WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING

Lecture 7: Frequency domain behaviour of Haar filter banks

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1 Introduction

So far we have looked at the structure of the Haar Analysis and synthesis filter bank. In this lecture, the frequency domain behaviour of the Haar MRA filter banks is explored.

2 Haar filter banks

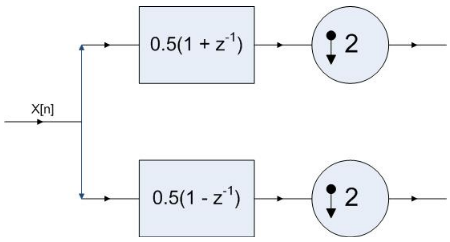

Figure 1: Haar analysis filter bank

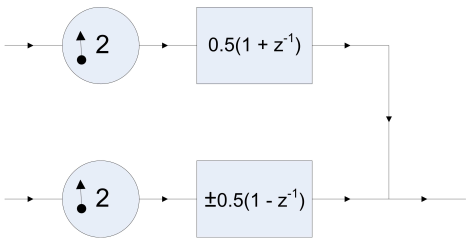

Figure 2: Haar synthesis filter bank

The reason we use $\pm$ in the synthesis filter bank is due to a slight ambiguity to determine where to place sum sample and difference sample. If '+' sign in used, sum sample would get placed at even location and difference sample at odd location. If '-' sign is used, the reverse would happen.

Important: Note the the analysis and synthesis filter banks are almost the same(except for the scaling factor).

Haar filter banks are not the ideal filter banks. It will soon be understood why. However, understanding Haar filter banks leads to clarification of many concepts of Multiresolution analysis.

3 Frequency domain behaviour

The frequency domain behaviour can be determined if we substitute $z = e^{j\omega}$. To do this , we must first ensure that the unit circle lies within the **Region of Convergence (ROC)** the Z transform.

3.1 Region of Convergence

The region of convergence of any Z-transform lies within two concentric circles of radius R_1 and R_2 as shown in the figure 3.

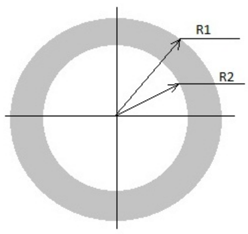

Figure 3: The ROC for any Z-transform

In general, R_1 could be ∞ and R_2 could be 0. The boundary circles may or may not be included in the ROC. If the circle with radius 1, *i.e* unit circle, is included in the ROC, the system is said to have a frequency response *i.e.*, we say the sequence has a DTFT(Discrete time fourier transform). To determine the frequency response, we substitute $z = e^{j\omega}$. Notice that $|z| = 1$, *i.e.* we are evaluating the Z-transform on the unit circle.

3.2 Analysis filter bank

Substitute $z = e^{j\omega}$ in $\frac{1+z^{-1}}{2}$, we get

$$\begin{aligned} \frac{1+e^{-j\omega}}{2} &= e^{\frac{-j\omega}{2}} \frac{e^{\frac{j\omega}{2}} + e^{\frac{-j\omega}{2}}}{2} \\ &= e^{\frac{-j\omega}{2}} cos(\frac{\omega}{2}) \end{aligned} \quad (1)$$

The magnitude of this response is given by $|cos(\frac{\omega}{2})|$ (because $|e^{\frac{-j\omega}{2}}| = 1$). Similarly the phase response is given by $\frac{-\omega}{2}$ as the $cos(\frac{\omega}{2})$ doesn't contribute to phase as it is real and positive. The graph of magnitude response is shown in the figure.

Important: We plot only for positive ω noting that magnitude is a response is an **even function** of ω and hence the complete spectrum will also involve a mirror image of the spectrum in figure about the Y-axis. Similarly, the phase response is an odd function of ω and thus $\angle H(-\Omega_0) = -\angle H(\Omega_0)$.

We see that this response approximates a crude low pass filter. For comparison, the frequency response of an ideal lowpass filter is shown in figure 5.

If we plot the phase response of the above filter we will get a filter as shown in figure 6.

We see that the phase response is a **straight line passing through the origin**.

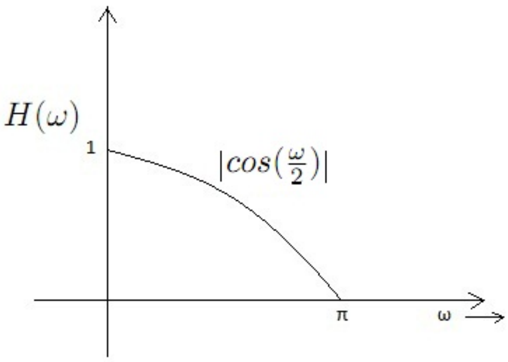

Figure 4: Magnitude response of filter

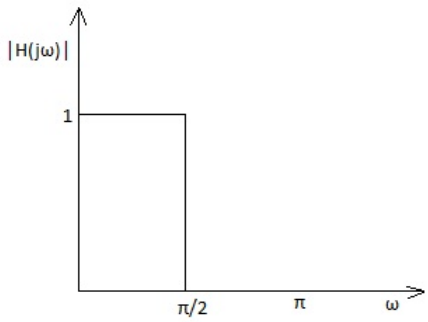

Figure 5: Magnitude response of ideal lowpass filter

3.2.1 Importance of having a linear phase

If we apply a signal $A_0cos(\Omega_0 t + \phi_0)$ to a system with frequency response given by $H(\Omega)$, then the output is given by

$$\begin{aligned} Output &= |H(\Omega_0)|A_0cos(\Omega_0 t + \phi_0 + \angle H(\Omega_0)) \\ &= |H(\Omega_0)|A_0cos\left[\Omega_0(t + \frac{\angle H(\Omega_0)}{\Omega_0}) + \phi_0\right] \end{aligned} \tag{2}$$

where $\angle H(\Omega_0)$ is the angle introduced by system function $H(\Omega_0)$. We see that this has resulted in a time shift in signal, which is dependent on signal frequency Ω_0. Thus here we see that phase is a **necessary evil**. For without phase, the system would not be causal, and phase introduces a time shift, which is, in general, different for different frequencies. If we want to preserve shape of the waveform, we can at least try that all frequencies are shifted by the same time *i.e.*

$$\frac{\angle H(\Omega_0)}{\Omega_0} = \tau_0 \ (independent\ of\ \Omega_0)$$

This implies that

$$\angle H(\Omega_0) = \Omega_0 \tau_0$$

This is an equation of a straight line passing through origin, hence called as linear phase.

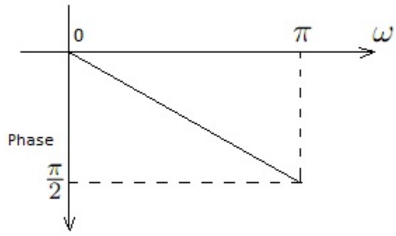

Figure 6: Phase response of ideal lowpass filter

3.3 Second filter in analysis filter bank

Substitute $z = e^{j\omega}$ in the expression $\frac{1}{2}(1 - z^{-1})$, we get

$$\frac{1}{2}(1 - e^{-j\omega}) = je^{\frac{-j\omega}{2}} sin(\frac{\omega}{2})$$

The magnitude and phase responses are shown in the figures 7 and 8 respectively.

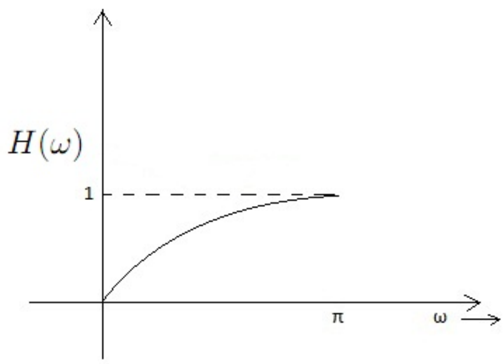

Figure 7: Magnitude response of second analysis filter

Note that in calculating phase response we get an extra term of $\frac{\pi}{2}$ due to presence of $\jmath$. The expression for phase response is thus given by

$$phase(\omega) = \frac{\pi}{2} - \frac{\omega}{2}$$

It is thus seen that although the graph is still a straight line, it **no longer passes through the origin**. This phase response is thus called **pseudo-linear** response. To summarize, we will again see the complete phase and magnitude responses of both filters. Figure 9 shows the response of the first filter and figure 10 refers to the response of the second filter. The phase response is antisymmetric about origin because the expression is an **odd** function of ω(due to presence of $sin(\frac{\omega}{2})$).
Note that in the the phase response of the second filter, at $\omega = 0$ has two values namely $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. However, since value of magnitude at $\omega = 0$ is zero, the phase doesn't matter.
If we add the functions of both filters, we get

$$\frac{1}{2}(1 + z^{-1}) + \frac{1}{2}(1 - z^{-1}) = 1$$

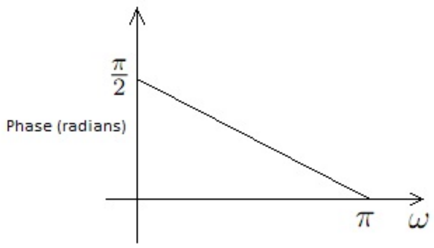

Figure 8: Phase response of second analysis filter

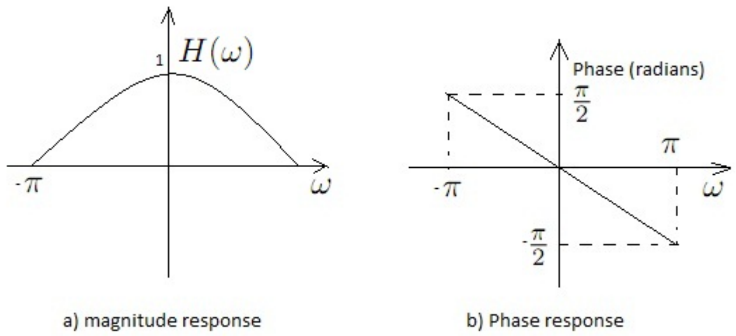

Figure 9: Magnitude and Phase response of first analysis filter

The meaning of this result is that if we pass a sine wave through both filters and add the filter outputs, we get back the original sine wave.This property is called the **Magnitude complimentarity** property of the filters

If we pass a wave of frequency ω_0 through a filter of transfer function $H(\omega)$, the power output is given by $|H(\omega_0)|^2$. If we add the power outputs of both filters, we get

$$|cos(\frac{\omega_0}{2})|^2 + |sin(\frac{\omega_0}{2})|^2 = 1$$

Thus, we see that even the sum of powers from both filters is conserved and and on addition of the filter outputs, we get the same power back. This property is known as the **Power complimentarity** property.

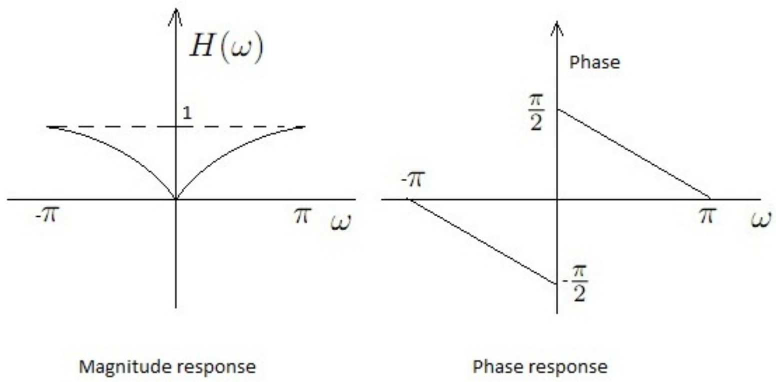

Figure 10: Magnitude and Phase response of second analysis filter