WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 13: Conjugate Quadrature Filter Bank

Prof.V.M.Gadre, EE, IIT Bombay

1 Introduction

We continue in this lecture to build upon the particular class of filter bank which we have introduced in the previous lecture called a Conjugate Quadrature Filter(CQF) bank.

2 Conjugate Quadrature Filter bank

For the perfect reconstruction system we must first do away aliasing. The alias cancellation equation for the two band filter bank is given by

$$G_0(Z)H_0(-Z) + G_1(Z)H_1(-Z) = 0$$

$$\frac{G_1(Z)}{G_0(Z)} = -\frac{H_0(-Z)}{H_1(-Z)}$$

Equating the numerator and denominator we get the relation between $G_0(Z), H_1(-Z), G_1(Z)$ and $H_0(-Z)$ as

$$\begin{aligned} G_1(Z) &= -H_0(-Z) \\ G_0(Z) &= H_1(-Z) \end{aligned}$$

The relation between the analysis HPF (high pass filter) and analysis LPF (low pass filter) called a conjugate quadrature relationship, is given by

$$H_1(Z) = z^{-D}H_0(-Z^{-1})$$

Here z^{-D} term is used to introduce causality. Putting $Z = e^{j\omega}$ in the above equation we get the frequency response equation as

$$\begin{aligned} H_1(Z) &= z^{-D}H_0(-Z^{-1})|_{Z=e^{j\omega}} \\ H_1(e^{j\omega}) &= e^{-j\omega D}H_0(-e^{-j\omega}) \end{aligned}$$

The magnitude response is given by

$$\begin{aligned} |H_1(e^{j\omega})| &= |e^{-j\omega D}H_0(-e^{-j\omega})| \\ |H_1(e^{j\omega})| &= |e^{-j\omega D}||H_0(-e^{-j\omega})| \\ |H_1(e^{j\omega})| &= |H_0(-e^{-j\omega})| \end{aligned}$$

$H_0(Z)$ is a Low pass filter with a real impulse response (real coefficients), therefore

$$H_0(e^{-j\omega}) = \overline{H_0(e^{j\omega})}$$

The magnitude response of LPF $H_0(Z)$ is symmetric along the magnitude axis and phase response is anti-symmetric along the frequency axis ω.

$$H_0(-e^{-j\omega}) = H_0(e^{-j(\omega\pm\pi)})$$

NOTE: LPF with cutoff frequency $\frac{\pi}{2} \overset{\text{(With shift by } \pi \text{ on } \omega)}{\rightleftharpoons}$ HPF with cutoff frequency $\frac{\pi}{2}$

We have shown,

$$H_1(Z) = z^{-D}H_0(-Z^{-1})$$

For the perfect reconstruction the equation must satisfy,

$$\begin{aligned}
G_0(Z)H_0(Z) + G_1(Z)H_1(Z) &= C_0z^{-D} \\
H_1(-Z)H_0(Z) - H_0(-Z)H_1(Z) &= C_0z^{-D} \\
(-1)^{-D}z^{-D}H_0(Z^{-1})H_0(Z) - H_0(-Z)z^{-D}H_0(-Z^{-1}) &= C_0z^{-D}
\end{aligned}$$

We need the following for perfect reconstruction systems,

$$(-1)^{-D}H_0(Z^{-1})H_0(Z) - H_0(-Z)H_0(-Z^{-1}) = C_0$$

If we consider the Haar filter then the relationship between $H_0(Z)$ and $H_1(Z)$ is given by,

$$\begin{aligned}
H_0(Z) &= 1 + z^{-1} \\
H_0(-Z^{-1}) &= 1 - z
\end{aligned}$$

The above equation is non-causal so to make it causal by inserting delay, we get the below equation,

$$z^{-D}H_0(-Z^{-1}) = z^{-D}(1 - z)$$

Here z^{-D} retains causality.
If D is odd,

$$\begin{aligned}
H_0(Z)H_0(Z^{-1}) + H_0(-Z)H_0(-Z^{-1}) &= -C_0 \\
H_0(Z)H_0(Z^{-1}) + H_0(-Z)H_0(-Z^{-1}) &= \text{Constant}
\end{aligned}$$

Putting $Z = e^{j\omega}$, we get the above equation in the frequency domain as,

$$H_0(e^{j\omega})H_0(e^{-j\omega}) + H_0(-e^{j\omega})H_0(-e^{-j\omega}) = \text{Constant}$$

For real impulse response we have,

$$\begin{aligned}
H_0(e^{-j\omega}) &= \overline{H_0(e^{j\omega})} \\
H_0(e^{j\omega})\overline{H_0(e^{j\omega})} + H_0(e^{j(\omega\pm\pi)})\overline{H_0(e^{j(\omega\pm\pi)})} &= \text{Constant} \\
|H_0(e^{j\omega})|^2 + |H_0(e^{j(\omega\pm\pi)})|^2 &= \text{Constant}
\end{aligned}$$

Above equation is called the power complementary equation.
For perfect reconstruction system,

$$H_0(Z)H_0(Z^{-1}) + H_0(-Z)H_0(-Z^{-1}) = \text{Constant}$$

Lets assume $\kappa_0(Z) = H_0(Z)H_0(Z^{-1})$

$$\kappa_0(Z) + \kappa_0(-Z) = \text{Constant}$$

We are going to choose even length of $H_0(Z)$,*i.e.* $D \longrightarrow$ Odd

$$\underset{n}{\underset{\uparrow}{h[n]}} : \quad \underset{0}{\underset{\uparrow}{h_0}}\ h_1\ h_2\ \ldots \underset{D}{\underset{\uparrow}{h_D}}$$

Similarly, $H_0(Z^{-1})$ is given by,

$$\underset{n}{\underset{\uparrow}{h[n]}} : \quad \underset{-D}{\underset{\uparrow}{h_D}} \ldots\ h_2\ h_1\ \underset{0}{\underset{\uparrow}{h_0}}$$

Here $H_0(Z)H_0(Z^{-1})$ corresponds to their convolution in time domain

$$(\underset{0}{\underset{\uparrow}{h_0}}\ h_1\ h_2\ \ldots \underset{D}{\underset{\uparrow}{h_D}}) \bigotimes (\underset{-D}{\underset{\uparrow}{h_D}} \ldots\ h_2\ h_1\ \underset{0}{\underset{\uparrow}{h_0}})$$

Let impulse response $h[k]$ be as given below

$$h[k] : \quad \underset{0}{\underset{\uparrow}{h_0}}\ h_1\ h_2\ \ldots \underset{D}{\underset{\uparrow}{h_D}}$$

And impulse response $g[k]$ is given below which is mirror image of $h[k]$, that means $g[k] = h[-k]$

$$g[k] : \quad \underset{-D}{\underset{\uparrow}{h_D}} \ldots\ h_2\ h_1\ \underset{0}{\underset{\uparrow}{h_0}}$$

Similarly $g[n-k]$ is shown below

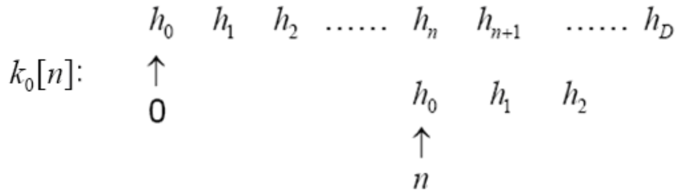
$$g[n-k] : \quad \underset{n}{\underset{\uparrow}{h_0}}\ h_1\ h_2\ \ldots \underset{n+D}{\underset{\uparrow}{h_D}}$$

The convolution between $h[k]$ and $g[k]$ is given

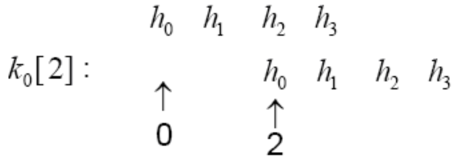
$$\kappa_0[n] = \sum_{k=-\infty}^{k=+\infty} h[k]g[n-k]$$

Here $h[k]$ is causal and filter length is $(D+1)$.
The convolution at the sample n is $y[n]$.

Shown below is the multiplication of $h[k]$ and $g[k]$ (which is shifted by n samples)

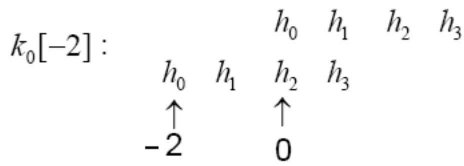


In Z-domain $\kappa_0(Z) = H_0(Z)H_0(Z^{-1})$.
The m^{th} sample of the filter $k_0[m]$ is $< h[k], h[k \pm m] >$
Let $m = 2$ and filter length 4 ($D = 3$)



$$k_0[2] = h_0h_2 + h_1h_3$$

If $m = -2$ and filter length 4 ($D = 3$)

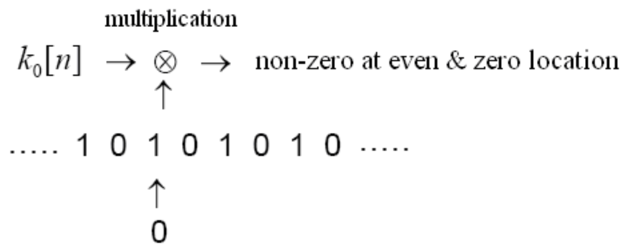


$$k_0[-2] = h_0h_2 + h_1h_3$$

That means the convolution between $h[n]$ and $h[-n]$ is symmetrical.

$$\begin{aligned} \kappa_0(Z) + \kappa_0(-Z) &= \text{Constant} \\ \frac{1}{2}\{\kappa_0(Z) + \kappa_0(-Z)\} &= \text{Constant} \end{aligned}$$

From the above equation the summation $\frac{1}{2}\{\kappa_0(Z) + \kappa_0(-Z)\}$ represents the nonzero sample value at even location and zero sample value at the odd location.
Let $\kappa_0(Z)$ correspond to the sequence $k_0[n]$, $\frac{1}{2}\{\kappa_0(Z) + \kappa_0(-Z)\}$ impulse response is shown below.



But from the equation $\frac{1}{2}\{\kappa_0(Z) + \kappa_0(-Z)\}$ = Constant, we want the non-zero sample value only at zero location and zero sample value for odd and even location.

So at the even location $m = 2l$ and $m \neq 0$ and ($l \in \mathbb{Z}$), we want zero sample value.

Let Daubechies filter with length 4 ($D = 3$)

$$h_0[n]: \quad \underset{\underset{0}{\uparrow}}{h_0}\ h_1\ h_2\ \ldots \underset{\underset{3}{\uparrow}}{h_3}$$

In the Haar case, $(1 - z^{-1})$ represents a High pass filter.

Here we consider the Daubechies filter with length 4 so two $(1 - z^{-1})$ in the High pass filter which means $(1 - z^{-1})^2$ factor in HPF.

Similarly, low pass filter has a factor $(1 - z^{-1})^2$.

A Daubechies low pass filter with length 4 is given by

$$H_0(Z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3}$$

We can write this equation in the factor of $(1 - z^{-1})^2$ *i.e.*

$$H_0(Z) = (1 + z^{-1})^2 (1 + B_0 z^{-1})$$

In the above equation, we need three zeros.

Two zeros are already chosen at unit circle which are $-1, -1$ and one zero is selected based on value of B_0. This value can be obtained by comparing the above two equations.
Expanding the above two equations

$$\begin{aligned} H_0(Z) &= (1 + 2z^{-1} + z^{-2})(1 + B_0 z^{-1}) \\ H_0(Z) &= 1 + (2 + B_0) z^{-1} + (1 + 2B_0) z^{-2} + B_0 z^{-3} \end{aligned}$$

The dot product of the impulse response of LPF with its even shifts must be zero. We will use this constraint to find the value of B_0 in the next lecture.