

## 1 Introduction

In this lecture we will learn how to relate  $\psi(t)$  and  $\phi(t)$  of the MRA to the filter bank by studying the generic structure of the analysis and synthesis filter bank. It is two band because of the dyadic nature of the scaling function i.e. for the scaling function we allow the scaling in powers of TWO. We will arrive at the general structure of the analysis and the synthesis filter banks based on studying the Haar MRA.

## 2 Haar analysis filter bank

The analysis filter bank is shown in figure 1. Recall that in the previous lecture, we had seen that the filter on the top is a crude low pass filter and filter in the bottom is a crude high pass filter. We call them as crude because ideally we would have wanted them as follows i.e. the passband of one filter should not overlap other filters stop band. We also saw that the filters together satisfy two very important properties, namely

- Magnitude complementarity, *i.e.* addition of amplitudes from both filters give the original amplitude of the signal.
- Power complementarity, *i.e.* addition of powers from individual filters gives back the original power of the signal

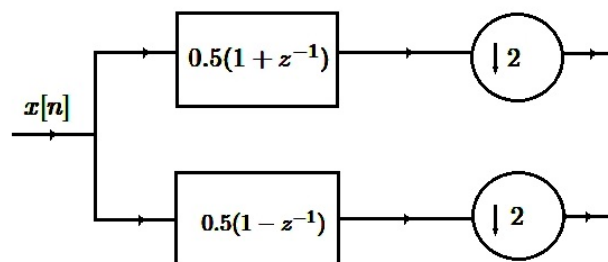


Figure 1: Haar analysis filter bank

## 3 Why is Haar filter not ideal?

To understand this, we should ask ourselves, what is the ideal response we would like to have? Let us have a look at both the filter responses of the Haar filter bank.

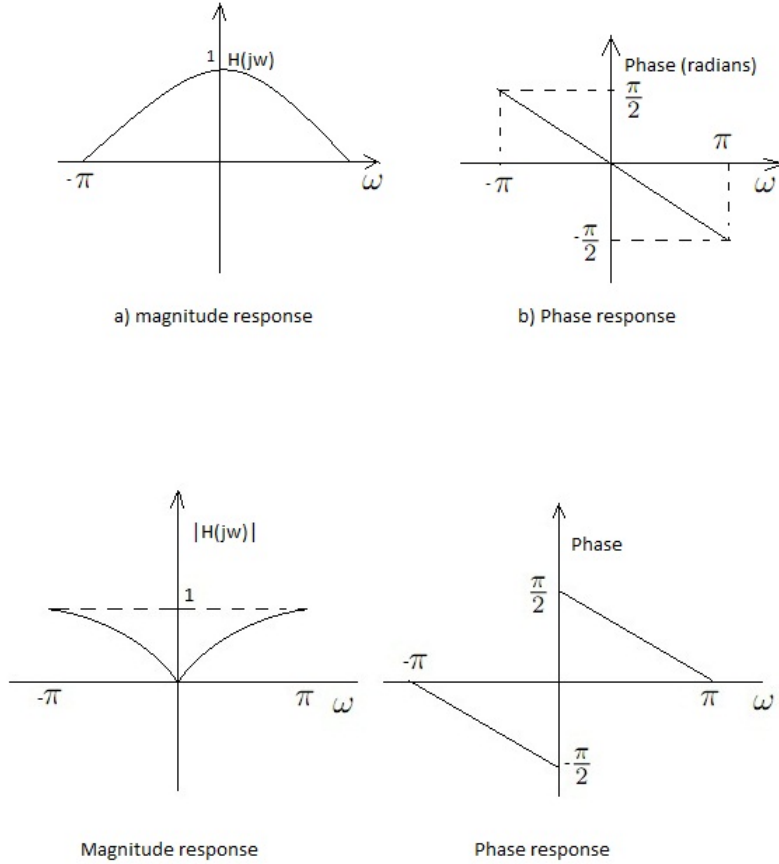


Figure 2: Magnitude and phase responses of analysis and synthesis filters of the Haar MRA

The value of magnitude response is same for both the filters at  $\omega = \frac{\pi}{2}$ . What we ideally want is a perfect high pass and a perfect low pass filter characteristics, as in the figure 3. It is clearly seen that the Haar filters are far away from ideal filter responses which we desire. The ideal filter bank would have the structure similar to that in figure 4. In ideal filter banks, the filters are identical. However the problem with ideal filters is that they are unrealizable and to reason it out we require the impulse responses of these ideal filters. This is not due to the limitations of technology at present but due to the nature of the system. The impulse response  $h[n]$  of a filter is obtained by taking the inverse Fourier transform of the filter frequency response as :

$$h_{ideal}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{ideal}[\omega] e^{j\omega n} d\omega$$

The ideal Low pass and High pass filter impulse responses are obtained by solving the above integral. The impulse responses of the ideal Low pass and High pass filters are :

$$\begin{aligned} h_{idealLPPF}[n] &= \frac{\sin(\pi n/2)}{(\pi n)}, & n \neq 0 \\ &= 1/2, & n = 0 \end{aligned}$$

and

$$h_{idealHPF}[n] = \frac{\sin(\pi n) - \sin(\pi n/2)}{(\pi n)}, \quad n \neq 0$$

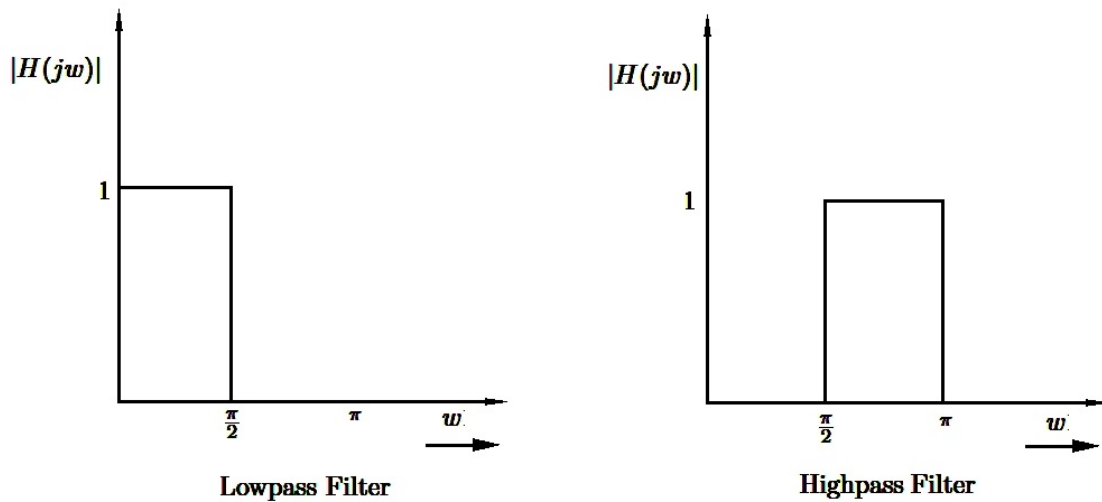


Figure 3: Ideal filters magnitude response

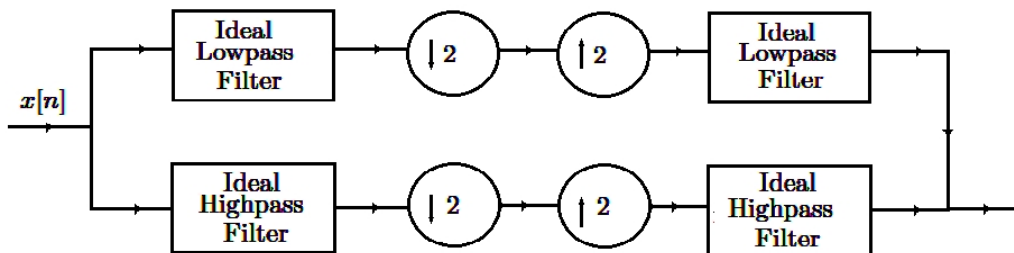


Figure 4: Ideal filter bank structure

$$= 1/2, \quad n = 0$$

The above impulse responses show that the ideal filters are :

- Infinitely non-causal:  
The impulse response extends infinitely towards left side i.e. for the negative values of index  $n$  which makes it non causal. For finite amount of non-causality, *i.e* only a finite number of  $t$  Hence the infinite non-causal nature of the ideal filters makes it impossible to realize them.
- Unstable:  
A linear time invariant system is said to be stable in BIBO (Bounded Input Bounded Output) sense if it produces a bounded output for a bounded input. The condition on the impulse response to attain stability is given as  $\sum_{n \in \mathbb{Z}} |h[n]| < \infty$  i.e. the absolute sum of the impulse response coefficients should be finite. For ideal filters this absolute sum of impulse response coefficients diverges and *may not* produce a bounded output for a bounded input and are therefore unstable. Intuitively the system may be stable for many

of the known input signal but for some peculiar signal the system may give unbounded output i.e. the system may become unstable. In other words for a bounded input the system may give unbounded output.

- Irrational:

The most important disqualification is that these systems are IRRATIONAL. Rationality and irrationality are the characteristics of linear time invariant (LTI) systems which have System function i.e. their  $Z$ -transforms exist in some finite region of convergence. A filter system function is said to be rational if it can be expressed in terms of the ratio of two finite series in powers of  $z$ . Ideal filters cannot be expressed as a ratio of two **finite** series in powers of  $z$  and are therefore irrational or unrealizable. By non-realizability we mean that the system cannot be realized by physical means. There are no known techniques to realize irrational filters today as they require infinite resources. Suppose we want to generate exponential signals they can easily be generated by using R-C components. Irrationality must not be confused with stability and causality. Let us see an example of an irrational system function.

The function  $e^{z^{-1}}$ ,  $|z| > 0$  is irrational and can be expanded as:

$$e^{z^{-1}} = \sum_{n=0}^{\infty} \frac{(z^{-1})^n}{n!}$$

Thus, the inverse  $Z$ -transform for the above equation can be obtained as

$$h[n] = \frac{1}{n!} u[n]$$

This is a convergent sum. Thus we see that not all irrational systems are unstable, but it is impossible to realize this as it requires infinite resources. The ideal is not achievable many times, but we can get as close as possible to the ideal by investing more and more resources.

## 4 Realizable two band filter bank

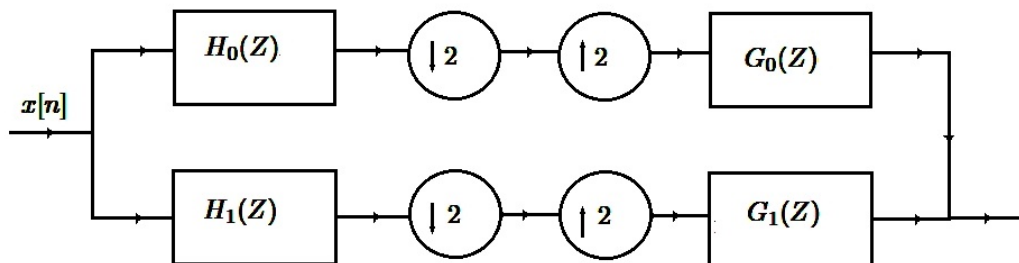


Figure 5: Haar analysis and synthesis filter banks

The above filter bank is realizable if  $H_0(Z)$ ,  $H_1(Z)$ ,  $G_0(Z)$ ,  $G_1(Z)$  are all rational system functions. If the filters must not satisfy any of the above mentioned disqualifications. Hence

the system responses  $H_0(Z), H_1(Z), G_0(Z), G_1(Z)$  must be rational, stable, finitely causal.  $H_0(Z), G_0(Z)$  aspire to be ideal low pass filters with  $\omega_c = \frac{\pi}{2}$  and  $H_1(Z), G_1(Z)$  aspire to be high pass filters with cutoff frequency  $\omega_c = \frac{\pi}{2}$  as ideal filters are not achievable.

## 5 Relation between Haar MRA and filter bank

In this section, we will try to establish a relation between the functions  $\phi(t), \psi(t)$  and the filter banks. Let us focus our attention on Haar MRA. As discussed earlier the Haar MRA is not ideal. But we get a lot of concepts that can be learned from the shortcomings of Haar MRA that can be utilized to construct better families of Dyadic Multiresolution analysis.

### 5.1 Relation between the function $\phi(t)$ and filter banks

For Haar MRA,  $\phi(t)$  is the basis of  $V_0$  i.e.  $\phi(t) \in V_0$ . Also recall that there exists a ladder of spaces in MRA which states that  $V_0 \subset V_1$ .  $\phi(t)$  should therefore be expressible in the basis of  $V_1$  i.e.  $\phi(2t - n), n \in \mathbb{Z}$ . From the figure we see that  $\phi(t)$  is expressible in its own dilates and translates.

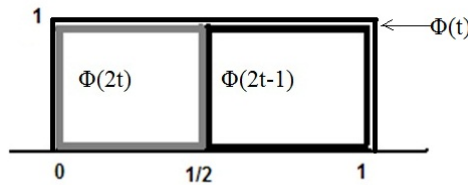


Figure 6: Expressing  $\phi(t)$  in term of dilates and translates

$$\phi(t) = \phi(2t) + \phi(2t - 1)$$

The above equation is called as the **dilation equation**. If we write down the coefficients of  $\phi(2t - n)$  as a sequence, we get the impulse responses corresponding to the Lowpass filter.

$$\begin{array}{c} 1 \quad 1 \\ \uparrow \\ 0 \end{array}$$

The dilation equation can thus be modified in a generalized way as follows:

If  $h[n]$  is the impulse response of the low pass filter in the two band filter bank then,

$$\phi(t) = \sum_{n \in \mathbb{Z}} h[n] \phi(2t - n)$$

### 5.2 Relation between the function $\psi(t)$ and filter banks

$\psi(t) \in V_0$  should also be expressible in terms of basis of  $V_1$  i.e.  $\phi(2t - n), n \in \mathbb{Z}$ .

Graphically,

From the figure, we can see that the dilation equation will be of the form

$$\psi(t) = \phi(2t) - \phi(2t - 1)$$

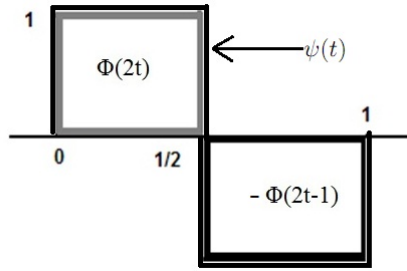


Figure 7: Expressing  $\psi(t)$  in term of dilates and translates

$$\begin{array}{cc} 1 & -1 \\ \uparrow & \\ 0 & \end{array}$$

If we write down the coefficients of  $\phi(2t - n)$  as a sequence, we get the impulse responses corresponding to the Highpass filter.

The dilation equation can thus be modified in a generalized way as follows.

If  $g[n]$  is the impulse response of the high pass filter in the two band filter bank then,

$$\psi(t) = \sum_{n \in \mathbb{Z}} g[n] \phi(2t - n)$$

This implies that, if we know the impulse responses, we can go the reverse way round to generate  $\phi(t)$  and  $\psi(t)$ .

Thus in short,

Lowpass filter  $\rightarrow$  scaling function expansion

Highpass filter  $\rightarrow$  wavelet expansion

Therefore, we can completely characterize the system  $\phi(t)$  and  $\psi(t)$  if we know their dilation equation.

## 6 The strategy for future analysis

Once we have obtained the dilation equations, our next aim is to use these equations to find the wavelet and the scaling function given the highpass and the lowpass filter responses. This is accomplished in two steps as follows:

- Take Fourier transform on both sides of the dilation equation to get a recursive equation in Fourier domain which completely characterizes  $\mathcal{F}\{\phi(t)\}$  in terms of DTFT of  $h[n]$ .
- Relate Fourier transform of wavelet function to Fourier transform of scaling function by using DTFT of  $g[n]$  to obtain the wavelet function.