WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 12: Perfect Reconstruction

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1 Introduction

In the previous lecture we have dealt with Two Channel Filter Bank in detail. Some of the important points of it are noted below.

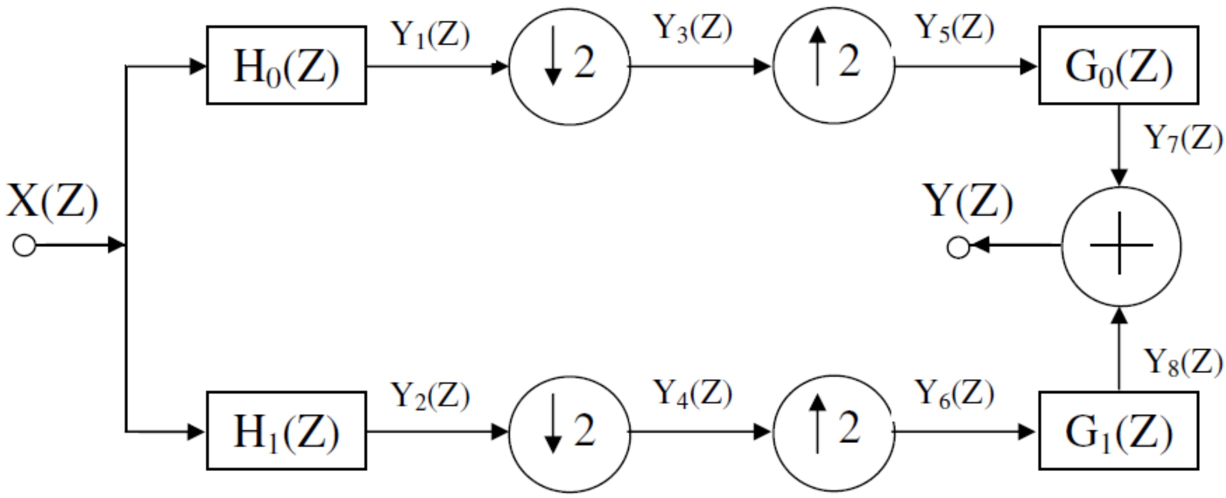

Figure 1: The Haar Filter Bank

$$Input\colon x[n] \xrightarrow{Z} X(Z)$$

$$Output\colon y[n] \xrightarrow{Z} Y(Z)$$

Analysis side has Low Pass Filter and High Pass Filter with system functions $H_0(Z)$ and $H_1(Z)$. Synthesis side has Low Pass Filter and High Pass Filter with system functions $G_0(Z)$ and $G_1(Z)$. Output is

$$Y(Z) = \tau_0(Z)X(Z) + \tau_1(Z)X(Z) \tag{1}$$

$\tau_1(Z)$ is called Alias System Function. In fact, this term is not correct in the sense that system is not shift invariant with $\tau_1(Z)$. For Alias cancelation we want $\tau_1(Z)$ to be equal to zero. Then system becomes Linear Shift Invariant(LSI).

$$\tau_1(Z) = \frac{1}{2}\{G_0(Z)H_0(-Z) + G_1(Z)H_1(-Z)\} \tag{2}$$

For $\tau_1(Z) = 0$

$$\frac{G_1(Z)}{G_0(Z)} = -\frac{H_0(-Z)}{H_1(-Z)} \tag{3}$$

Simply equating numerator and denominator we get

$$G_1(Z) = \pm R(Z)H_0(-Z) \tag{4}$$

$$G_0(Z) = \mp R(Z)H_1(-Z) \tag{5}$$

Here, if $H_1(-Z)$ is a High Pass Filter, $G_0(Z)$ becomes Low Pass Filter. If $\tau_1(Z)$ becomes zero, the system becomes LSI as $Y(Z) = \tau_0(Z)X(Z)$ with $\tau_0(Z)$ being a system function.

2 Condition for Perfect Reconstruction

We want to decompose a signal and reconstruct it back with ideal case as $X(Z) = Y(Z)$. Even tolerable changes in the output can be accepted. *What could be tolerable or acceptable changes?*

In case of time systems, time delays are tolerable since finite time is needed to process the signal at analysis and synthesis side. Even an output multiplied by constant is acceptable. So, in perfect reconstruction process $\tau_0(Z)$ takes the form of

$$\tau_0(Z) = C_0 z^{-D} \tag{6}$$

where C_0 is a constant. Ideally, we would like to have $\tau_0(Z) = 1$ for all z. But this makes the system non causal. So, a factor Z^{-D} is allowed to take care of causality.

In Haar Filter Bank shown in Figure (1) we have following

$$\begin{aligned}
\tau_1(Z) &= \frac{1}{2}\{G_0(Z)H_0(-Z) + G_1(Z)H_1(-Z)\} \\
&= \frac{1}{2}\left\{(1+z^{-1})\left(\frac{1-z^{-1}}{2}\right) \pm (1-z^{-1})\left(\frac{1+z^{-1}}{2}\right)\right\} \qquad (7)
\end{aligned}$$

For alias cancelation we need $\tau_1(z)$ to be equal to zero. This gives $G_1(z) = -(1-z^{-1})$. System functions of Haar Filter Bank are as follows

$$H_0(Z) = \frac{1+z^{-1}}{2}$$

$$H_1(Z) = \frac{1-z^{-1}}{2}$$

$$G_0(Z) = 1 + z^{-1}$$

$$G_1(Z) = -(1+z^{-1})$$

We find the value of $\tau_0(Z)$

$$\begin{aligned}
\tau_0(Z) &= \frac{1}{2}\{G_0(Z)H_0(Z) + G_1(Z)H_1(Z)\} \\
&= \frac{1}{2}\left\{(1+z^{-1})\left(\frac{1+z^{-1}}{2}\right) - (1-z^{-1})\left(\frac{1-z^{-1}}{2}\right)\right\} \\
&= z^{-1} \qquad (8)
\end{aligned}$$

This represents a delay of one sample. This delay is required on account of causality. If we try to avoid this delay we must have non-causality either on synthesis side or analysis side. Simplest possibilty for alias cancellation is $G_0(Z) = \pm H_1(Z)$. In case of Haar Filter Bank

$$H_1(Z) = \frac{1 - z^{-1}}{2}$$

$$H_1(-Z) = \frac{1 + z^{-1}}{2}$$

Essentially $G_0(Z) = \pm R(Z)H_1(-Z)$ condition should be satisfied. More generally for alias cancelation we need

$$G_0(Z) = \pm R(Z)H_1(-Z)$$
$$G_1(Z) = \mp R(Z)H_0(-Z)$$

In particular, for Haar case we have chosen $R(Z) = 2$ and $G_1(Z)$ as

$$\begin{aligned} G_1(Z) &= -2H_0(Z) \\ &= \{-2[\frac{1}{2}(1 + z^{-1})]\} \\ &= -(1 - z^{-1}) \end{aligned}$$

3 Polynomial As An Input

Haar MRA has many concepts hidden it. But we need to study what is the problem associated with Haar MRA. In other way, we should learn how Haar is a beginning of a family of Multiresolution Analysis. To have more insight we look at low pass and high pass filters in different perspective. It can be understood by taking an example of constant sequence as input and noting output at different points in filter bank.

$$x[n] = C_1 \quad \text{for all n}$$
$$H_1(Z) = \frac{1}{2}(1 - z^{-1})$$
$$\frac{x[n] - x[n-1]}{2} = 0 \text{ for all n}$$

If there is a constant component in the input sequence of Haar filter Bank, it is destroyed by High pass filter.

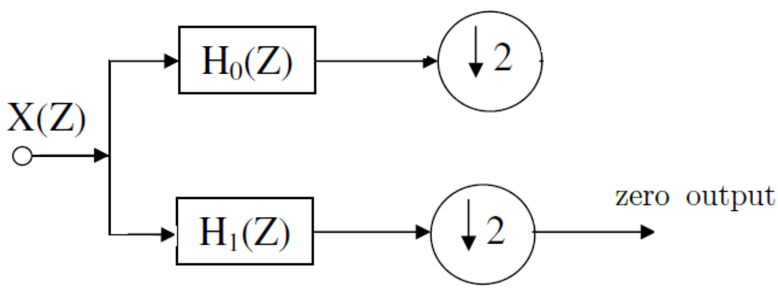

Figure 2: Output for constant input

Figure 3: Cascade system

In Haar Filter Bank we have a term $(1-z^{-1})$. *What could be the situation in case of multiple or cascaded* $(1-z^{-1})$ *?* If we think an input having polynomial components, every instance of $(1-z^{-1})$ reduces degree of polynomial by one. A Taylor series is polynomial expansion of input and if we subject some terms in the polynomial expansion to $(1-z^{-1})$ we have interpretation like this

$$a_0 n^M + a_1 n^{M-1} + a_2 n^{M-2} + \ldots + a_M = x[n]$$

where $x[n]$ is polynomial input sequence. Every time this is subjected to $1-z^{-1}$ what happens is

$$a_0 n^M + a_1 n^{M-1} + a_2 n^{M-2} + \ldots + a_M - \{a_0(n-1)^{M-1} + a_1(n-1)^{M-2} + \ldots + a_M\}$$

When we expand this, coefficient of n^M is $a_0 - a_0$ i.e. zero.
Each time we subject this polynomial to action of $(1-z^{-1})$, we are reducing degree by one. For example, consider sequence of polynomial degree one

$$\begin{aligned} output &= (3n+5) - [3(n-1)+5] \\ &= 3n + 5 - 3n + 3 - 5 \\ &= 3 \qquad\qquad for\ all\ n \end{aligned}$$

Hence coefficient of highest power of n vanishes. These terms are $(1-z^{-1})$ on high pass branch and can not be on low pass branch. We build up whole family of MRA with more and more $(1-z^{-1})$ terms on high pass branch. We intend to build **Daubechies MRA**. As we go to increase seniority in this family, there is more and more $(1-z^{-1})$ terms on high pass branch. Effectively we are reducing degree of higher order polynomial on high pass branch. We are killing them on high pass branch *i.e.* we are transferring them on low pass branch. We are retaining more smoothness on low pass branch. Addition to this we want same filters on analysis and synthesis side of filter bank. So we get one class of filter bank as *Conjugate Quadrature Filter bank.* Describing equation of this is simple.We start from aliasing cancelation condition

$$G_0(Z) = \pm H_1(Z)$$

$$G_1(Z) = \mp H_0(Z)$$

We choose (inspired by Haar)

$$G_0(Z) = +H_1(Z)$$

$$G_1(Z) = -H_0(-Z)$$

Here we keep away a factor of 2 that can be absorbed by constant C_0. We need $\tau_1(Z) = 0$ (by construction) and $\tau_0(Z)$ as

$$\tau_0(Z) = \frac{1}{2}\{H_1(-Z)H_0(Z) + (-H_0(-Z))H_1(Z)\} \tag{9}$$

As $H_1(Z)$ is a high pass filter (in synthesis side) $H_1(-Z)$ becomes a low pass filter with cut off $\frac{\pi}{2}$. Therefore first term in above expression represents cascade of two low pass filters and second term represents a cascade of two high pass filters with cut off $\frac{\pi}{2}$.

In case of Haar there is relation between H_0 and H_1. For perfect reconstruction, $\tau_0(Z)$ should be equal to delay and some multiplying constant. In Haar case,

$$H_1(-Z) = \frac{1 + z^{-1}}{2} = H_0(Z)$$

We shall in general note that $H_0(Z)$ should be related to $H_1(-Z)$. Choosing $H_1(Z)$ to be slightly modified from $H_0(-Z)$ as

$$H_1(Z) = z^{-D} H_0(-Z^{-1})$$

For Haar case,

$$z^{-1} H_0(-Z^{-1}) = \frac{Z^{-1} - 1}{2}$$

In general case for $H_1(Z) = z^{-D} H_0(-Z^{-1})$, $\tau_0(Z)$ becomes

$$\begin{aligned} \tau_0(Z) &= \frac{1}{2}\{H_0(Z)(-Z^{-D})H_0(Z^{-1}) - H_0(-Z)(Z^{-D})H_0((-Z)^{-1})\} \\ &= \frac{1}{2}\{H_0(Z)(Z^{-D})H_0(Z^{-1})(-1^{-D}) - H_0(-Z)(Z^{-D})H_0((-Z)^{-1})\} \qquad (10) \end{aligned}$$

In above expression, we choose value of D and put condition on H_0, then this becomes perfect reconstruction situation. We look at it in more detail in the next lecture.