1   Introduction

In last lecture we have derived ideal two band filter bank. Frequency response of ideal filter is obtained for HAAR filter bank. In this lecture we will study the relation between filter coefficients and basis vectors. HAAR filter bank case is particularly considered for calculations.

2   Iterating the filter bank for $\phi(\cdot)$, $\psi(\cdot)$

Generic dilation equation relates filter bank to scaling function and filter bank to wavelets.
If $h[n]$: low pass filter impulse response.
Now we will relate the operation of LPF $H(z)$ on basis of input signal $\phi(t)$ as

$$\phi(t) = \sum_{n=0}^{\infty} h[n]\phi(2t - n)$$
$$= \phi(2t) + \phi(2t - 1)$$

Let the **fourier transform of** $\phi(t)$ be $\hat{\phi}(\Omega)$

$$\hat{\phi}(\Omega) = \int_{-\infty}^{+\infty} \phi(t)e^{-j\Omega t}dt$$
$$= \int_{-\infty}^{+\infty} \sum_{n=0}^{\infty} h[n]\phi(2t - n)e^{-j\Omega t}dt$$
$$= \sum_{n=0}^{\infty} h[n] \int_{-\infty}^{+\infty} \phi(2t - n)e^{-j\Omega t}dt$$
$$= \sum_{n=0}^{\infty} h[n]\frac{1}{2}e^{\frac{-j\Omega n}{2}} \int_{-\infty}^{+\infty} \phi(\lambda)e^{-j\Omega\lambda/2}d\lambda$$
$$= \sum_{n=0}^{\infty} h[n]\frac{1}{2}e^{\frac{-j\Omega n}{2}}\hat{\phi}\left(\frac{\Omega}{2}\right)$$
$$\hat{\phi}(\Omega) = \frac{1}{2}H\left(\frac{\Omega}{2}\right)\hat{\phi}\left(\frac{\Omega}{2}\right)$$

Recursively we can write,

$$\hat{\phi}\left(\frac{\Omega}{2}\right) = \frac{1}{2}H\left(\frac{\Omega}{4}\right)\hat{\phi}\left(\frac{\Omega}{4}\right)$$

Hence after N recursions,

$$\hat{\phi}(\Omega) = \left\{\prod_{m=1}^{N} \frac{1}{2}H\left(\frac{\Omega}{2^m}\right)\right\}\hat{\phi}\left(\frac{\Omega}{2^N}\right) \qquad \text{where} N \in \mathbb{Z}$$

As,

$$N \xrightarrow{lim} \infty \frac{\Omega}{2^N} = 0$$

Therefore,

$$\hat{\phi}\left(\frac{\Omega}{2^N}\right) = \hat{\phi}(0)$$

$$\hat{\phi}(\Omega) = \left\{\prod_{m=1}^{N} \frac{1}{2} H\left(\frac{\Omega}{2^m}\right)\right\}\hat{\phi}(0)$$

where $\hat{\phi}(0)$ = constant. Therefore we can express the fourier transform $\hat{\phi}(\Omega)$ completely in terms of product of dilated LPF transfer function $H(\Omega)$ as above.

The product of dilated terms in frequency domain implies the convolution of impulse response $h[n]$, subject to above dual equation of Fourier transform.

Now consider the case of HAAR wavelet here a $= \frac{1}{2}$

$h[n]$ and its fourier transform is shown in Figure 1.

Figure 1: $h[n]$ and $H(\Omega)$

$$\text{As } h[n] = \{1, 1\} \stackrel{F.T}{\Longleftrightarrow} H(\Omega) = (1 + e^{-j\Omega n})$$

$$\text{since } h[n] \stackrel{F.T}{\Longleftrightarrow} H(\Omega)$$
$$h\left[\frac{n}{a}\right] \stackrel{F.T}{\Longleftrightarrow} |a|.H(a\Omega)$$

Here $\delta \rightarrow 0$

Now consider a term

$$\prod_{m=1}^{N} \frac{1}{2} H\left(\frac{\Omega}{2^m}\right)$$

let us take first N=3 for our convenience.

$$H\left(\frac{\Omega}{2}\right).H\left(\frac{\Omega}{4}\right).H\left(\frac{\Omega}{8}\right)$$

Now put $\frac{\Omega}{8} = \lambda$

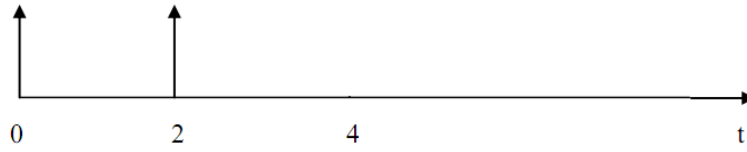Hence the product be $H(4\lambda).H(2\lambda).H(\lambda)$
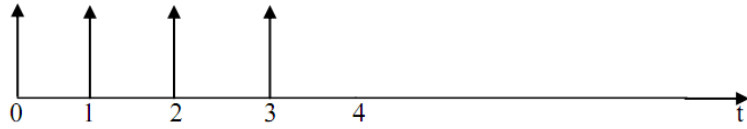
Figure 2: $h[n]$ upsampled by 2

Figure 3: $h[n]$*($h[n]$ upsampled by2)

In time domain $H(4\lambda)$ means $h[n]$ upsampled by 4 and $H(2\lambda)$ is $h[n]$ upsampled by 2.
Now find out the convolution $h[n]$ *($h[n]$ upsampled by 2)* ($h[n]$ upsampled by 4).
As we have $h[n]$ shown in above Figure 1.
($h[n]$ upsampled by 2) is shown in Figure 2.
$h[n]$*($h[n]$ upsampled by 2) is shown in Figure 3.
$h[n]$ upsampled by 4 is shown in Figure 4.
Now the convolution shown in figure 3 convolves with impulse response ($h[n]$ upsampled by 4)
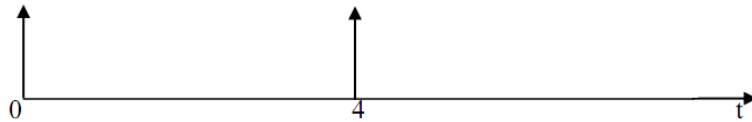which is shown in Figure 5.

Figure 4: $h[n]$ upsampled by 4

Now replacing $\dfrac{\Omega}{8} = \lambda$ as previous assumption. Since $\dfrac{\Omega}{8}$ means expansion in frequency domain,
hence in time domain we need to contract signal by the same factor hence contracting the
resulting convolution term by factor of 8 we get as shown in Figure 6.
If we consider N=8 $i.e.$, considering infinite iterations we get continuous signal from 0 to 1 in
time domain as shown in Figure 7.
Similarly we can construct the wavelet function as it is a function of basis function.

$$\psi(t) = \sum_{n=0}^{\infty} g[n]\phi(2t - n)$$

In terms of haar wavelet $g[n] = \{1, -1\}$

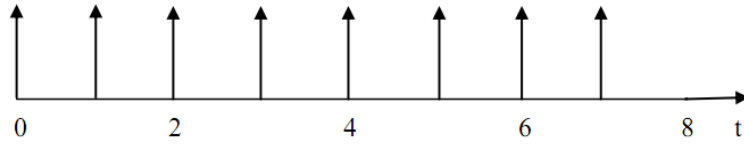$$\psi(t) = \phi(2t) - \phi(2t - 1) \qquad\qquad for N = 1$$

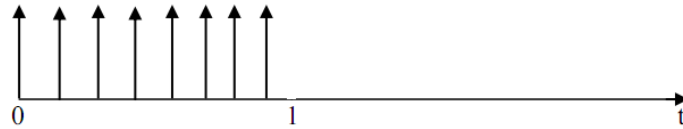9 - 3

Figure 5: $h[n]*(h[n]$ upsampled by 2)*$(h[n]$ upsampled by 4)

Figure 6: after substituting $\Omega/8 = \lambda$

Let the **fourier transform of $\psi(t)$** be $\hat{\psi}(\Omega)$

$$\hat{\psi}(\Omega) = \int_{-\infty}^{+\infty} \psi(t)e^{-j\Omega t}dt$$

$$= \int_{-\infty}^{+\infty} \sum_{n=0}^{\infty} g[n]\phi(2t-n)e^{-j\Omega t}dt$$

$$= \sum_{n=0}^{\infty} g[n] \int_{-\infty}^{+\infty} \phi(2t-n)e^{-j\Omega t}dt$$

$$= \sum_{n=0}^{\infty} g[n]\frac{1}{2}e^{\frac{-j\Omega n}{2}} \int_{-\infty}^{+\infty} \phi(\lambda)e^{-j\Omega\lambda/2}d\lambda$$

$$= \sum_{n=0}^{\infty} g[n]\frac{1}{2}e^{\frac{-j\Omega n}{2}} \hat{\phi}\left(\frac{\Omega}{2}\right)$$

$$\hat{\psi}(\Omega) = \frac{1}{2}G\left(\frac{\Omega}{2}\right)\hat{\phi}\left(\frac{\Omega}{2}\right)$$

Recursively we can write,

$$\hat{\psi}\left(\frac{\Omega}{2}\right) = \frac{1}{2}G\left(\frac{\Omega}{4}\right)\hat{\phi}\left(\frac{\Omega}{4}\right)$$
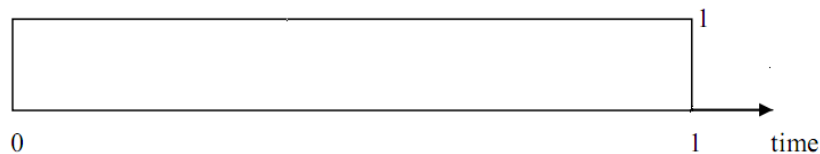

Figure 7: after infinite iterations
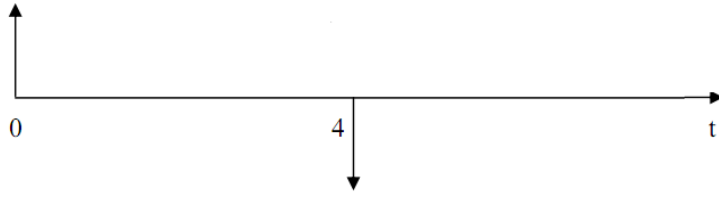
9 - 4

Figure 8: $g[n]$ upsampled by 4

Figure 9: $h[n]$ upsampled by 2

Hence the recursive equation be,

$$\hat{\psi}(\Omega) = \frac{1}{2}G\left(\frac{\Omega}{2}\right)\frac{1}{2}H\left(\frac{\Omega}{4}\right)\frac{1}{2}H\left(\frac{\Omega}{8}\right)\ldots\ldots\hat{\phi}(0)$$

$$\psi(\Omega) = \frac{1}{2}G\left(\frac{\Omega}{2}\right)\left\{\prod_{m=2}^{N}\frac{1}{2}H\left(\frac{\Omega}{2^m}\right)\right\}\hat{\phi}(0)$$

Now, let N=3.

$$G\left(\frac{\Omega}{2}\right).H\left(\frac{\Omega}{4}\right).H\left(\frac{\Omega}{8}\right)$$

Now put $\dfrac{\Omega}{8} = \lambda$

Hence the product be $G(4\lambda).H(2\lambda).H(\lambda)$

In time domain $G(4\lambda)$ means $g[n]$ upsampled by 4 and $H(2\lambda)$ is $h[n]$ upsampled by 2.

Now find out the convolution $h[n] * (h[n]$ upsampled by 2) * ($g[n]$ upsampled by 4) for wavelet function.

Now let us see this convolution.

$g[n]$ upsampled by 4 is shown in Figure 8.

$h[n]$ upsampled by 2 is shown in Figure 9.

$h[n]$ * ($h[n]$ upsampled by 2) is shown in Figure 10.

The convolution output obtained in Figure 10 (*i.e.*, $h[n]*h[n]$ upsampled by 2) is convolved with $g[n]$ upsampled by 4 is shown in Figure 11.

Now again considering the assumption $\dfrac{\Omega}{8} = \lambda$, contract the signal by factor 8 as explained
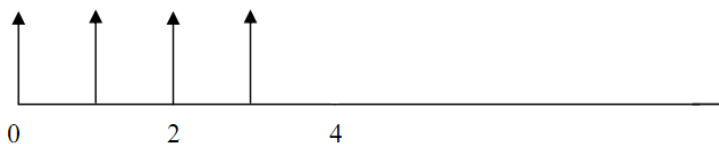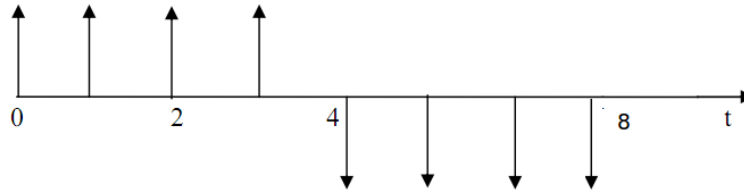

Figure 10: $h[n]*h[n]$ upsampled by 2

9 - 5

Figure 11: $h[n]$*($h[n]$ upsampled by 2)*($g[n]$ upsampled by 4)
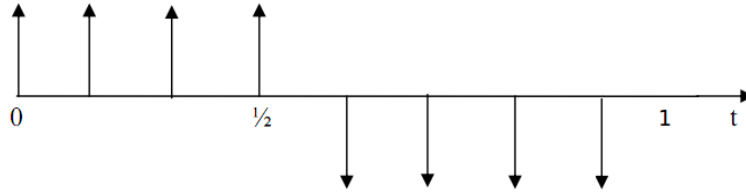

Figure 12: after substituting $\Omega/8 = \lambda$

above.

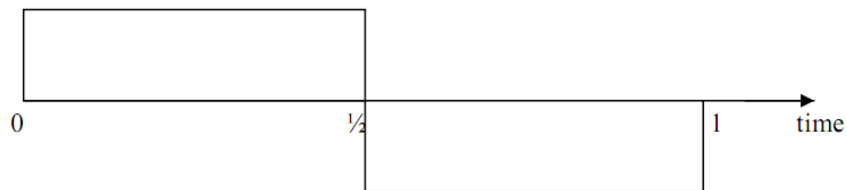Hence the resulted signal is shown in Figure 12. Hence after infinite iterations $N = \infty$ we reconstruct the $\psi(t)$ as shown in Figure 13.

Figure 13: After infinite iterations