WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 10: Analysis of Up and Downsampling

Prof.V.M.Gadre, EE, IIT Bombay

1 Two Channel Filter Bank:

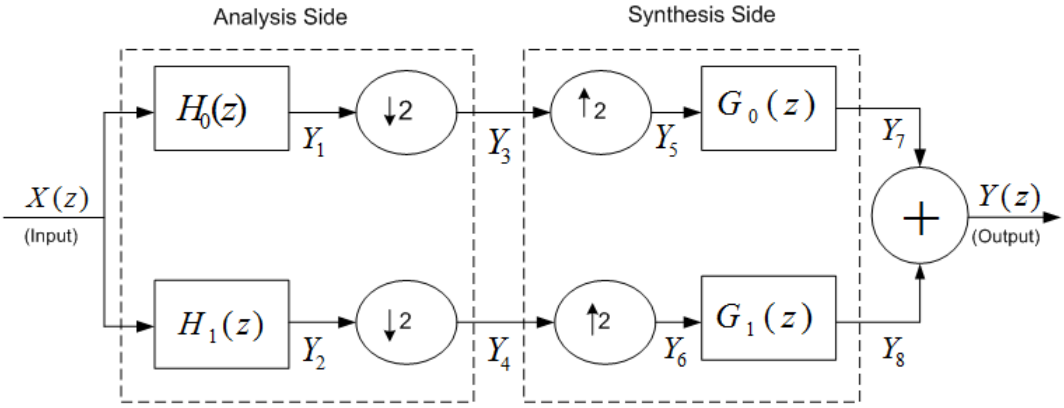
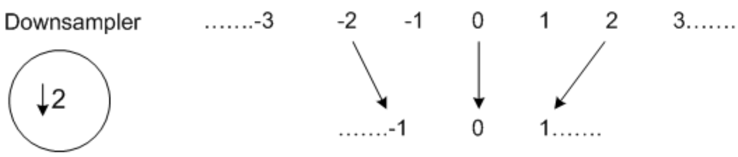


Figure 1: Two Channel Filter Bank

$H_0(z)$: Discrete time analysis Lowpass filter with angular cutoff frequency $\pi/2$
$G_0(z)$: Discrete time synthesis Lowpass filter with angular cutoff frequency $\pi/2$
$H_1(z)$: Discrete time analysis Highpass filter with angular cutoff frequency $\pi/2$
$G_1(z)$: Discrete time synthesis Highpass filter with angular cutoff frequency $\pi/2$

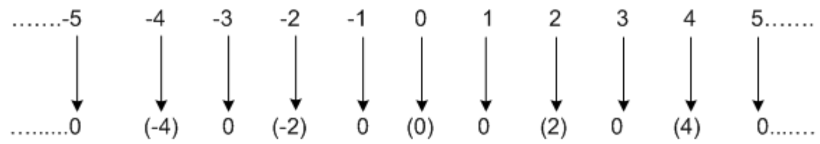
The signals $X, Y_1, Y_2, Y_5, Y_6, Y_7, Y_8, Y$ are all of sequences of sampling rate x. The signal Y_3, Y_4 are sequences of sampling rate $x/2$.
The unusual and new blocks in the above shown block diagram are the upsampler and downsampler.

Analysis of Downsampler operation:
Let us consider the downsampler block in detail. A basic downsampling operation is as follows:



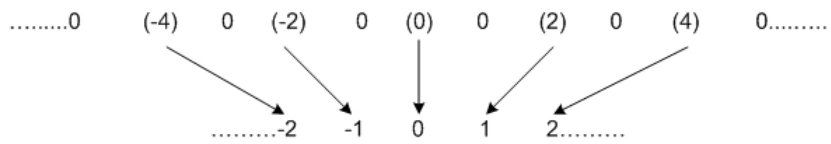
The above sequence of numbers represent the indices of the samples of a signal prior to downsampling, the bottom sequences of numbers represents the resultant indices of the signal after subjecting it to downsampling operation. Hence a downsampling operation can be viewed as the combination of 2 steps as follows:

1. It first "kills" some samples as explained below:

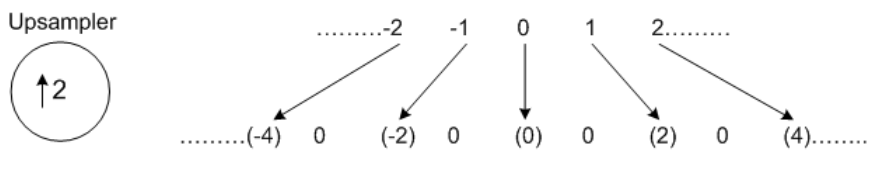


where $(\cdot)$ represent the indices of the samples which are retained without killing and those with no braces represent the killed samples of the given sequence.

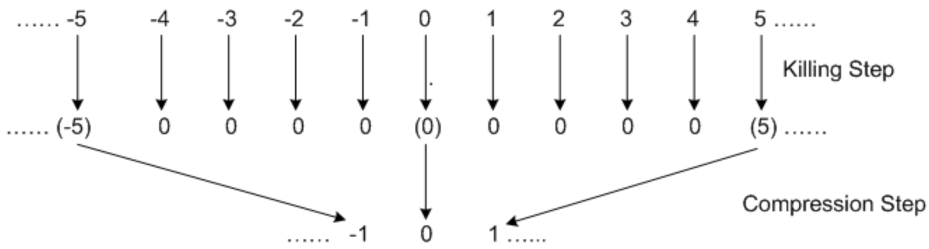
2. Then it compresses the resultant as shown below:



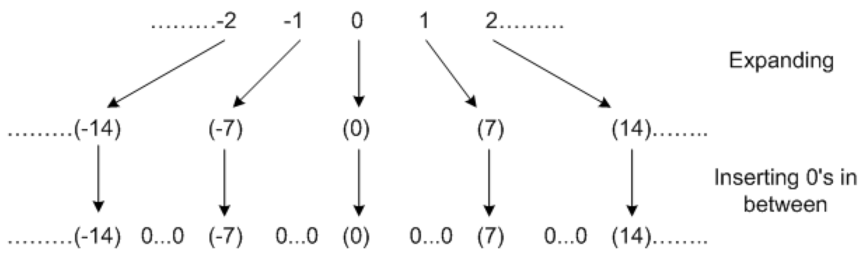
Analysis of Upsampler operation:



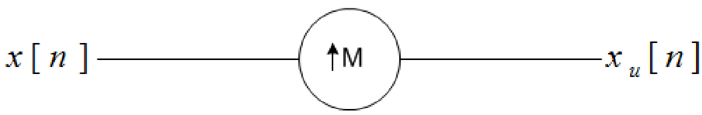
The upsampler outdoes the last 'compression' step of a downsampler. Hence it is also called Expander. The upsampler operation is invertible, we can get the original sequence i.e. the sequence prior the upsampling it converters a sequence of lower sampling rate to a sequence of higher sampling rate, whereas the downsampling operation is not invertible. The downsampler converts a sequence of higher sampling rate to a sequence of lower sampling rate. Downsampler and Upsampler operations are taken care of by clocking rates at different points in the system. Consider the downsampling a sequence by M and let $M = 5$, this is as shown below:



Consider the Upsampling operation by M. Assume $M = 7$. Then it is as follows:



Z-Domain effect of Upsampling:

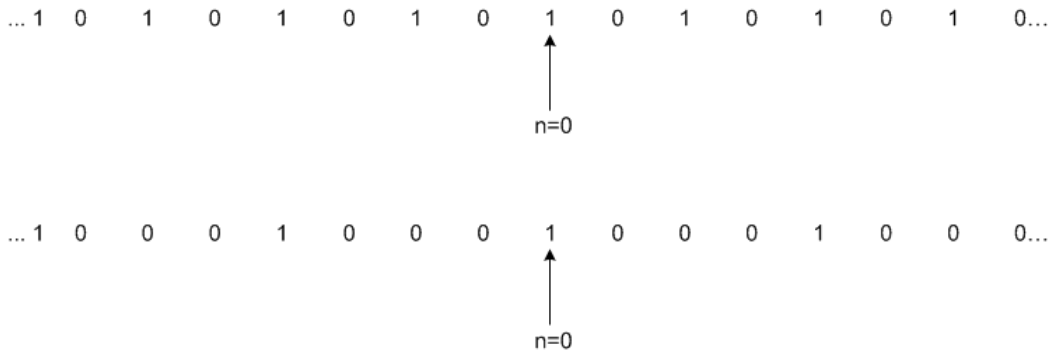


$$X_u[z] = \sum_{-\infty}^{+\infty} x_u[k] z^{Mk}$$
$$X_u[z] = X[z^M]$$
$$X[z] = X_u[z^{\frac{1}{M}}]$$

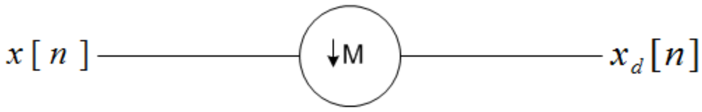
Therefore it is clearly evident that the upsampling operation is invertible.

Z-Domain effect of Downsampling:

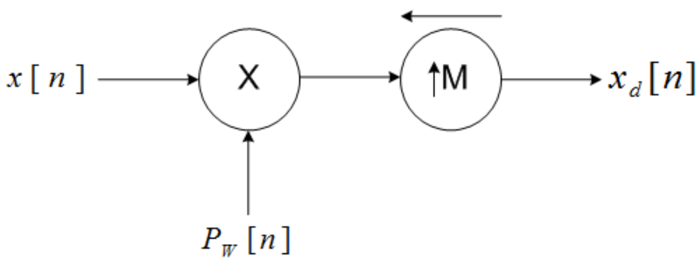
First "killing step" can be viewed as a "modulation" or "multiplication by killing sequence" (a window). We pass those values through the window which are retained. Window periodic sequence: $P_W[n]; P$ represents periodic and W represents windowed.



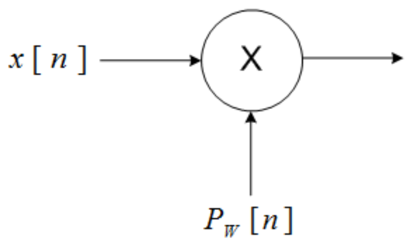
The process of killing is essentially multiplication by $P_W[n]$ for the appropriate M.



Can be written as



where the bar on upsampler denotes the inverse upsampling operation.
Multiplying a sequence by a sequence in $Z-$domain is best done by exponentials. So, the non-invertible part of the downsampling operation must be replaced.



In other words, we need to express $P_W[n] = \sum_{k=1}^{Q} C_k \alpha_k^n$ DFT represents time limited periodic sequences as the combination of exponentials,i.e., time limited sequence means $b_0, b_1, ..., b_{N-1}$.

Its Discrete Fourier Transform (DFT) is given as:

$$B[k] = \sum_{n=0}^{N-1} b[n] e^{-j\frac{2\pi}{N}nk} \quad where \quad k = 0, 1, ..., N-1$$

We can reconstruct $b[n]$ by the IDFT equation defined as

$$b[n] = \frac{1}{N} \sum_{k=0}^{N-1} B[k] e^{j\frac{2\pi}{N}nk} \quad where \quad n = 0, 1, ..., N-1$$

even if reconstructed for $n > N-1$ up to ∞, we can get a sequence periodic in N. The expression $\sum_{k=0}^{N-1} B[k] e^{j\frac{2\pi}{N}nk}$ generates a periodic sequence say $\tilde{b}[n]$.

$$\begin{aligned} \tilde{b}[n + lN] &= \sum_{k=0}^{N-1} B[k] e^{j\frac{2\pi}{N}(n+lN)k} \\ &= \sum_{k=0}^{N-1} B[k] e^{j\frac{2\pi}{N}nk} . e^{j2\pi l} \\ \tilde{b}[n + lN] &= \tilde{b}[n] \end{aligned}$$

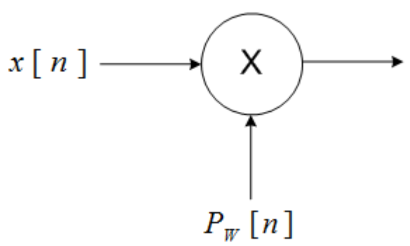
For M=2, we have a periodic sequence ...1 0 1 0 1 0 1 0 1 0 1...
where one period of the sequence is 1 0, taking DFT of the one period 1 0 is given as

$$B[k] = 1.e^{-j\frac{2\pi}{2}} + 0 = 1$$

$P_W[N]$ for $M = 2$ is

$$\begin{aligned} &= \frac{1}{2} \sum_{k=0}^{1} B[k] e^{j\frac{2\pi}{2}nk} \\ &= \frac{1}{2} \sum_{k=0}^{1} 1.e^{j\frac{2\pi}{2}nk} \\ &= \frac{1}{2}[1^n + (-1)^n] \end{aligned}$$

Hence



gives $Z-$transform as

$$
\begin{aligned}
&= \sum_{n=-\infty}^{+\infty} x[n]\left(\frac{1}{2}[1^n + (-1)^n]\right) z^{-n} \\
&= \frac{1}{2}\sum_{n=-\infty}^{+\infty} x[n]z^{-n} + \frac{1}{2}\sum_{n=-\infty}^{+\infty} x[n](-z)^{-n} \\
&= \frac{1}{2}\left(X[z] + X[-z]\right)
\end{aligned}
$$

This is the modulation operation. So, it is followed by an inverse upsampler operation for the downsampling operation to be completed.

$$X_d[n] = \frac{1}{2}\left(X\left[z^{\frac{1}{2}}\right] + X\left[-z^{\frac{1}{2}}\right]\right)$$

Additional information:
Downsampling operation in time domain corresponds to Aliasing in the frequency domain unless the given signal is sufficiently bandlimited in time. It can be pictorially explained as follows:

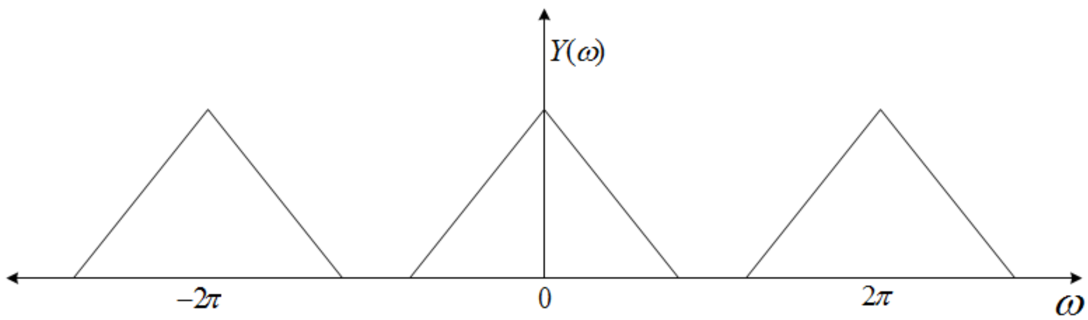


Figure 2: Signal spectrum before downsampling operation

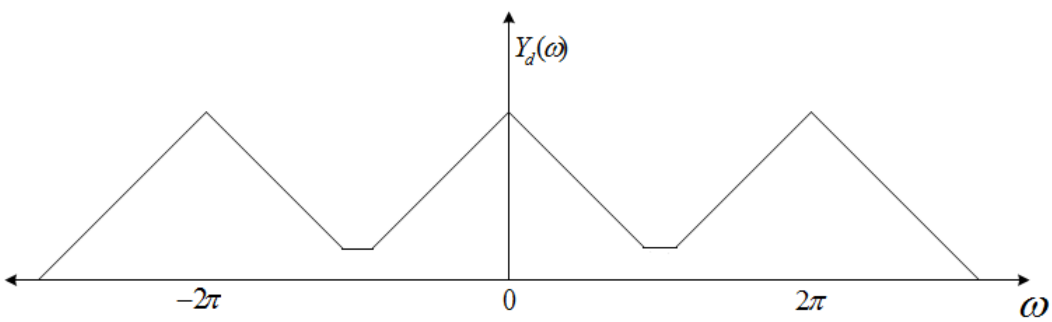


Figure 3: Signal spectrum after downsampling operation

The first figure shows the discrete time fourier transform of a signal without downsampling

operation. The signal then subjected to the downsampling operation may result in aliasing in the spectrum of the signal as shown in the second figure above. It may not be with every downsampling cases as explained below:
Example: Let $x[n]$ is a discrete time signal given by

$$x_1[n] = [-1, 2, 1, 0, 3, -2], \quad n = 0, 1, ..5$$

$$x_2[n] = [-1, 0, 1, 0, 3, 0], \quad n = 0, 1, ..5$$

if the above signals are subjected to downsampling operation by a factor of 2, then the signal $x_1[n]$ will suffer aliasing and signal $x_2[n]$ will not suffer aliasing.
Hence the downsampling operation is not invertible when it result in the aliasing of the spectrum in the frequency domain, while it is invertible when it doesnot result in aliasing.
Whereas the upsampling operation is always invertible.
The aliasing that is noticed in the 2-band filter bank is because of the downsampling operations involved in the process.