WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 11: Two Channel Filter Bank

Prof. V.M.Gadre, EE, IIT Bombay

1 Introduction

In the previous lecture we studied Z domain analysis of two channel filter bank. This structure is often used in the implementation of discrete wavelet transform. So in this lecture we intend to analyze it completely by relating the Z-Transform at every point of following structure and finding relationship between $H_0(Z)$, $H_1(Z)$, $G_0(Z)$ and $G_1(Z)$ which ensures perfect reconstruction at Y(Z).

2 Two Channel Filter Bank

Consider two channel filter bank as shown in Figure 1 In previous lecture we studied effect of

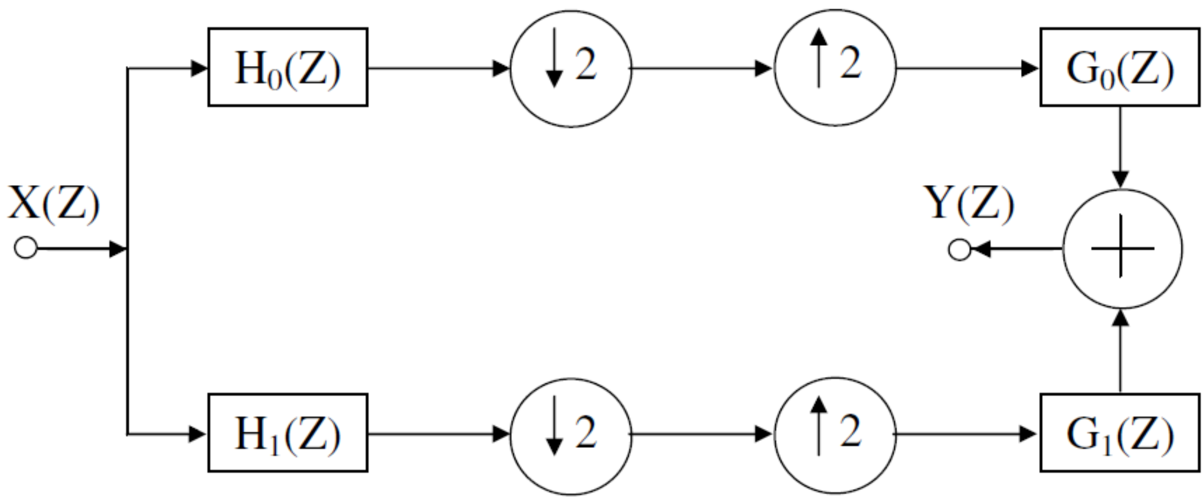

Figure 1: Two Channel Filter Bank

up and downsampling as:

1. **Effect of Upsampler**

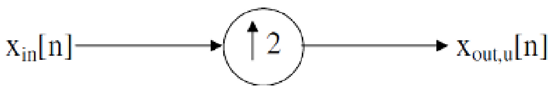

Figure 2: Upsampler

$$X_{out,U} = X_{in}(Z^2) \tag{1}$$

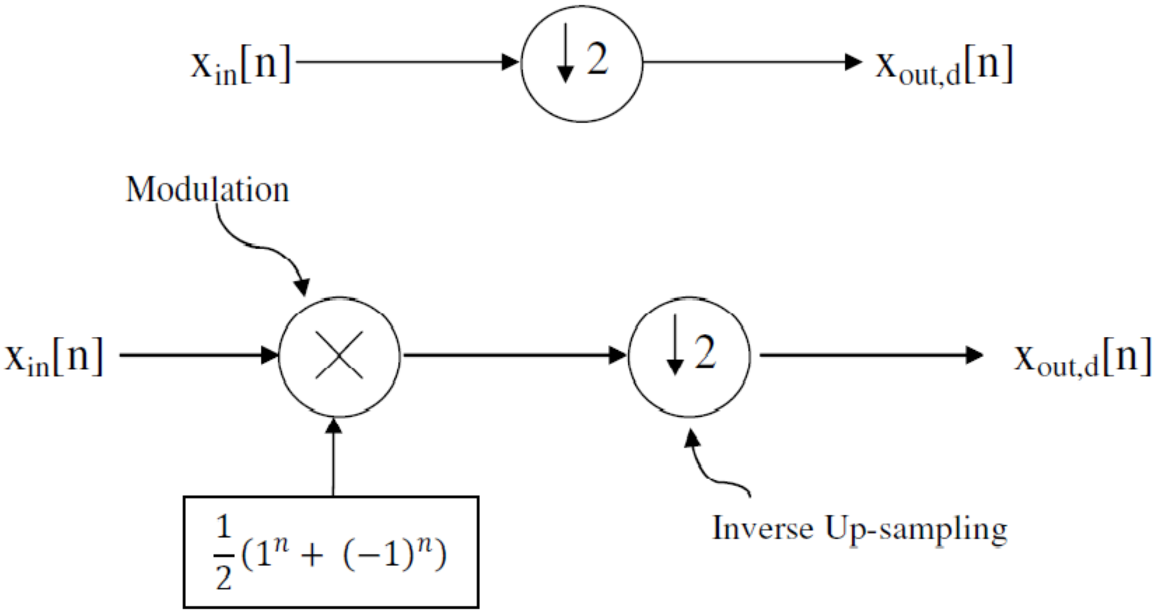

Figure 3: Downsampler

2. **Effect of Downsampler**

In Z domain,

$$X_{out,D} = \frac{1}{2}[X_{in}(Z)^{\frac{1}{2}} + X_{in}(-Z)^{\frac{1}{2}}] \tag{2}$$

As proved earlier, down-sampling by factor of 2 operation can be split into modulation by a sequence followed by inverse up-sampling by 2. This follows from the fact that, up-sampling by any factor is invertible operation which implies that inverse up-sampling is meaningful. Here, $(Z)^{\frac{1}{2}}$ appears due to inverse up-sampler. If inverse sampler is not considered, then power of $\frac{1}{2}$ will disappear.
Using this Z-transform, we can carry out the relation between Z-transform of input and Z-transform of output provided it exists throughout the process of analysis and synthesis.

3 Z-Domain Analysis of Filter Bank

To simplify the process, we name the output of each block as shown in Fig.4. With this notations, Z-transform at each point can be written quite easily. It is seen that $Y_1(Z)$ and $Y_2(Z)$ are simply X(Z) passed through filter $H_0(Z)$ and $H_1(Z)$ respectively.

$$Y_1(Z) = H_0(Z)X(Z) \tag{3}$$

$$Y_2(Z) = H_1(Z)X(Z) \tag{4}$$

We can write the relation between $Y_3(Z)$ and $Y_4(Z)$ in terms of $Y_1(Z)$ and $Y_2(Z)$ respectively. But if we notice the steps for down-sampling, then it is clear that inverse up-sampling operation needed for down-sampling cancels with up-sampler leaving only modulation part as a combined effect of down-sampling and up-sampling. Thus it becomes easy to jump from $Y_1(Z)$ and $Y_2(Z)$ to directly $Y_5(Z)$ and $Y_6(Z)$ respectively. This strategy is quite useful in analyzing multi-rate system particularly when down-sampler is followed by up-sampler. Thus it follows that,

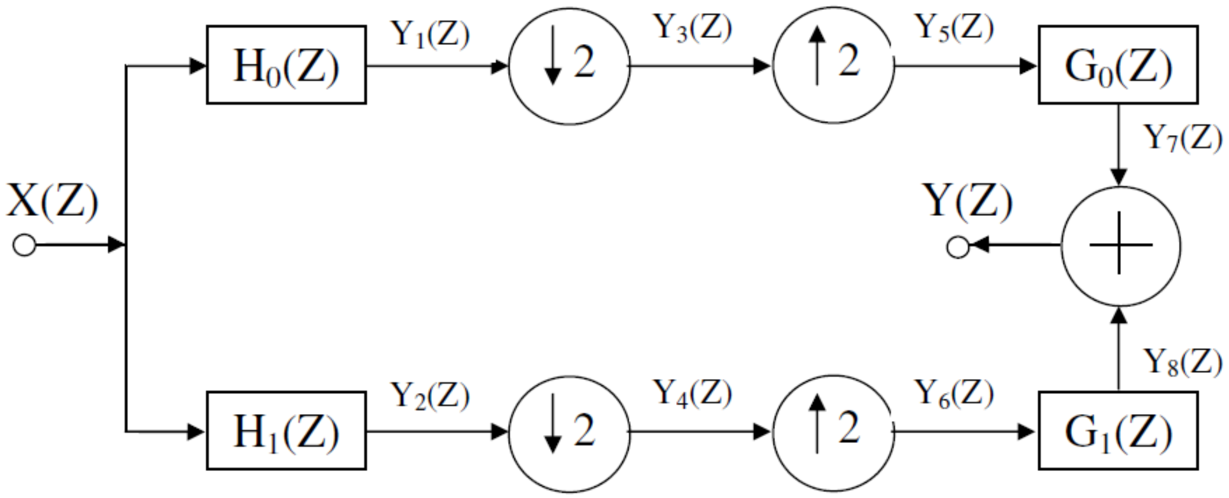

Figure 4: Filter bank with notations

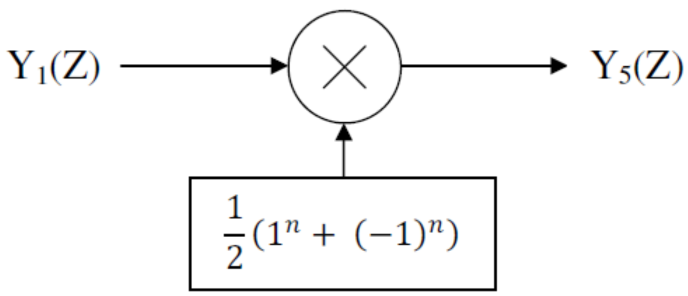

Figure 5: Jumping across Up and Down sampler

$$Y_5(Z) = \frac{1}{2}\{Y_1(Z) + Y_1(-Z)\} \tag{5}$$

$$Y_6(Z) = \frac{1}{2}\{Y_2(Z) + Y_2(-Z)\} \tag{6}$$

The important point is that from this point on, we have contribution from $X(Z)$ as well as $X(-Z)$. Significance of $X(-Z)$ will be dealt with later on. Once we have $Y_5(Z)$ and $Y_6(Z)$ we can easily go back and write for $Y_3(Z)$ and $Y_4(Z)$ as follows.

$$Y_3(Z) = Y_5(Z^{\frac{1}{2}}) \tag{7}$$

Jumping across upsampler and downsampler is useful since it brings us quickly to output. We are only one step away from output which can be easily achieved as

$$Y(Z) = Y_7(Z) + Y_8(Z) \tag{8}$$

where $Y_7(Z)$ and $Y_8(Z)$ are given by

$$Y_7(Z) = Y_5(Z)G_0(Z) \tag{9}$$

$$Y_8(Z) = Y_6(Z)G_1(Z) \tag{10}$$

where

$$Y_2(Z) = H_1(Z)X(Z) \tag{11}$$

$$Y_2(-Z) = H_1(-Z)X(-Z) \tag{12}$$

In total, we can write

$$Y_(Z) = \tau_0(Z)X(Z) + \tau_1(Z)X(Z) \tag{13}$$

where

$$\tau_0(Z) = \frac{1}{2}\{G_0(Z)H_0(Z) + G_1(Z)H_1(-Z)\} \tag{14}$$

$$\tau_1(Z) = \frac{1}{2}\{G_0(Z)H_0(-Z) + G_1(Z)H_1(-Z)\} \tag{15}$$

This implies that $Y(Z)$ is a linear combination of $X(Z)$ and $X(-Z)$ in Z domain. If the term $X(-Z)$ would not have been there, $Y(Z)$ would have simply be the $X(Z)$ passed though a filter with function $\tau_0(Z)$ like a LSI system. Dependence on $X(-Z)$ is what brings the trouble in the equation. To understand this, let us first interpret what the term X(-Z) spectrally means and what it reflects in time domain.

Effect of X(-Z)

To understand effect of $X(-Z)$ let us go back to frequency domain by substituting $z \leftarrow e^{jw}$. It implies

$$X(-Z) = X(e^{\jmath(\omega\pm\pi)})$$

i.e. we are shifting the spectrum of $e^{\jmath(\omega\pm\pi)}$ by $+\pi$ or $-\pi$. Due to periodicity on the ω axis of 2π, shifting by $+\pi$ or $-\pi$ is equivalent. The impact of this shift can be best understood by taking an example. Here $X(e^{\jmath(\omega)})$ is the Fourier transform of some sequence $x[n]$ figure 6. But

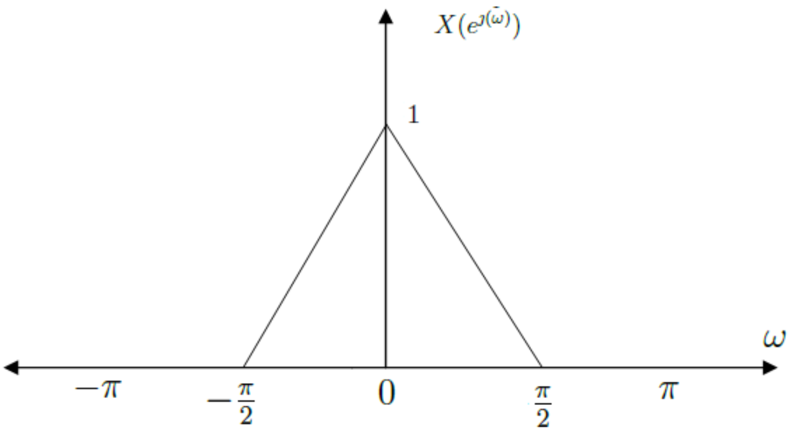

Figure 6: Spectrum of $X(\omega)$

this is just the principal interval of ω axis. Actually, $X(e^{\jmath(\omega)})$ looks as shown in figure 7.

After shifting by $+\pi$ or $-\pi$, spectrum becomes as shown in figure 8. Thus it is clearly seen that shifting by $+\pi$ or $-\pi$ is same. Notation A and $\bar{\text{A}}$ is justified since for real $x[n]$ which is a general case, $X(e^{\jmath(\omega)})$ has conjugate symmetry.

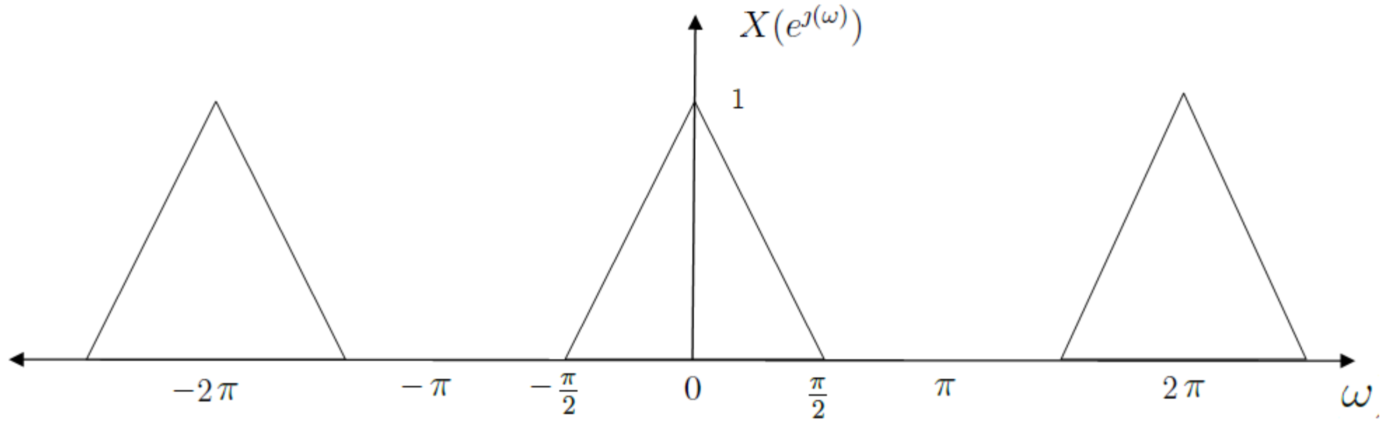

Figure 7: Periodic nature of spectrum of $X(\omega)$

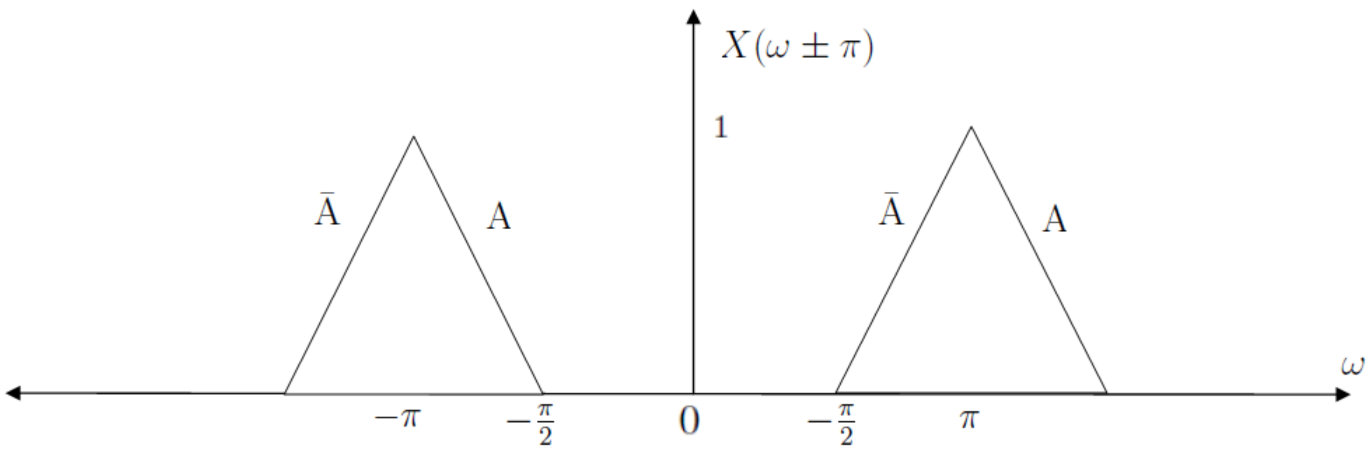

Figure 8: Spectrum of $X(\omega)$ after shift of $\pm\pi$

Redrawing the spectrum by shifting the $X(e^{\jmath(\omega)})$ by $+\pi$ or $-\pi$. For clarity in future, $X(e^{\jmath(\omega)})$ is written as $X(\omega)$. We mark the edges of spectrum as A and Ā carefully. A which has been

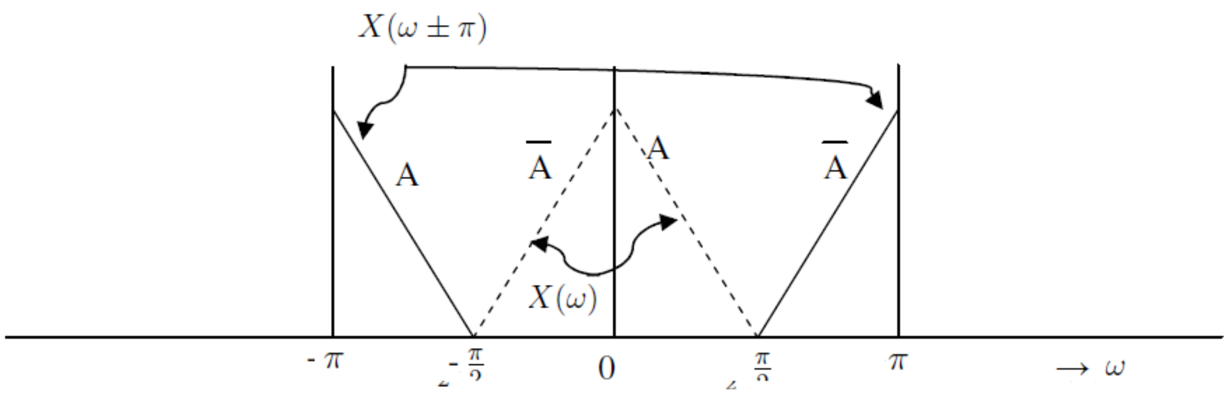

Figure 9: Spectrum of $X(\omega)$ in the principal interval

shifted to region π to $\frac{3\pi}{2}$ is repeated in region $-\pi$ to $-\frac{\pi}{2}$. In figure 9, solid line represents

$X(\omega \pm \pi)$ while dashed line represents original $X(\omega)$. From the above diagram, we see clearly how the position of A and $\overline{A}$ has been modified from its original position. The so called negative frequencies between $\frac{\pi}{2}$ to 0 has now appeared between $\frac{\pi}{2}$ to π. This has caused two things to happen.

- The order of frequency has been reversed. Frequency which was initially ordered as 0 to $\frac{-\pi}{2}$ has now been reordered between $\frac{\pi}{2}$ to π. Larger frequency has now become smaller and vice versa. For e.g. frequency $\frac{\pi}{4}$ which was larger than $\frac{\pi}{8}$ appears as a smaller frequency in shifted spectrum.

- The frequency themselves has changed. In other words, frequencies have attained a false identity.

This is exactly the same phenomenon as aliasing. It occurs in sampling if the input signal is not sampled with the adequate rate. It has occurred due to the down-sampler used in the process. By using the down-sampler, we have allowed the possibility of aliasing. The term $X(-Z)$ brings forth the consequences of that would be there if aliasing is allowed to be remain present. It is due to this fact that term $X(-Z)$ is called aliasing term. In Two Channel Filter Bank where we want perfect reconstruction, this aliasing should be absent. Perfect reconstruction means after you analyze (decompose), and finally reconstruct (synthesis), $Y(Z)$ is exact replica of $X(Z)$. When the filters are chosen properly for example as in case of Haar this condition is fulfilled. Thus in general term $X(-Z)$ is troublemaker. The first step to ensure the prefect reconstruction is to do away with aliasing term.

4 Aliasing Cancellation

When we say that we do not want aliasing to appear at the output it essentially means that $\tau_1(Z) = 0$. In terms of filter response we want

$$G_0(Z)H_0(-Z) + G_1(Z)H_1(-Z) = 0$$

If we explicitly express the synthesis filter in terms of the analysis filter we can easily ensure that $\tau_1(Z) = 0$. Expressing synthesis filter as required can be done by simply rearranging this equation.

$$\frac{G_1(Z)}{G_0(Z)} = -\frac{H_0(-Z)}{H_1(-Z)} \tag{16}$$

Very simple choice for this condition is:

$$G_1(Z) = \pm H_0(-Z)$$

$$G_0(Z) = \mp H_1(-Z) \tag{17}$$

This is a simple choice but definitely not the only choice. We can allow the factor common in both numerator as well as denominator which gets cancelled when we divide them. So more generally,

$$G_1(Z) = \pm R(Z)H_0(-Z) \tag{18}$$

$$G_0(Z) = \mp R(Z)H_1(-Z) \tag{19}$$

where R(Z) is factor cancelled. Interpretation of $G_1(Z) = \pm H_0(-Z)$ can be shown as following. Ideally, a $H_0(Z)$ is a low pass filter with cutoff frequency of $\frac{\pi}{2}$ with frequency spectrum as shown below. Then $H_0(-Z)$ has spectrum as shown in Figure 11. This is nothing but a spectrum of

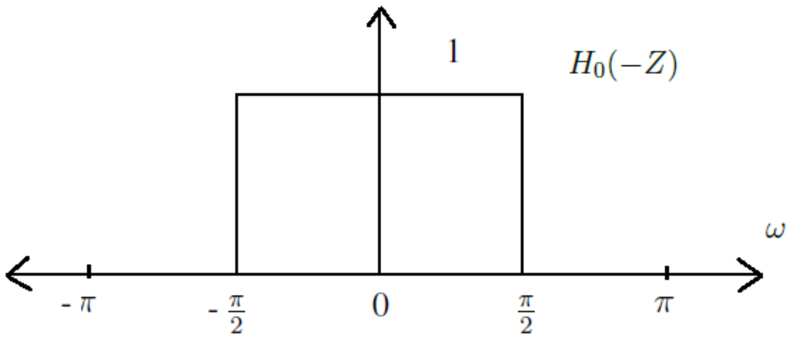

Figure 10: Spectrum of Ideal Low Pass Filter

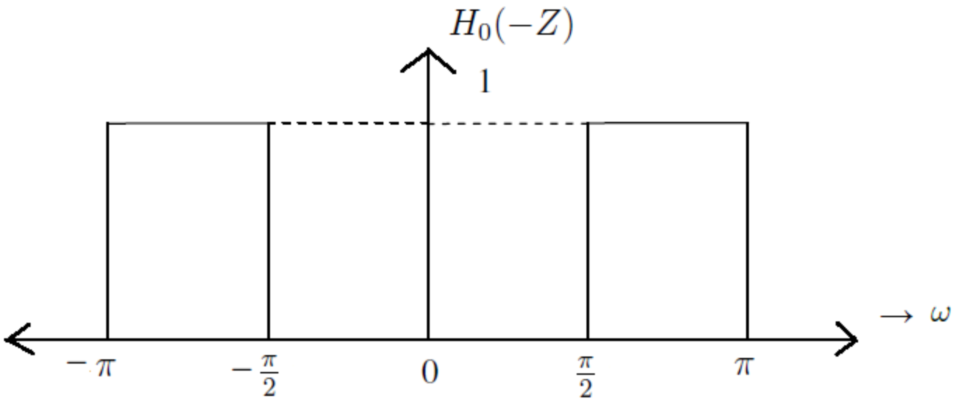

Figure 11: Spectrum of Ideal High Pass Filter

ideal high pass filter with cut-off $\frac{\pi}{2}$. Thus relationship $G_1(Z) = +H_0(-Z)$ makes a lot of sense for ideal filter. Same analogy goes for $G_0(Z) = -H_1(-Z)$. Since this is just a magnitude plot, effect of minus sign is not visible here. It only adds additional phase of $\pm\frac{\pi}{2}$ to the system.
Thus, we see that how this simple choice for $G_0(Z)$ and $G1_(Z)$ makes sense for ideal filter. If we generalize our choice with factor R(Z), there is slightly more modification than just low-pass to high-pass and vice-versa. With this choice we have completed the first step towards perfect reconstruction. Next step which includes perfect reconstruction property to be proved will be done in next class.