Introduction

In lecture 14, we studied steps required to design conjugate quadrature filter-banks. We set out with the aim to achieve systematic design steps for building higher order filter banks and learnt relationships between analysis and synthesis filters and between analysis low pass filter and analysis high pass filter which emanated from alias cancelation and perfect reconstruction conditions. Such analysis also enabled us to design higher and higher order filters of Daubechies family (as a special case of conjugate quadrature filter-bank) through incorporation of more and more $(1+z^{-1})$ terms in analysis low pass filter transfer function for achieving regularity required for convergence. Till this point, our analysis of scaling and wavelet functions has been either in time domain or in transform domain. However, several tasks require localization in both time and transform domain simultaneously and hence the need of such transform. Wavelet transforms arrive from family of transforms which provide such localization also known as multi-resolution analysis of the underlying signal. In this lecture, we analyze Haar scaling and wavelet functions for time and frequency localization. We analyze their frequency domain behaviors over the containment ladder. This very analysis gives us reason enough to move towards ideal aspirations for scaling and wavelet transforms and in turn towards the basic question of bound over simultaneous localization in time and frequency domain also known as the 'uncertainty principle' in coming lectures.

Description

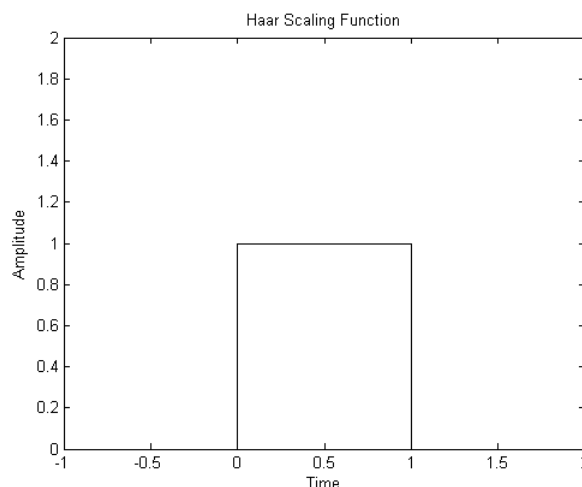We start our analysis by considering Haar scaling function, which is shown in figure 1.

Figure 1: Haar Scaling Function

If we denote Ω as analog angular frequency variable and $\widehat{\phi}(\Omega)$ as Fourier transform of scaling function then Fourier transform can be carried out in the following way:

$$\widehat{\phi}(\Omega) = \int_{-\infty}^{+\infty} \phi(t) e^{-j\Omega t} \, dt$$

$$\widehat{\phi}(\Omega) = \frac{1 - e^{-j\Omega}}{j\Omega}$$

$$\widehat{\phi}(\Omega) = e^{-0.5j\Omega} \left[\frac{\sin 0.5\Omega}{0.5\Omega} \right]$$

If we consider magnitude only for this scaling function then it may be represented and depicted as shown in figure 2.
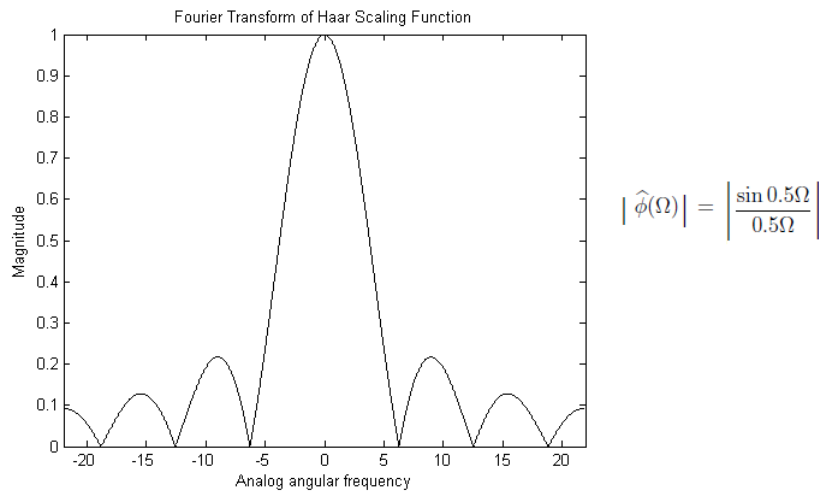

Figure 2: Magnitude of Fourier Transform for Haar Scaling Function

The above analysis of scaling function is applicable to subspace V_0. A space having piecewise constant approximation over standard unit interval [n, n+1]. Let us now analyze time and frequency domain behavior of scaling function as we move across the subspace ladder. In subsequent discussion, we use ... $\subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2$... relationship between subspaces and consider only magnitude of the Fourier transform. For general case, we note the following relationship:

$$\phi(t) \xleftrightarrow{F.T.} \widehat{\phi}(\Omega)$$
$$\phi(\alpha t) \xleftrightarrow{F.T.} \frac{1}{|\alpha|} \widehat{\phi}\left(\frac{\Omega}{\alpha}\right); \qquad \forall \ \alpha \in \mathbb{R} - \{0\}$$

As translation only affects phase in frequency domain, we consider scaling function without translation in time domain without loss of generality (as our analysis is limited to magnitude only). In this case, above relationship may be represented in the following manner.

$$\phi(t) \xleftrightarrow{F.T.} \left| \widehat{\phi}(\Omega) \right|$$

$$\phi(2^m t) \xleftrightarrow{F.T.} \left| \frac{1}{|2^m|} \widehat{\phi}\left(\frac{\Omega}{2^m}\right) \right|$$

Using this relationship, scaling function in time and frequency domain across various subspaces may be sketched as shown in figure 3. Note the changes in localization in time and frequency as we move from coarser subspaces to finer subspaces.

As evident from figure 3, as we go from subspace V_{-1} towards V_2, subspace localization in time improves by factor of 2, i.e. in V_0 subspace scaling function lies in [0 1] interval (nonzero values), whereas in V_1 subspace scaling function lies in [0 0.5] interval. In this sense we may say that time localization gets better and better as we move from $V_{-\infty}$ to $V_{+\infty}$. Similarly, localization in frequency gets poorer by factor of 2 as we go from subspace V_{-1} to V_2, i.e. width of main lobe doubles each time we move to higher subspace. In this sense, we may add that frequency localization gets poorer and poorer as we move from $V_{-\infty}$ to $V_{+\infty}$.

We shall now see how such localization affects projection of some signal $x(t) \in L_2(\mathbb{R})$ on various subspaces. We shall also check the relationship between taking dot product in time and frequency domains. For this purpose, we shall first consider orthonormal basis at some subspace V_m. We note that:

$$\Big\langle \phi(2^m t - n), \phi(2^m t - l) \Big\rangle$$

where, $< a, b >$ represents dot product between functions a and b. As mentioned, when $n \neq l$, scaling functions do not overlap with each other and hence yield zero dot product. We shall now check what happens when $n = l$. As evident, this case provides norm of the scaling function, which may be obtained in the following manner:

$$\left\| \phi(2^m \cdot) \right\|_2^2 = \int_0^{2^{-m}} (1)^2 \, dt = 2^{-m}$$

where, $\|\cdot\|_2^2$ represents square of L_2 norm and ' $\cdot$ ' represents corresponding argument. Here, argument is used to denote that this relationship is true for any valid translation. As can be seen, we need to make scaling function unit norm in order to form orthonormal basis for subspace V_m. Hence, $\sqrt{2^m}\phi(2^m - n) = 2^{\frac{m}{2}}\phi(2^m - n)$ forms an orthonormal spanning set for subspace V_m. This relationship may be represented in the following manner:

$$V_m = span\Big\{ 2^{\frac{m}{2}} \phi(2^m t - n) \Big\} \qquad \forall \quad n, m \in \mathbb{Z}$$

Projection of signal $x(t)$ over this subspace may be denoted as $\Big\langle x(\cdot), 2^{\frac{m}{2}}\phi(2^m \cdot - n) \Big\rangle$, where again '$\cdot$' denotes any valid argument. For understanding relationship between dot product in time and frequency domain we shall interpret what happens when we take Fourier transform and inverse Fourier transform.

If, $x(t) \leftrightarrow \widehat{x}(\Omega)$ then $\widehat{x}(\Omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\Omega t}dt$. During this process, we take components of $x(t)$ along directions provided by complex exponentials $e^{-j\Omega t}$. Also, inverse Fourier transform gives $x(t)$ back from $\frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{x}(\Omega)e^{j\Omega t}d\Omega$, which may be interpreted as reconstruction of $x(t)$ from components along $e^{j\Omega t}$. Now, from Parseval's theorem we have the following relationship:

$$< x(t), y(t) >= \Big\langle \widehat{x}(\Omega), \widehat{y}(\Omega) \Big\rangle$$

Which states that dot product is independent of basis we select and is equal in time and angular frequency domain within factor of 2π. Using this relationship we may denote the relationship between projection of function $x(t)$ over subspace V_m in the following manner:

$$\left\langle x(t), 2^{\frac{m}{2}}\phi(2^m \cdot -n)\right\rangle = \frac{1}{2\pi}\left\langle \widehat{x}(\Omega), 2^{\frac{m}{2}}\widehat{\phi(2^m - n)}\right\rangle$$

In figure 4, solid line shows Fourier transform of Haar scaling function and dotted line shows Fourier transform of signal under consideration, namely $x(t)$. As we can see, contribution of side lobes towards overall dot product is unsubstantial compared to contribution from main lobe. In other words, frequencies of $x(t)$ inside main lobe are emphasized with respect to frequencies outside the main lobe. As we move across the subspace ladder different amount of frequencies are emphasized. To be more precise, as we go from $V_{-\infty}$ to $V_{+\infty}$, more and more frequencies are emphasized as peak always remains on zero frequency.

Let us now analyze Haar wavelet function in time and frequency domain. Haar wavelet function is shown in figure 5. Haar wavelet may also be represented in terms of basis of subspace V_1 as it is clearly in V_1. This relationship may be represented as $\psi(t) = \phi(2t) - \phi(2t - 1)$. Using this relationship we may find Fourier transform of Haar wavelet function. Clearly,

$$\phi(t) \quad \overset{F.T.}{\longleftrightarrow} \quad \widehat{\phi}(\Omega)$$
$$\phi(2t) \quad \overset{F.T.}{\longleftrightarrow} \quad \frac{1}{2}\widehat{\phi}\left(\frac{\Omega}{2}\right)$$
$$\phi(2t - 1) \quad \overset{F.T.}{\longleftrightarrow} \quad \frac{1}{2}e^{\frac{-j\Omega}{2}}\widehat{\phi}\left(\frac{\Omega}{2}\right)$$

Using these relationships we may get

$$\begin{aligned}
\psi(t) &= \phi(2t) - \phi(2t - 1) \\
\widehat{\psi}(\Omega) &= \frac{1}{2}\widehat{\phi}\left(\frac{\Omega}{2}\right) - \frac{1}{2}e^{\frac{-j\Omega}{2}}\widehat{\phi}\left(\frac{\Omega}{2}\right) \\
\widehat{\psi}(\Omega) &= \frac{1}{2}\left(1 - e^{\frac{-j\Omega}{2}}\right)\widehat{\phi}\left(\frac{\Omega}{2}\right) \\
\widehat{\psi}(\Omega) &= je^{\frac{-j\Omega}{2}}\frac{\sin^2\left(\frac{\Omega}{4}\right)}{\frac{\Omega}{4}}
\end{aligned}$$

If we consider magnitude only then we may write

$$|\widehat{\psi}(\Omega)| = \left|\frac{\sin^2\left(\frac{\Omega}{4}\right)}{\frac{\Omega}{4}}\right|$$

Using this relationship we may plot Haar wavelet function across various subspaces. Figure 6 shows Haar wavelet in subspaces W_{-1} to W_2.

Obviously, as in the case of Haar scaling function, here also localization in time improves as we move from W_{-1} to W_2 by factor of 2. Similarly, localization in frequency degrades as we go in similar direction by factor of 2. However, important thing to note here is along with the bandwidth (as we may call it) centre frequency also shifts, unlike in the case of Haar scaling function. If we now interpret this result by using Parseval's theorem then we may state that in case of Haar wavelet different bands with increasing bandwidth are emphasized as we go from $W_{-\infty}$ to $W_{+\infty}$. This characteristic of Haar wavelet function is similar to aspirant band pass filter. Again, similar notion of time and frequency localization applies here as in case of Haar scaling function. Namely, localization in time gets better and better and localization in frequency gets poorer and poorer as we go from $W_{-\infty}$ to $W_{+\infty}$.

Summary

In this lecture we analyzed Haar scaling and wavelet functions in detail. Rather than analyzing functions in either time domain or in frequency domain, we consider them in joint domain, i.e. time and frequency domains simultaneously. Analysis reveals that time and frequency domain localization improves and deteriorates respectively as we go from left to right in subspace ladder and vice versa. Also, Haar scaling function aspires to become low pass filter and wavelet function aspires to be band pass filter. Through this analysis we may consider ideal frequency responses of scaling and wavelet functions, which may improve overall response and projections over approximation and detail subspaces (V and W subspaces respectively). Findings of this sort allow us to consider fundamental question of bound over simultaneous localization in time and frequency domain, namely the 'uncertainty principle', which we shall look into in subsequent lectures.
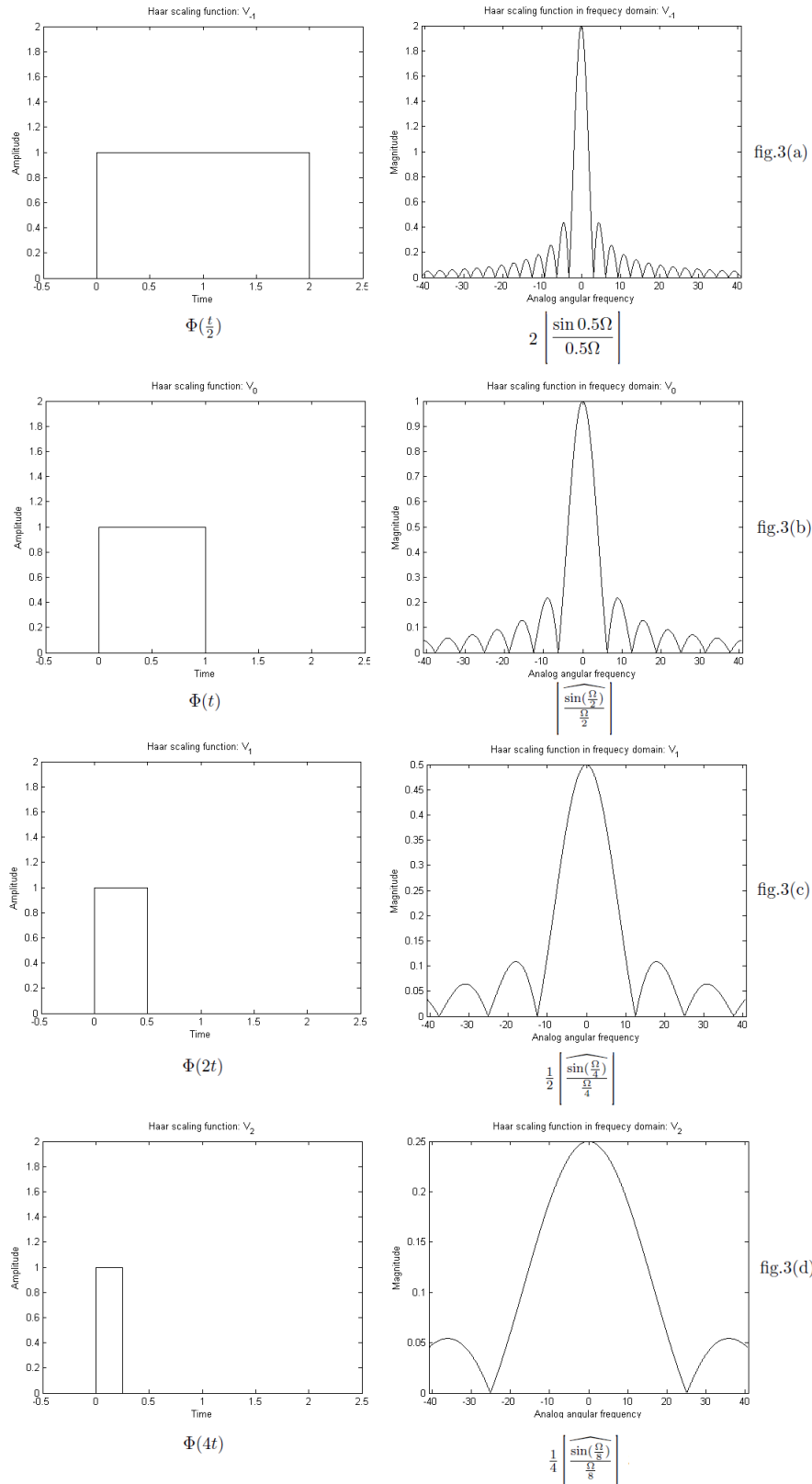
Figure 3: Haar scaling function at various time and frequency resolutions. 3(a): scaling function and frequency domain characteristic in V_{-1} subspace; 3(b): scaling function and frequency domain characteristic in V_0 subspace; 3(c): scaling function and frequency domain characteristic in V_1 subspace and 3(d): scaling function and frequency domain characteristic in V_2 subspace. Note how localization varies across different subspaces starting from coarse subspace V_{-1}.

Figure 4: Pictorial representation of dot product in frequency domain of a signal $x(t)$ with haar scaling function in some subspace
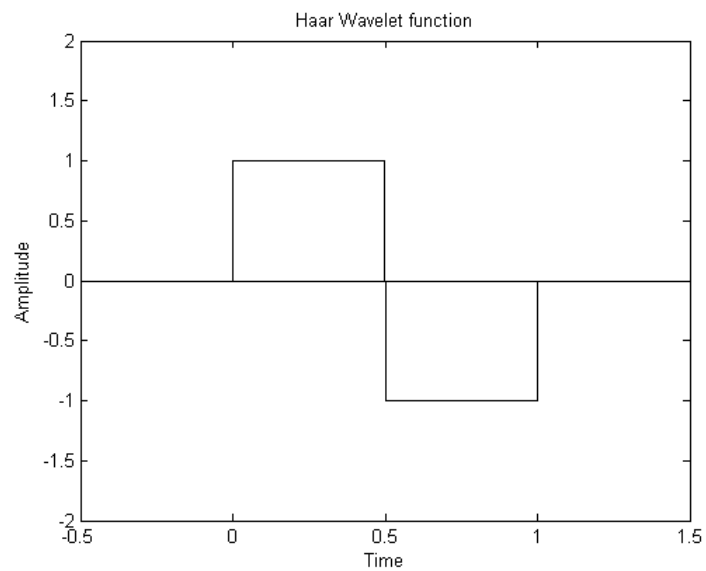

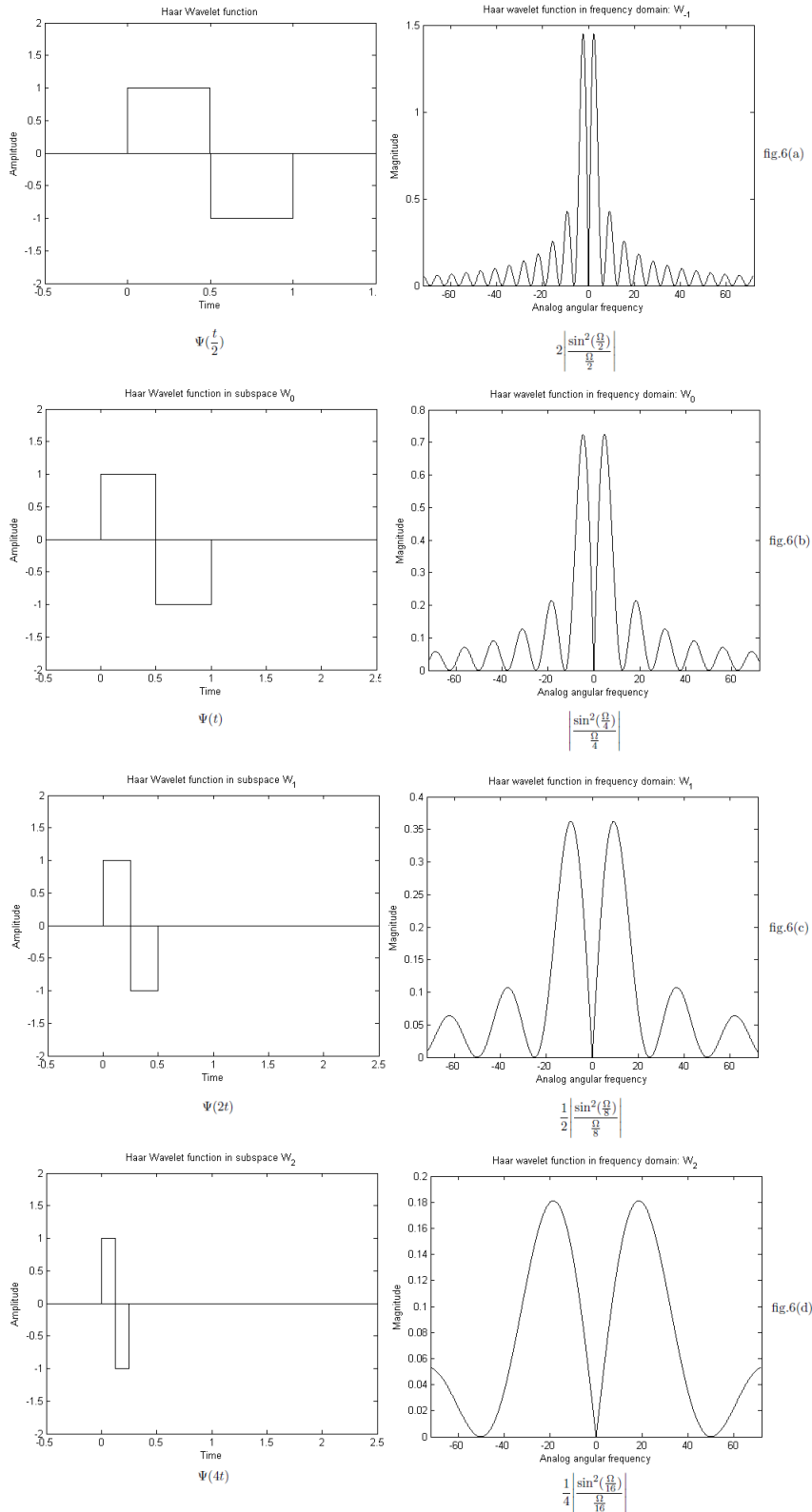Figure 5: Haar wavelet function in time domain

Figure 6: Haar wavelet function at various time and frequency resolutions. 6(a): wavelet function and frequency domain characteristic in W_{-1} subspace; 6(b): wavelet function and frequency domain characteristic in W_0 subspace; 6(c): wavelet function and frequency domain characteristic in W_1 subspace and 6(d): wavelet function and frequency domain characteristic in W_2 subspace.