforms. Function $|\sin\left(\frac{\Omega}{4}\right)|$ has a monotonically incresing characteristics between 0 to

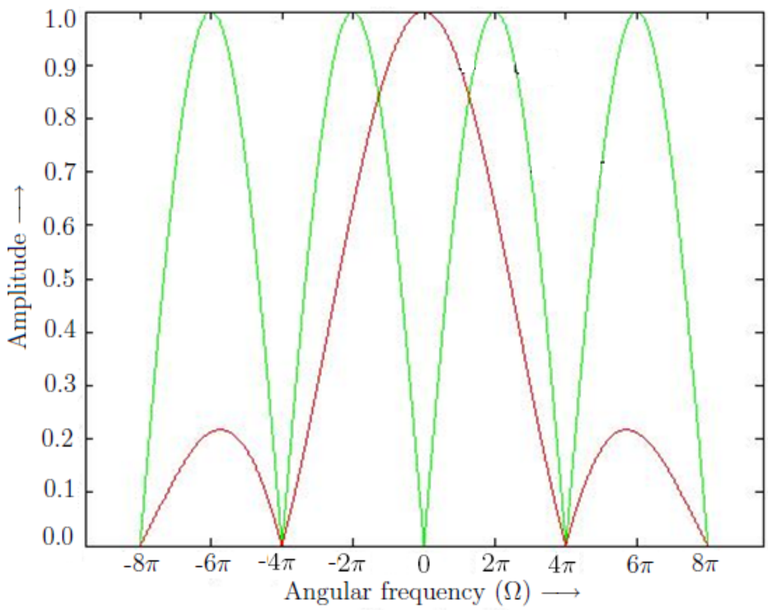

Figure 2: Magnitude plots of $\frac{\sin\left(\frac{\Omega}{4}\right)}{(\frac{\Omega}{4})}$ (red) and $\sin\left(\frac{\Omega}{4}\right)$ (green)

2π and decreasing characteristics between 2π to 4π. So one cannot possibly have a value of the product of $|\sin\left(\frac{\Omega}{4}\right)|$ and $\frac{\sin\left(\frac{\Omega}{4}\right)}{(\frac{\Omega}{4})}$ higher in the range 2π to 4π than the value of it at 2π. We can see the product is zero at $\Omega = 0$ and what after 2π is less than what is at 2π. So, somewhere in between 0 to 2π that product is having maximum and after that the product monotonically decreases. Also the function is not symmetric in the range 4π to 8π around 6π. So the maximum of the product in 4π to 8π is not at 6π, but somewhere around 6π. Finally, the product would look like as shown in Figure 3. In range -4π to 4π, $\widehat{\phi}(\Omega)$ and $\widehat{\psi}(\Omega)$ look like as shown in Figure 4.

From Figure 4, it is observed that $\widehat{\phi}(\Omega)$ and $\widehat{\psi}(\Omega)$ emphasizes those frequencies lying around zero and those frequencies laying around its maximum in the band 0 to 4π respectively and de-emphasizes frequencies on either sides. It is clear that $\widehat{\psi}(\Omega)$ has a bandpass characteristic and hence acts as a bandpass function. A band pass function emphasizes frequencies somewhere around its center frequency, where its value is maximum and de-emphasizes both sides around zero and around infinity. We note that when we contract scaling or wavelet function in time, we go up the ladder in Haar MRA and when we expand, we go down the ladder. Thus, when we go up the ladder, we are expanding in frequency domain and contracting in time domain. And when we go down the ladder, we are expanding in time and therefore contracting in frequency.

When we go down the ladder, we are contracting in frequency and we are emphasizing smaller and smaller frequency band around zero and also, as we are contracting $\widehat{\psi}(\cdot)$, we are empha-

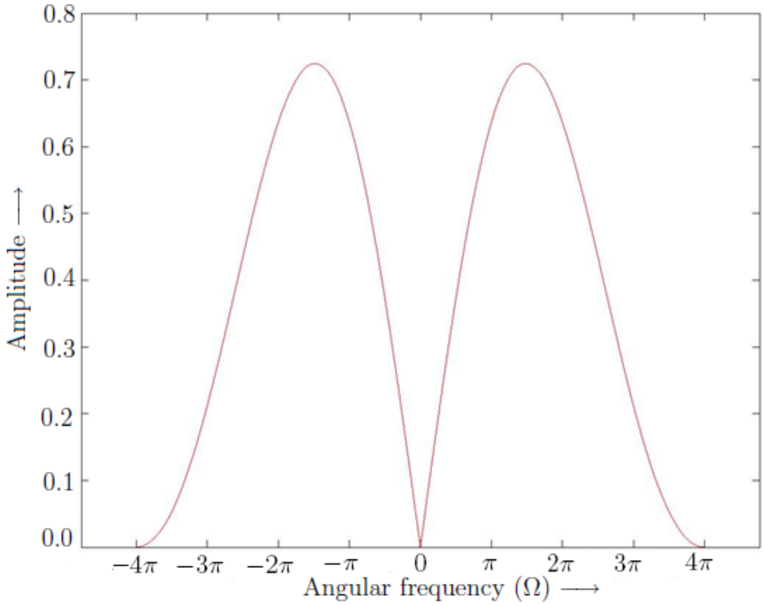

Figure 3: Magnitude plot of $\frac{\sin^2\left(\frac{\Omega}{4}\right)}{\left(\frac{\Omega}{4}\right)}$

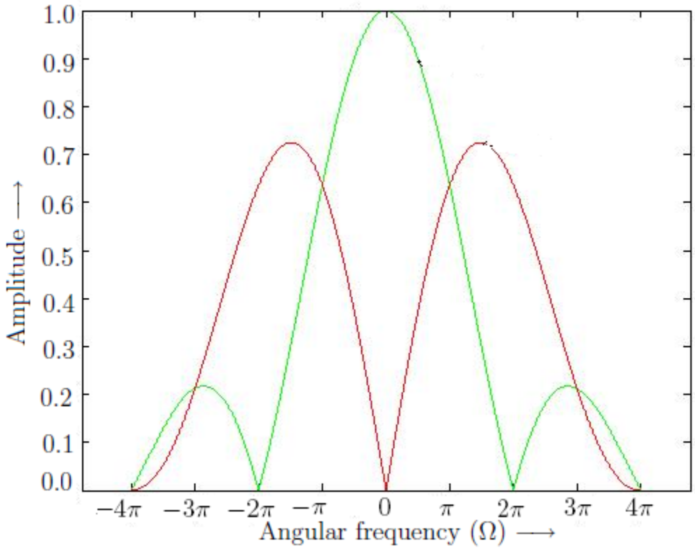

Figure 4: Magnitude of Fourier Transforms of $\widehat{\phi}(\Omega)$ (green) and $\widehat{\psi}(\Omega)$ (red) in the range -4π to 4π

sizing frequencies around smaller and smaller center frequency. The center frequency of $\widehat{\psi}(\cdot)$ decreases geometrically or logarithemically as we go down the ladder in the Haar MRA and width of the band of $\widehat{\psi}(\cdot)$ also decreases geometrically or logarithmically. Here the ratio of bandwidth to center frequency remains constant. We call this as constant quality factor. For a bandpass filter or bandpass function the quality factor can be defined as

$$Quality\ Factor = \frac{centre\ frequency}{bandwidth}$$

Typically the term bandwidth is used to denote that range of frequencies within which the magnitude remains within a certain percentage of maximum magnitude. More specifically, we often use half power bandwidth, where the magnitude falls to $\frac{1}{\sqrt{2}}$ of its maximum value, is considered as the cutoff point of that signal. The ratio $\frac{1}{\sqrt{2}}$ has a significance, at that point, where magnitude is $\frac{1}{\sqrt{2}}$ of maximum, power of a sine wave falls to $\frac{1}{2}$ of the maximum value.

Therefore two important observations can be made as

- The ratio of of bandwidth to center frequency of $|\widehat{\psi}(\cdot)|$ remains constant.
- As we go up the ladder of MRA, we deal with $|\widehat{\psi}(\cdot)|$ having higher center frequency and larger bandwidth. Similarly, as we go down the ladder, $|\widehat{\psi}(\cdot)|$ possesses lower center frequency and smaller bandwidth.

3 Time localization and frequency localization

Now, we use bandwidth as a measure of the range of frequencies that are emphasized by the function. This is because in finding dot product of $x(t)$ with and translate of $\psi(t)$ or any stretched or compressed version of it, Parseval's theorem says that we are, in fact multiplying Fourier transform of $x(t)$ and Fourier Transform of particular translate or dilate of $\psi(t)$ in frequency domain. The same argument is valid for $\phi(t)$ also.

Now translate does not have any effect on magnitude, but dilate has. So when we take dot product of $\psi(t)$ with $x(t)$, we are multiplying the part of $|\widehat{x}(\Omega)|$ which lie within the band, by a larger number and other part by a smaller number. So in effect a filtering effect is also being observed. Effectively $\phi(t)$ is doing a lowpass filtering operation and $\psi(t)$ is doing a bandpass filtering operation.

If we take $\phi(\cdot)$ itself and we focus on main lobe of Fourier Transform then, we are emphasizing on signals ranging between $0 - 2\pi$, and we are doing it by an operation in time domain and time restriction can be said precisely. $\phi(\cdot)$ and $\psi(\cdot)$ are very precisely localized in time. So the product of $\phi(\cdot)$ and $\psi(\cdot)$ or any of their dilate or translate with $x(t)$ is also localized in time.

In signal processing, we observe conflict between time and frequency. In this case, time localization is precise, but localization in frequency is somewhat suspended. We can roughly say that (focusing on the main lobe) these are in some sense localized. But there are side lobes also.

Ideally, we would like to have precise time as well as frequency localization simultaneously. To find out frequency localization consider dot product of $x(\cdot)$ with perpendicular translates of $\psi(\cdot)$.

So the product is (assuming both functions are real):

$$\int_{-\infty}^{\infty} x(t)\phi(t+\tau)dt$$

From Parseval's theorem

$$\int_{-\infty}^{\infty} x(t)\phi(t+\tau)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{x}(\Omega)\widehat{\phi(t+\tau)}d\Omega$$

$$\int_{-\infty}^{\infty} x(t)\phi(t+\tau)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{x}(\Omega)\widehat{\phi}(\Omega)e^{j\Omega\tau}d\Omega$$

This is the inverse Fourier Transform of $\widehat{x}(\Omega)\widehat{\phi}(\Omega)$ at the point τ.

So when we multiply by $\widehat{\phi}(\Omega)$, we are in effect doing some kind of lowpass filtering operation and when we take Inverse Fourier Transform, we are taking what comes out of the crud lowpass filter whose impulse response is $\phi(t)$ or very close to $\phi(t)$.

Now we sample this, at $\tau = n$ where $n \in \mathbb{Z}$. i.e., if we take a function $y(t)$ and sample it ideally for $n \in \mathbb{Z}$, we get

$$C_0 \sum_{k\in\mathbb{Z}} \widehat{y}(\Omega + 2\pi k)$$

C_0 is a constant, this constant relates to sampling process. We can ignore this constant for this moment. So, in order to reconstruct $y(t)$ from its samples these translates must not interfere with the original. So, we have to ensure that these $\widehat{y}(\Omega + 2\pi k)$ are non overlapping with the original and that is ensured by ensuring that the lowpass filter cuts off at $\Omega = \pi$. Had $\widehat{\phi}(\Omega)$ been an ideal lowpass function with a cut off of π, then this aliasing process $C_0 \sum_{k\in\mathbb{Z}} \widehat{y}(\Omega+2\pi k)$ would leave $\widehat{y}(\Omega)$ unaffected. So that is the ideal situation we are looking for.

Now we need to look at what is the ideal towards we are driving, as far as $\psi(t)$ goes. When we go from V_0 to V_1, we have noted that V_1 is just like V_0, but compressed by a factor of 2 in time, and therefore expanded by a factor of 2 in frequency domain. So for V_1 (Haar MRA ladder), we expanded by two 2 in frequency, that means we are asking for a lowpass filter whose cut off is 2π, instead of π. Now we have interpretation for incremented subspace. Obviously, V_0 is going to contain information between 0 and π and V_1 between 0 and 2π. Then W_0 will contain information between π to 2π. This is shown in Figure 5. So $\psi(\cdot)$ is aspiring to be a bandpass

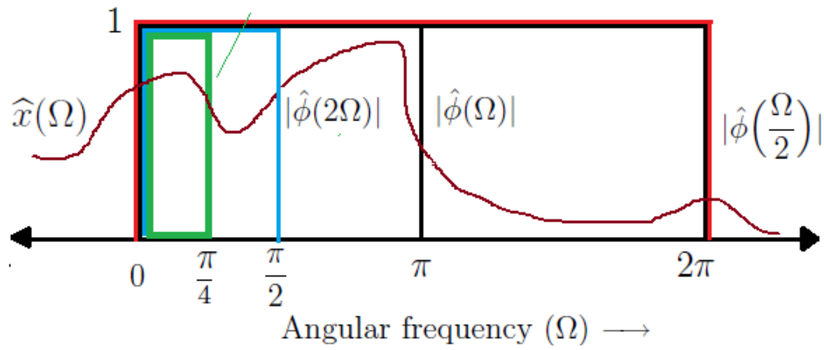

Figure 5: $\phi(\cdot)$ aspires to become low pass function. Ideally, information captured by it in different subspaces V_{-1}(blue), V_0(black) and V_1(red)

function between π and 2π. Similarly, from going V_{-1} to V_0, we use corresponding dilate of $\psi(\cdot)$ that aspires to be a bandpass function between $\frac{\pi}{2}$ and π and when we go from V_1 to V_2, we use dilate of $\psi(\cdot)$ that aspires to be a bandpass function between 2π and 4π and so on. This is illustrated in Figure 6. Now, we want to confine ourselves in a certain region of time and also want to focus or confine on a particular region of frequency. The first question that arises is, whether it is possible or not. Can we be compactly supported in time and frequency

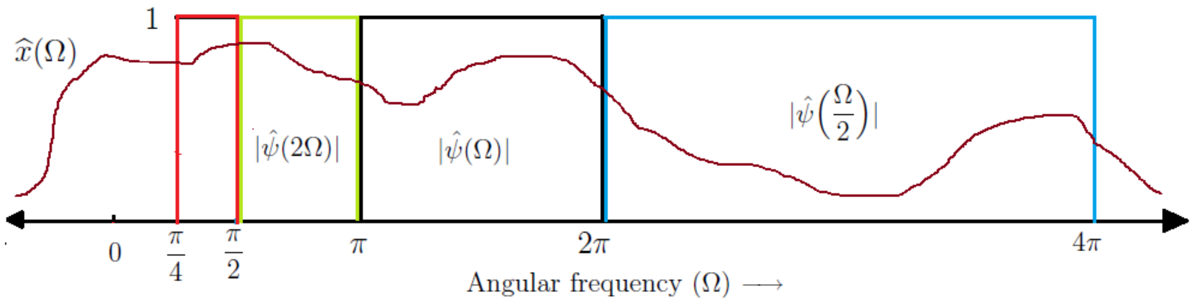

Figure 6: $\psi(\cdot)$ aspires to become band pass function. Ideally, information captured by it in different subspaces W_{-1}(green),W_0(black) and W_1(blue)

simultaneously? The answer is no. We cannot be compactly supported in both the domains.

However, if we do not ask for compact support in both domains, it is possible to have a function whose most of the energy is contained in the finite interval over time as well as frequency. Such function can be said to have a compact support in a weaker sense. $\phi(\cdot)$ and $\psi(\cdot)$ are bounded in both domains in a weaker sense as we focus on main lobe. Main lobe has certain amount of energy. Then $\phi(\cdot)$ and $\psi(\cdot)$ are localized in time and frequency both. Variance is important statistical property that is very useful in calculating spread of a given function, which is indicative of concentration of energy of a function within certain band (in time as well as frequency domain).

The question arises that is it possible to have finite variances in both frequency as well as time domain simultaneously? The answer is yes. We can have a both the variances of finite value. Now how small these variances can be? To answer this question we introduce time-frequency uncertainty. In case of Haar wavelet, it is somewhat concentrated in frequency, but well concentrated in time. Daubechies function, as we go at higher order, we get a somewhat better filtering operation that is better frequency localization.

In next lecture we shall investigate the concept of uncertainty deeply.